$\hbar$-expansion of integrable hierarchies
— a recursive construction of solutions—

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§0. Introduction

- $\hbar$-dependent KP and Toda appear in, e.g.,
  - string theory,
  - random matrix theory,
  - algebraic geometry (generating function in the enumerative geometry)

- **dispersionless** KP $= \lim_{\hbar \to 0} \hbar$-dependent KP
  - i.e., lowest order part (ord $\hbar = -1$), quasi-classical limit of KP
  - Explicit solutions: Kodama, Gibbons,...
  - Related to Laplacian Growth, Riemann’s mapping theorem:
    Mineev-Weinstein, Wiegmann, Zabrodin, Krichever, Teo, Gibbons, Tsarev, Takasaki, T,...
Problem: construct a solution of $\hbar$-KP (with lower order terms), starting from a solution of dKP (i.e., the highest order part)!

But we have thrown away information on lower order terms by $\lim_{\hbar \to 0}$.

How can we recover such information?

Solution of dispersionless hierarchies $\leftrightarrow$ canonical transformation

$\implies$ Use quantized canonical transformation!

Plan:

1. Dispersionless & $\hbar$-KP hierarchy
2. Recursive construction of dressing (wave) operator
3. $\hbar$-expansion of wave function
4. $\hbar$-expansion of $\tau$-function
§1. Dispersionless & $\hbar$-dependent KP hierarchy

Dispersionless KP hierarchy:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \quad n = 1, 2, \ldots,$$

where

- $\mathcal{L} = \xi + \sum_{n=1}^{\infty} u_{0,n+1}(t)\xi^{-n}$, $t = (t_1, t_2, \ldots)$ ($x = t_1$).
- $\mathcal{P}_{\geq 0}$: truncation to the polynomial part.
- $\{A(x, \xi), B(x, \xi)\} = \frac{\partial A}{\partial \xi} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial \xi}$: Poisson bracket
Dispersionless KP = “quasi-classical analog” of the KP hierarchy

KP hierarchy:

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0}, \quad n = 1, 2, \ldots,$$

where

- \( L = \partial + \sum_{n=1}^{\infty} u_{n+1}(t) \partial^{-n}, \quad t = (t_1, t_2, \ldots) \) \( (x = t_1, \partial = \partial/\partial x). \)
- \( P_{\geq 0}: \) truncation to the differential operator part.
- \([A(x, \partial), B(x, \partial)] = [A, B]: \) commutator
Dispersionless KP = "\( \hbar \rightarrow 0 \) limit" of the KP hierarchy

\( \hbar \)-KP hierarchy:

\[
\hbar \frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0}, \quad n = 1, 2, \ldots ,
\]

where

- \( L = \hbar \partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar, t)(\hbar \partial)^{-n}, t = (t_1, t_2, \ldots) \) \( (x = t_1, \partial = \partial/\partial x) \).
- \( P_{\geq 0} \): truncation to the differential operator part.
- \([A(x, \hbar \partial), B(x, \hbar \partial)] = [A, B] : \) commutator
To obtain the dispersionless KP, $L$ should be “regular”:

$$L = \hbar \partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar, t)(\hbar \partial)^{-n},$$

$$u_n(\hbar, x, t) = \sum_{m=0}^{\infty} \hbar^m u_{m,n}(x, t): \text{regular with respect to } \hbar.$$

Convenient tools: $\hbar$-order, $\hbar$-symbol

$\hbar$-order: $\text{ord}^\hbar \left( \sum_{m,n} a_{n,m}(x, t) \hbar^n \partial^m \right) \overset{\text{def}}{=} \max\{m - n \mid a_{n,m}(x, t) \neq 0\}.$

$\text{ord}^\hbar \hbar = -1$, $\text{ord}^\hbar \partial = 1$, $\text{ord}^\hbar \hbar \partial = 0$.

$u_n$: regular with respect to $\hbar \iff \text{ord}^\hbar(L) = 0$. 
\( \hbar \)-principal symbol of \( A = \sum a_{n,m}(x, t) \hbar^n \partial^m (\sim \text{highest order terms}) : \)

\[
\sigma^\hbar(A) \overset{\text{def}}{=} \sum_{m-n=\text{ord}(A)} a_{n,m}(x, t) \xi^m
\]

\[
\sigma^\hbar(AB) = \sigma^\hbar(A)\sigma^\hbar(B), \quad \sigma^\hbar([A, B]) = \{\sigma^\hbar(A), \sigma^\hbar(B)\}.
\]

\( \sigma^\hbar(\hbar\text{-KP}) = \text{dispersionless KP (dKP)} \)

\[
B_n = (L^n)_{\geq 0} \implies \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0},
\]

\[
\hbar \frac{\partial L}{\partial t_n} = [B_n, L] \implies \frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}.
\]
Dressing operator $W$ of $\hbar$-KP: $L = \text{Ad} W (\hbar \partial) = W (\hbar \partial) W^{-1}$

$\text{ord}^\hbar(L) = 0 \implies W$ has a specific form:

$$W = \exp \left( \frac{X}{\hbar} \right) (\hbar \partial)^{\alpha(\hbar)/\hbar},$$

$$X = \sum_{k=1}^{\infty} \chi_k(\hbar, x, t) (\hbar \partial)^{-k}, \quad \text{ord}^\hbar X = 0,$$

$$\alpha(\hbar) : \text{const.}, \quad \text{ord}^\hbar \alpha(\hbar) = 0.$$

(Hereafter, we assume $\alpha(\hbar) = 0$ for simplicity.)

Dressing operation for $\mathcal{L}$ of dKP:

$$\mathcal{L} = \exp(\text{ad}_{\{,\}} X_0) \xi,$$

$$X_0 := \sigma^\hbar(X), \text{ ad}_{\{,\}}(f)(g) := \{f, g\}$$
Orlov-Schulman operator $M$ of $\hbar$-KP:

\[ M = \text{Ad} \left( W \exp \left( \hbar^{-1} \zeta(t, \hbar \partial) \right) \right) x \]

\[ = W \left( \sum_{n=1}^{\infty} nt_n (\hbar \partial)^{n-1} + x \right) W^{-1} \]

\[ = \sum_{n=1}^{\infty} nt_n L^{n-1} + x + \sum_{n=1}^{\infty} v_n(\hbar, x, t) L^{-n-1}, \]

\[ (\zeta(t, \hbar \partial) = \sum_{n=1}^{\infty} t_n (\hbar \partial)^n) \]

- \( \text{ord}_{\hbar}(M) = 0 \)

- Canonical commutation relation: \( [L, M] = \hbar \)

- Lax equations: \( \hbar \frac{\partial M}{\partial t_n} = [B_n, M], (n = 1, 2, \ldots). \)
Principal symbol of $M = \text{OS-function of } dKP$:

$$M = \exp(\text{ad}_{\{,\}} X_0) \exp(\text{ad}_{\{,\}} \zeta(t, \xi)) x$$

$$= \exp(\text{ad}_{\{,\}} X_0) \left( \sum_{n=1}^{\infty} nt_n \xi^{n-1} + x \right)$$

$$= \sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-1} + x + \sum_{n=1}^{\infty} v_{0,n}(t) \mathcal{L}^{-n-1}$$

$$(\zeta(t, \xi) = \sum_{n=1}^{\infty} t_n \xi^n, \ v_{0,n} := \sigma^h(v_n)).$$

- Canonical commutation relation: $\{\mathcal{L}, M\} = 1$
- Lax equations $\frac{\partial M}{\partial t_n} = \{B_n, M\}, (n = 1, 2, \ldots)$. 
Riemann-Hilbert type construction of solutions of dKP ([TT91])

(i) Assumption: \( f_0(x, \xi), g_0(x, \xi), \mathcal{L}, \mathcal{M} \) satisfy:

- \( \{ f_0, g_0 \} = 1 \) (canonical transformation);
- \( \mathcal{L} \) and \( \mathcal{M} \) have expansion of the same form as those of dKP.
- \( f_0(\mathcal{M}, \mathcal{L}), g_0(\mathcal{M}, \mathcal{L}) \) do not contain negative powers of \( \xi \):
  \[
  (f_0(\mathcal{M}, \mathcal{L}))_{<0} = (g_0(\mathcal{M}, \mathcal{L}))_{<0} = 0.
  \]

\( \Rightarrow \mathcal{L}: \) solution of dKP, \( \mathcal{M} \): corresponding OS-function

(ii) \( (\mathcal{L}, \mathcal{M}) \): solution of dKP \( \Rightarrow \exists (f_0, g_0) \) satisfying the conditions in (i).
Riemann-Hilbert type construction of solutions of $\hbar$-KP ([TT95])

(i) Assumption: $f(\hbar, x, \hbar\partial), g(\hbar, x, \hbar\partial), L, M$ satisfy:

- $\text{ord}^\hbar f = \text{ord}^\hbar g = 0$, $[f, g] = \hbar$ (quantized canonical transformation);
- $L$ and $M$ have the same form as those of $\hbar$-KP and $[L, M] = \hbar$;
- $f(\hbar, M, L)$ and $g(\hbar, M, L)$ are differential operators:

$$ (f(\hbar, M, L))_{<0} = (g(\hbar, M, L))_{<0} = 0. $$

$\implies L$: solution of $\hbar$-KP, $M$: corresponding OS-operator

(ii) $(L, M)$: solution of $\hbar$-KP $\implies \exists (f, g)$ satisfying the conditions in (i).

Problem: $f, g$: given $\implies$ When do $L$ and $M$ exist?
In terms of the dressing operator $W$,

Condition “$(f(\hbar, M, L))_{<0} = (g(\hbar, M, L))_{<0} = 0$” $\iff$ Condition

\[
\begin{align*}
  f(\hbar, M, L) &= \text{Ad} \left( W \exp \left( \hbar^{-1} \zeta(t, \hbar \partial) \right) \right) f(\hbar, x, \hbar \partial), \\
  g(\hbar, M, L) &= \text{Ad} \left( W \exp \left( \hbar^{-1} \zeta(t, \hbar \partial) \right) \right) g(\hbar, x, \hbar \partial)
\end{align*}
\]

are both differential operators”.

Recall $W$ should have the form ($\alpha(\hbar)$ omitted)

\[
W = \exp \left( \frac{X(\hbar, x, t, \hbar \partial)}{\hbar} \right),
\]

\[
X(\hbar, x, t, \hbar \partial) = \sum_{n=0}^{\infty} \hbar^n X_n(x, t, \hbar \partial), \quad \text{ord}^\hbar (X_n(x, t, \hbar \partial)) = 0.
\]

**Idea:** construct $X_n$ (instead of $L, M$) recursively!
Main theorem ([TT09])

Assumptions:

- \((f, g)\): quantized canonical transformation given.
  
  \[ f_0 := \sigma^\hbar(f), \quad g_0 := \sigma^\hbar(g). \]

- Solution of dKP corresponding to \((f_0, g_0)\) is known.
  
  Let \((X_0)\) be its dressing functions.

\[ \Rightarrow X_n \text{ are recursively constructed so that} \]

\[ W = \exp \left( \frac{X}{\hbar} \right) \quad \left( X = \sum_{n=0}^{\infty} \hbar^n X_n \right). \]

is a dressing operator of the \(\hbar\)-dependent KP hierarchy.
Explicit algorithm: Assume \( X_0, \ldots, X_{i-1} \) and \( \alpha_0, \ldots, \alpha_{i-1} \) are given.

\[
W^{(i-1)} := \exp \left( \frac{1}{\hbar} \sum_{n=0}^{i-1} h^n X_n \right) \exp \left( \frac{1}{\hbar} \sum_{n=0}^{i-1} h^n \alpha_n \log h \hbar \right)
\]

- (Step 1) Set

\[
P^{(i-1)} := W^{(i-1)} \text{Ad} \left( e^{h^{-1} \zeta(t, h \hbar)} \right) f, \quad Q^{(i-1)} := W^{(i-1)} \text{Ad} \left( e^{h^{-1} \zeta(t, h \hbar)} \right) g,
\]

Expand \( P^{(i-1)} \) and \( Q^{(i-1)} \) with respect to the \( h \)-order as

\[
P^{(i-1)} = P_0^{(i-1)} + h P_1^{(i-1)} + \cdots + h^k P_k^{(i-1)} + \cdots
\]

\[
Q^{(i-1)} = Q_0^{(i-1)} + h Q_1^{(i-1)} + \cdots + h^k Q_k^{(i-1)} + \cdots
\]

(\( \text{ord}^h P_k^{(i-1)} = \text{ord}^h Q_k^{(i-1)} = 0. \))

- (Step 2)

\[
P_0 := \sigma^h(P_0^{(i-1)}), \quad Q_0 := \sigma^h(Q_0^{(i-1)}), \quad P_i^{(i-1)} := \sigma^h(P_i^{(i-1)}), \quad Q_i^{(i-1)} := \sigma^h(Q_i^{(i-1)}),
\]

\[
\alpha_i \log \xi + \tilde{X}_i := \int_0^\xi \left( \frac{\partial Q_0}{\partial \xi} P_i^{(i-1)} - \frac{\partial P_0}{\partial \xi} Q_i^{(i-1)} \right) \, d\xi.
\]
• (Step 3) Define a series \( \mathcal{X}_i(x, t, \xi) = \sum_{k=1}^{\infty} \mathcal{X}_{i,k}(x, t)\xi^{-k} \) by

\[
\mathcal{X}_i = \tilde{\mathcal{X}}_i' - \frac{1}{2}\{\sigma^h(X_0), \tilde{\mathcal{X}}_i'\} + \sum_{p=1}^{\infty} K_{2p}(\text{ad}_\{\} (\sigma^h(X_0)))^{2p} \tilde{\mathcal{X}}_i',
\]

\[
\tilde{\mathcal{X}}_i' := \alpha_i \log \xi + \tilde{\mathcal{X}}_i(x, \xi) - \exp(\text{ad}_\{\} \sigma^h(X_0))(\alpha_i \log \xi).
\]

\[K_{2p} = B_{2p}/(2p)!, \quad (B_{2p}: \text{Bernoulli number})\]

• (Step 4) \( X_i(x, t, \hbar \partial) = \) operator with the principal symbol \( \mathcal{X}_i \):

\[
X_i = \sum_{k=1}^{\infty} \mathcal{X}_{i,k}(x, t)(\hbar \partial)^{-k}.
\]
§2 \( \hbar \)-expansion of wave functions

Wave function of \( \hbar \)-KP: \( \Psi(\hbar, x, t; z) = W e^{(xz + \zeta(t, z))/\hbar} \)

= solution of linear equations

\[
L \Psi = z \Psi, \quad M \Psi = \hbar \frac{\partial \Psi}{\partial z}, \quad \hbar \frac{\partial \Psi}{\partial t_n} = B_n \Psi \quad (n = 1, 2, \ldots)
\]

Prop ([TT09])

(i) \( \Psi \) has the WKB form: \( \Psi(\hbar, x, t; z) = \exp(S(\hbar, x, t, z)/\hbar) \),

\[
S(\hbar, x, t; z) = \sum_{n=0}^{\infty} \hbar^n S_n(x, t; z) + \zeta(t, z).
\]

(ii) \( S_n \) is determined by \( X_0, \ldots, X_n \).

(iii) Conversely, \( X_n \) is determined by \( S_0, \ldots, S_n \).
Combining the recursion algorithm for $X_n$, we have, in principle, recursion algorithm: $(S_0, \ldots, S_{n-1}) \rightarrow S_n$.

Remark:

$$a(x, t)(\hbar \partial)^ne^{(xz+\zeta(t, z))/\hbar} = a(x, t)z^ne^{(xz+\zeta(t, z))/\hbar},$$

i.e., $\hbar \partial \rightarrow z$. Since $\Psi = \exp \left( \frac{X}{\hbar} \right) (\hbar \partial)^{\alpha/\hbar} e^{(xz+\zeta(t, z))/\hbar}$, Proposition is obvious? NO!

$$e^{X/\hbar} = \sum_{n=0}^{\infty} \frac{X^n}{n!\hbar^n} : \text{non-commutative product}$$

$$e^{S/\hbar} = \sum_{n=0}^{\infty} \frac{S^n}{n!\hbar^n} : \text{commutative product}$$

In fact, the proof and the algorithm of computation $X_i \leftrightarrow S_i$ is much complicated.
Simplest example: $X = x(h\partial)^{-1}$ (Time variables $t$ are omitted.)

$$
\exp \frac{1}{\hbar} x(h\partial)^{-1} e^{xz/\hbar} = \sum_{n=0}^{\infty} \frac{(x(h\partial))^n}{n!\hbar^n} e^{xz/\hbar}
$$

$$
= \left(1 + \frac{1}{1!\hbar} xz^{-1} + \frac{1}{2!\hbar^2} (x^2 z^{-2} - \hbar xz^{-3})
\right.
\left. + \frac{1}{3!\hbar^3} (x^3 z^{-3} - 3\hbar x^2 z^{-4} + 3\hbar^2 xz^{-5})
\right.
\left. + \frac{1}{4!\hbar^4} (x^4 z^{-4} - 6\hbar x^3 z^{-5} + 15\hbar^2 x^2 z^{-6} - 15\hbar^3 xz^{-7}) + \cdots \right) e^{xz/\hbar}
$$

$$
= \exp \frac{1}{\hbar} \left( xz^{-1} - \frac{xz^{-3}}{2} + \frac{xz^{-5}}{2} - \frac{5xz^{-7}}{8} + \cdots \right) e^{xz/\hbar}.
$$

Why no more negative powers of $\hbar$ in the last expression?!
§3 $\hbar$-expansion of $\tau$-function

$\tau$-function:

$$\Psi(t; z) = \frac{\tau(t - \hbar[z^{-1}])}{\tau(t)} e^{\hbar^{-1} \zeta(t, z)}, \quad ([z^{-1}] = (1/z, 1/2z^2, 1/3z^3, \ldots)).$$

By taking logarithm,

$$\hbar^{-1} \sum_{n=0}^{\infty} \hbar^n S_n(t; z) = \left(e^{-\hbar D(z)} - 1\right) \log \tau(t), \quad D(z) = \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial t_j}.$$

$\log \tau$ has an expansion: $\log \tau = \hbar^{-2} \sum_{n=0}^{\infty} \hbar^n F_n(t)$.

$\implies F_n$ is determined from $S_{n-1}$ and $S_n$

& vice versa, $S_n$ is determined from $F_0, \ldots, F_n$.

$\hbar$-expansion of $\tau$-function

$\implies F_n$ is determined recursively from $(f, g)$ and $F_0$.  

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§4 Concluding remarks

- We have established the algorithm to compute lower order terms in principle. But in practice the algorithm is much complicated.

- Similar results for $\hbar$-Toda hierarchy.

- Relation to “topological recursion”?:
  - Our recursion is applied to ANY solution of $\hbar$-KP.
  - Top. rec. is for GOOD solutions coming from matrix models etc.

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<thead>
<tr>
<th></th>
<th>initial data</th>
<th>data for recursion</th>
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<tbody>
<tr>
<td>Top. rec.</td>
<td>spectral curve</td>
<td>geometric data on spec. curve</td>
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<tr>
<td></td>
<td>$\downarrow?$</td>
<td>$(\text{Bergmann kernel, etc.})$</td>
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<tr>
<td>T-T</td>
<td>canonical transf.</td>
<td>quantized canon. transf.</td>
</tr>
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<td>$\downarrow?$</td>
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Ideologically the situation is parallel to the Krichever map:

<table>
<thead>
<tr>
<th>Algebro-geometric solution</th>
<th>data for solution</th>
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<tbody>
<tr>
<td>Riemann surface, line bundle, point, etc.</td>
<td>Krichever map ↓</td>
</tr>
<tr>
<td>Sato theory of KP</td>
<td>element of the Grassmann manifold</td>
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Thank you for your attention.