Poisson reduction on the space of curves in $\mathbb{R}^n$

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$S_\nu$ denotes the space of scalar shift-operators $L$ of order $\nu$, with periodic coefficients:

$$L = D^\nu + u_{\nu-1}D^{\nu-1} + \cdots + u_1D + u_0,$$

$$(Df)_m = f_{m+1} \text{ for } f \in Fun(\mathbb{Z}, \mathbb{R}) = \text{ infinite sequences in } \mathbb{R},$

$$u_k \in Fun(\mathbb{Z}/N\mathbb{Z}, \mathbb{R}) = \text{ periodic sequences in } \mathbb{R}.$$

The objective is to describe $S_\nu$ as a Poisson space.
This has already been done by Frenkel, Reshetikhin and Semenov-Tian-Shansky, by a procedure mimicking the reduction of Drinfeld and Sokolov. The point of view I will describe reflects their setting, but differs in a small detail. It seems evident that in the low-rank cases which have so far been worked through and which will be shown now, the present construction is more general than that of FRS.
To justify the claim, consider the \( \nu = 2 \) case:

\[
S_2 = \{ L = D^2 - \mu D + \rho \}
\]

\( \triangleright \) familiar as the standard setting for the Toda lattice system.

\( \triangleright \) after fixing \( \rho \equiv 1 \) it’s a discrete analogue of the space of periodic Schrödinger operators, on which the standard Poisson structure - found by FRS - is identified with the Faddeev-Takhtajan-Volkov structure; a discrete analogue of the Virasoro algebra, which is appropriate for the continuous limit of a second order differential operator of the restricted form \( L = \partial^2 + u \).

We’ll see a family of Poisson structures on \( S_2 \) which includes both of the above. The Toda model was excluded in the FRS analysis.
Poisson Geometry

– has its origins firmly based in Classical Mechanics: this is not only aesthetically appealing, but also useful, as any result no matter how sophisticated is likely to have a manifestation in the context of the motion of a rigid body, or a number of coupled rigid bodies, or a system of springs and rigid bodies...
– is involved in modern methods of quantisation
– is related to Representation Theory
– is the twin of symplectic geometry, to which all of the above are equally related
– is a useful tool: e.g. used to single out certain classes of numerical integration methods; application to stability analysis; analysis of symmetries.
In classical mechanics the principle of least action leads to Hamilton’s equation on $\mathbb{R}^{2n}$. There is a function $H \in C^\infty(\mathbb{R}^{2n})$ and in “local coordinates” $(q, p)$,

$$
\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p).
$$

If the system evolves in a space which is understood as being a $2n$-dimensional subspace of a larger space we’ll have rather local coordinates in $\mathbb{R}^{2n+r}$ say $(q, p, w)$ and

$$
\dot{q} = \frac{\partial H}{\partial p}(q, p, w), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p, w), \quad \dot{w} = 0.
$$

Suppose we make a change of coordinates on $\mathbb{R}^{2n+r}$, 

$$(q, p, w) \mapsto x.$$

Then we are led to

$$
\dot{x} = P(x)\nabla_x H \text{ or, for any } F \in C^\infty(M), \quad \frac{dF}{dt}(x) = \frac{\partial F}{\partial x_i} \dot{x}_i
$$

$$
\frac{dF}{dt} = \nabla F^T P \nabla H =: \{F, H\}.
$$
We have arrived at the coordinate free version of Hamilton’s equation. Let $M$ be a differentiable manifold. A Poisson bracket is a map $\{\ ,\ \} : C^\infty(M) \wedge C^\infty(M) \to C^\infty(M)$ satisfying

$$\{F, HK\} = \{F, H\}K + \{F, K\}H,$$

$$\{\{F, H\}, K\} + c.p. = 0.$$

A physical model is determined by specifying $M$, $\{\ ,\ \}$ and $H \in C^\infty(M)$. Evolution of the physical system is

$$\dot{x} = X_H(x) \quad X_H = \{\cdot, H\},$$

$$\Leftrightarrow \frac{dF}{dt} = \{F, H\}.$$
Symmetries

From a physical point of view it is crucial that, if necessary, freedom in the choice of reference frame can be incorporated into the model: Suppose a Lie group $G$ acts on $M$. If the model is to represent a system for which the evolution does not depend on the $G$ action, we’ll have $H \in C^\infty(M)^G$. If we measure a $G$-invariant quantity, we expect it to remain $G$-invariant for all time, i.e. $F \in C^\infty(M)^G \Rightarrow \dot{F} \in C^\infty(M)^G$. Thus

$$F, H \in C^\infty(M)^G \Rightarrow \{F, H\} \in C^\infty(M)^G. \quad (*)$$

A sufficient way to get this basic condition is

$$\forall \phi, \psi \in C^\infty(M), \forall g \in G, \{\phi(g \cdot ), \psi(g \cdot )\} = \{\phi, \psi\}(g \cdot ).$$

This leads to the moment map $J : M \to \mathfrak{g}^*$ and a theory of reduction (with lots of results and many papers!)
More generally, suppose that there is a Poisson bracket on $G$ and

$$\forall \phi, \psi \in C^\infty(M),$$

$$\{\phi(g \cdot \cdot), \psi(g \cdot \cdot)\}_M(p) + \{\phi(\cdot p), \psi(\cdot p)\}_G(g) = \{\phi, \psi\}_M(g \cdot p).$$

This leads also to a theory of reduction, also many results and articles, but less well developed and less well-known. $G$ is called a Poisson Lie Group. Practically speaking this subject involves the $r$-matrix, Lie bialgebras, Drinfeld double, Heisenberg double, etc.

Poisson Lie groups are especially important in applications to integrable systems. Essentially speaking the subject was born in this context.
Let $\mathcal{M}$ be a Poisson manifold with a Poisson action of a PLG $G$, i.e. there is a PB on $G$ compatible with the group structure s.t., with $F \in C^\infty(\mathcal{M}) \Rightarrow \hat{F} \in C^\infty(\mathcal{M}, G)$, defined by

$$\hat{F}(m, g) := F(g \cdot m)$$

there holds, for any $F, H \in C^\infty(\mathcal{M})$,

$$\{F, H\}_{\mathcal{M}}(g \cdot m) = \{\hat{F}(m, \cdot), \hat{H}(m, \cdot)\}_G(g) + \{\hat{F}(\cdot, g), \hat{H}(\cdot, g)\}_{\mathcal{M}}(m).$$

Then $\mathcal{M}/G$ inherits a Poisson structure from the Poisson structure on $\mathcal{M}$:

$$F, H \in C^\infty(\mathcal{M})^G \Rightarrow \{F, H\}_\mathcal{M} \in C^\infty(\mathcal{M})^G.$$

**Main new feature** is admissibility: If $K$ is a subgroup of $G$ we may sometimes get an induced PB on $\mathcal{M}/K$ but not always.
Let $N \in \mathbb{N}$ be fixed ("big enough"). Denote by $\mathcal{W}$ the space of twisted polygons of length $N$ in $\mathbb{R}^\nu = \{\text{row-vectors}\}$.

An element of $\mathcal{W}$ is a pair, $(V, M)$, where $V : \mathbb{N} \to \mathbb{R}^\nu$ is a sequence in $\mathbb{R}^\nu$ and $M$ is an element in $GL_\nu$. $V$ and $M$ are related by the condition of quasi-periodicity or twisting, $V_{n+N} = V_n M \ \forall n$:

$$\mathcal{W} = \{ (V, M) \in Fun(\mathbb{Z}, \mathbb{R}^\nu) \times GL(\nu, \mathbb{R}) | \ V_{k+N} = V_k M \ \forall n \}$$
Introduce the group \( C = \text{Fun}(\mathbb{Z}/N\mathbb{Z}, \mathbb{R}^\times) \) of periodic sequences of non-zero real numbers, with \((pq)_m = p_m q_m\) for \( p, q \in C \). The groups \( GL_\nu \) and \( C \) both have natural actions on \( \mathcal{W} \),

\[
(p, g) \cdot (V, M) = (pV g^{-1}, gM g^{-1}) \quad p \in C, \quad g \in GL_\nu.
\]

The two actions are not disjoint. For \( k \in \mathbb{R} \), the actions of \( k \in C \) and \( k \text{Id} \in GL_\nu \) have the same effect. It is convenient to replace \( GL_\nu \) by \( SL_\nu \) and leave \( C \) intact. It's also natural to restrict \( M \) to lie in \( SL_\nu \). Denote \( SL(\nu, \mathbb{R}) \) by \( G \) and \( sl(\nu, \mathbb{R}) \) by \( g \).
Poisson structure on $\mathcal{W}$

Let $R \in \wedge^2 g$ and let $C \in S^2 g$. Set $R_{\pm} = \frac{1}{2}(R \pm C)$. Let

$\phi \in \text{Fun}(\mathbb{Z}/N\mathbb{Z}, \mathbb{R})$ be an odd periodic function. Denote by $\sigma$ the “discrete sign function”. Define the bracket on $\mathcal{W}$

$$\{V^1_m, V^2_n\} = V_m \otimes V_n[R + \sigma_{m-n}(C + Id \otimes Id) + \phi_{m-n}],$$
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**Proposition** The above formulae define a Poisson bracket on $\mathcal{W}$ if and only if $C$ is the Casimir element and if $R$ is a classical $r$-matrix, i.e. $R$ satisfies the Yang-Baxter equation

$$[R^{12}, R^{13}] + \text{c.p.} = -[C^{12}, C^{13}].$$
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\[
\{V^1_m, V^2_n\} = V_m \otimes V_n[R + \sigma_{m-n}(C + Id \otimes Id) + \phi_{m-n}],
\]
\[
\{V^1_m, M^2\} = V^1_m[M^2 R_+ - R_+ M^2],
\]

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$$\{V_m^1, V_n^2\} = V_m \otimes V_n [R + \sigma_{m-n} (C + \text{Id} \otimes \text{Id}) + \phi_{m-n}],$$

$$\{V_m^1, M^2\} = V_m^1 [M^2 R_- - R_+ M^2],$$

$$\{M^1, M^2\} = (M \otimes M) R + R (M \otimes M) - M^1 R_+ M^2 - M^2 R_- M^1.$$

**Proposition** The above formulae define a Poisson bracket on $\mathcal{W}$ if and only if $C$ is the Casimir element and if $R$ is a classical $r$-matrix, i.e. $R$ satisfies the Yang-Baxter equation $[R^{12}, R^{13}] + c.p. = -[C^{12}, C^{13}].$
One may prove:

**Proposition** The actions of $C$ and $G$ on $\mathcal{W}$ are Poisson actions in the Poisson Lie group sense, when $G$ has the Sklyanin Poisson structure and $C$ has the zero Poisson structure.

**Proposition** The map $(V, M) \mapsto M$ is a momentum map for the action of $G$ (in the Poisson Lie group sense). The map $(V, M) \mapsto w(V, M)$ is a momentum map for the action of $C$.

Here $w_m(V) := |V_m \ldots V_{m+\nu-1}|$, the discrete Wronskian.
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Proof:

\[ \{w_m, V_n\} = \ldots \]

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One may prove:

**Proposition**  The actions of $\mathcal{C}$ and $G$ on $\mathcal{W}$ are Poisson actions in the Poisson Lie group sense, when $G$ has the Sklyanin Poisson structure and $\mathcal{C}$ has the zero Poisson structure.

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**Proof:**

\[
\{w_m, V_n\} = \ldots = \left(\sigma_{m-n} + \sum_{k=0}^{\nu-1} (\phi_{m+k-n} - \delta_{m+k-n})\right) w_m V_n.
\]

Here $w_m(V) := |V_m \ldots V_{m+\nu-1}|$, the discrete Wronskian.
It follows that there are Poisson structures on the factor spaces
\[ C\backslash W, \quad W/G, \quad C\backslash W/G \]
such that the projections
\[
\begin{array}{c}
\downarrow \\
C\backslash W \quad \longrightarrow \quad C\backslash W/G
\end{array}
\]
are all Poisson maps.
A formula for the Poisson bracket on $\mathcal{C}\setminus\mathcal{W}$

In local coordinates $v \in \mathbb{R}^{\nu-1}$ on $\mathcal{C}\setminus\mathcal{W}$ we get the

**Proposition**  The Poisson structure obtained by projection $\mathcal{W} \to \mathcal{C}\setminus\mathcal{W}$ has the form

$$\{v_1^1, v_2^2\} = (v_m \otimes v_n) \cdot R - \sigma_{m-n}(v_m - v_n) \otimes (v_m - v_n).$$

*Important remark:* There’s no $\phi$ in this formula.
We shall restrict to *nondegenerate* polygons, i.e. quasi-periodic sequences $V$ for which $w(V) \neq 0$. The $\nu + 1$ vectors $V, V', \ldots V^{(\nu)} \in \mathbb{R}^\nu$ are linearly dependent and $w(V) \neq 0 \Rightarrow \exists u_{\nu-1}, \ldots, u_0$ s.t.

$$V^{(\nu)} + u_{\nu-1}V^{(\nu-1)} + \cdots + u_1V' + u_0V = 0$$

It is obvious that the fields $u_k$, which depend on $V$, are $G$-independent, i.e. $u_k(g \cdot V) = u_k(V) \ \forall k$. Hence $(u_0, u_1, \ldots, u_{\nu-1})$ is a good set of coordinates on $W/G$. Also the quasi-periodicity property of $V$ implies that all the $u_k$ are periodic.
For any quasi-periodic sequence $V$, let us suppose that $f \in C$ is such that $w(fV) = 1$. For $\tilde{V} = fV$,

$$
\tilde{V}^{(\nu)} + \tilde{u}_{\nu-1} \tilde{V}^{(\nu-1)} + \cdots + \tilde{u}_1 \tilde{V}' + \tilde{u}_0 \tilde{V} = 0,
$$

but we easily check that $\tilde{u}_0 = (-1)^\nu$. Hence, as $C\backslash \mathcal{W} \ni [V] = [\tilde{V}]$, i.e. $\tilde{V} \sim V$ w.r.t. $C$-action, we have a set of coordinates on $C\backslash \mathcal{W}/G$ and a strategy for how to compute the projection $\mathcal{W} \to C\backslash \mathcal{W}/G$ in terms of these coordinates

1. given the sequence $V$, find $f \in C$ s.t. $w(fV) = 1$,

2. compute $V \mapsto \tilde{u}_k(V) = u_k(fV)$ for $k = 1, \ldots, \nu - 1$. 

– and the Poisson bracket on $\mathcal{C}\backslash \mathcal{W}/G$

The representation of the Poisson structure in terms of the coordinates $\tilde{u}_k$ on $\mathcal{C}\backslash \mathcal{W}$ can then be computed directly via the expressions $(\tilde{u}_1(V), \ldots, \tilde{u}_{\nu-1}(V))$.

Alternatively, one may choose to take the Poisson algebra generated by the functions $V \mapsto (u_0(V), u_1(V), \ldots, u_{\nu-1}(V))$ and then to perform Dirac reduction to the constrained subspace $u_0 \equiv (-1)^{\nu}$.

The role of the arbitrary function $\phi$ is an interesting one in the context of the Dirac reduction argument.
From now on we’ll look at examples:

Let’s apply our setup to the cases $\nu = 2$ and $\nu = 3$
\( \nu = 2 \)

We have

\[
V'' = \mu V' - \rho V \Rightarrow |VV''| = \mu(V)w(V), \quad w(V)' = \rho(V)w(V)
\]

and we may compute the Poisson bracket algebra (on \( \mathcal{W}/G \)) in terms of the coordinates \((\mu, \rho)\):

\[
\{\mu_m, \mu_n\} = \left( 2\phi_{m-n} - \phi_{m-n+1} - \phi_{m-n-1} \\
- \delta_{m-n+1} + \delta_{m-n-1} \right) \mu_m \mu_n + 2\delta_{m-n+1} \rho_n - 2\delta_{m-n-1} \rho_m,
\]

\[
\{\mu_m, \rho_n\} = \left( \phi_{m-n} + \phi_{m-n+1} - \phi_{m-n-1} - \phi_{m-n+2} \\
- \delta_{m-n} + \delta_{m-n-1} + \delta_{m-n+1} - \delta_{m-n+2} \right) \mu_m \rho_n,
\]

\[
\{\rho_m, \rho_n\} = \left( 2\phi_{m-n} - \phi_{m-n+2} - \phi_{m-n-2} \\
- \delta_{m-n+2} + \delta_{m-n-2} \right) \rho_m \rho_n.
\]
Thinking of $\phi$ as the kernel of an operator, we may write

\[
\begin{pmatrix}
\dot{\mu} \\
\dot{\rho}
\end{pmatrix}
= P(\mu, \rho) \cdot 
\begin{pmatrix}
\delta_\mu H \\
\delta_\rho H
\end{pmatrix}
\]

with $P =$

\[
\begin{pmatrix}
\mu [(2 - D - D^{-1}) \phi - D] + D^{-1}] \mu + 2 D \rho - 2 \rho D^{-1} & \mu [(1 + D - D^{-1} - D^2) \phi - 1 + D^{-1} + D - D^2] \rho \\
\rho [(1 + D^{-1} - D - D^{-2}) \phi + 1 - D - D^{-1} + D^{-2}] \mu & \rho [(2 - D^2 - D^{-2}) \phi - D^2 + D^{-2}] \rho
\end{pmatrix}
\]
Choices for $\phi$

For example:

(a) Choose $\phi$ in such a way that the Dirac reduction procedure is as simple as possible.

(b) Choose $\phi$ in such a way that the unreduced PB is as interesting as possible.
Choosing
\[ \phi = \frac{1 + D^2}{1 - D^2} \]
we obtain
\[
\begin{pmatrix}
\dot{\mu} \\
\dot{\rho}
\end{pmatrix} = \begin{pmatrix}
\mu \frac{1 - D}{1 + D} + D\rho - \rho D^{-1} & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\delta_{\mu} H \\
\delta_{\rho} H
\end{pmatrix}
\]
so there is nothing to reduce. The coordinate \( \rho \) has been rendered a Casimir and with \( \rho \equiv 1 \) we get the FRS formula for the reduced Poisson structure on \( \mathcal{C}\backslash \mathcal{W}/G \),
\[
P(u) = u \left( \frac{1 - D}{1 + D} \right) u + D - D^{-1}
\]
(This is an analogue of “\( P(u) = \partial^3 + u \partial + \partial u \)”)

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Choosing

\[ \phi = \frac{1 + D}{1 - D} \]

we get

\[
\left( \begin{array}{c}
\dot{\mu} \\
\dot{\rho}
\end{array} \right) = \left( \begin{array}{cc}
D\rho - \rho D^{-1} & \mu(D - 1)\rho \\
\rho(1 - D^{-1})\mu & \rho(D - D^{-1})\rho
\end{array} \right) \cdot \left( \begin{array}{c}
\delta\mu H \\
\delta\rho H
\end{array} \right)
\]

which is the “second Toda lattice” Poisson structure.
Lattice Virasoro is constrained second Toda

It follows directly from the general discussion that applying the constraint \( \rho \equiv 1 \) to the second Toda lattice Poisson structure results in the lattice Virasoro formula of FRS. Indeed (with slightly more generality)

\[
(D\rho - \rho D^{-1}) - (\mu(D - 1)\rho)\left[\rho(D - D^{-1})\rho\right]^{-1}(\rho(1 - D^{-1})\mu)\bigg|_{\rho\equiv\beta} = D\beta - \beta D^{-1} - \mu(D - 1)(D - D^{-1})^{-1}(1 - D^{-1})\mu
\]

\[
= D\beta - \beta D^{-1} + \mu \frac{1 - D}{1 + D} \mu
\]

– a formula of Veselov and Shabat 1992
The form of the Poisson tensor for $\nu = 3$:

$$
\begin{pmatrix}
\dot{a} \\
\dot{b} \\
\dot{\rho}
\end{pmatrix} = P \cdot 
\begin{pmatrix}
\delta_a H \\
\delta_b H \\
\delta_\rho H
\end{pmatrix}
$$

where

$$P = 
\begin{pmatrix}
 aK[(D - 1)\phi + D + 1]a & aK[(D^2 - 1)\phi + D^2 + 1]b & aK[(D^3 - 1)\phi + D^3 + 1]\rho \\
+2Db - 2bD^{-1} & +2D^2\rho - 2\rho D^{-1} & +D^3 + 1]\rho \\

bK_2[(D^2 - 1)\phi + D^2 + 1]b & bK_2[(D^3 - 1)\phi + D^3 + 1]\rho \\
+2aD\rho - 2\rho D^{-1}a & +D^3 + 1]\rho \\

\rho K_3[(D^3 - 1)\phi + D^3 + 1]\rho \\
\end{pmatrix}
$$

with

$$K = D^{-1} - 1, \quad K_2 = D^{-2} - 1, \quad K_3 = D^{-3} - 1$$
Choices for $\phi$ for $\nu = 3$

$$\phi = \frac{1 + D^3}{1 - D^3}, \quad \phi = \frac{1 + D^2}{1 - D^2}, \quad \phi = \frac{1 + D}{1 - D}$$

produce various standard interesting Poisson structures, related to so-called “extended Toda” lattice systems.
Part Two

- you don’t have to be Ramanujan...
...to see a pattern

The different choices of $\phi$ of especial interest have a natural form:

for $\nu = 2$ we made the choices

$$\phi = \frac{1 + D^2}{1 - D^2} \quad \text{and} \quad \phi = \frac{1 + D}{1 - D};$$

for $\nu = 3$ the choices were

$$\phi = \frac{1 + D^3}{1 - D^3}, \quad \phi = \frac{1 + D^2}{1 - D^2}, \quad \text{and} \quad \phi = \frac{1 + D}{1 - D}.$$  

Does this pattern show itself for larger values of $\nu$?

To answer this question we need to reformulate it. That is we must identify the qualities endowed by these special choices.
Proposition

Let $M$ be a Poisson manifold with Poisson tensor $P$. Let $\xi$ be a vector field on $M$. Suppose that $(L_\xi)^2 P = 0$. Then $L_\xi P$ is also a Poisson tensor, from which it follows that the pair $(P, L_\xi P)$ defines a bi-hamiltonian structure on $M$.

Proof. As $P$ is Poisson, we have $[P, P] = 0$. Applying $L_\xi$ to this condition gives

$$0 = L_\xi [P, P] = 2 [L_\xi P, P].$$  \hfill (*)

Applying $L_\xi$ again,

$$0 = L_\xi [L_\xi P, P] = [(L_\xi)^2 P, P] + [L_\xi P, L_\xi P].$$  \hfill (**)  

It follows from (**) that $L_\xi P$ is Poisson and it follows from (*) that $P$ and $L_\xi P$ are compatible.
$\mathbb{R}^\nu \ni V, V', V'', \ldots, V^{(\nu)}$ obey some lin.dep. relation and $w(V) \neq 0$ allows us to write

$$V^{(\nu)} - a^{(\nu-1)}V^{(\nu-1)} + \cdots + (-1)^{(\nu-r)}V^{(\nu-r)} + \cdots + (-1)^\nu a^{(0)}V = 0.$$  

i.e. $V \in \mathcal{W}$ engenders

$L = D^\nu - a_{\nu-1} D^{\nu-1} + \cdots + (-1)^{\nu-1} a_1 D + (-1)^\nu a_0 \in S_{\nu}$ and

$a = (a^{(0)}, a^{(1)}, \ldots, a^{(\nu-1)})$ are a good set of coordinates on $\mathcal{W}/G$.

For $k = 1, \ldots, \nu - 1$, define the functions $\alpha^{(k)}$ on $\mathcal{W}$ by

$$\alpha^{(k)} = |V \ldots V^{(\nu)}|^k, \text{ i.e. } \alpha_m^{(k)} = |V_m \ldots V_{m+k-1} V_{m+k+1} \ldots V_{m+\nu}|$$

Then, for $1 \leq k \leq \nu - 1$,

$$a^{(k)} = \frac{\alpha^{(k)}}{w} \quad \text{ and } \quad a^{(0)} = \frac{w'}{w}.$$
For $k \in \{1, \ldots, \nu - 1\}$, is there a choice of $\phi$ such that the Poisson tensor $P_{\phi}(a)$ on $\mathcal{W}/G$, when represented in the coordinates $a$, is linear in the coordinate $a^{(k)}$: so that

$$\frac{d^2}{dt^2} P(a^{(0)}, \ldots, a^{(k)} + t, \ldots, a^{(\nu-1)}) = 0?$$

Is there a choice of $\phi$ such that $a^{(0)}$ is a Casimir?
Theorem

For the family of $\phi$–dependent Poisson brackets on $S_{\nu} = \mathcal{W}/G$, with respect to the natural coordinates $a$ on $S_{\nu}$, given by $S_{\nu} \ni L = D_{\nu}^\nu - a_{\nu-1}D_{\nu}^{\nu-1} + \cdots + (-1)^{\nu-1}a_1D + (-1)^\nu a_0$

(i) the choice

$$\phi = \frac{1 + D_{\nu}}{1 - D_{\nu}}$$

forces $a_0$ to be a Casimir;

(ii) for $0 < k < \nu$, the choice

$$\phi = \frac{D_{k} + D_{\nu}}{D_{k} - D_{\nu}}$$

forces the Poisson bracket to have linear dependence on $a_k$. 

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Poisson reduction on the space of curves in $\mathbb{R}^n$
It is interesting to ask about what kind of integrable systems emerge from this picture. It is easy to see that various families of bi- or multi-hamiltonian systems can be uncovered just by applying the canonical procedures. It is also not hard to perceive how to make use of the monodromy matrix in order to generate commuting integrals. All of this will correspond to more or less standard results. The continuous analogue is delivered almost directly and will be seen to amount to the same as the DS version of things.
However when one comes to the area of discrete integrable maps or relations the situation is very far from clear. A system which remains a tantalising and frustratingly inaccessible example to place in the context of the setup presented here is the Pentagram Map. In particular, this is a system which is known to be defined on the phase space $S_3|_{w=1}$ and which also corresponds to a standard case from the DS family in the continuous limit. It preserves a Poisson bracket on $S_3|_{w=1}$ and a complete family of commuting integrals is generated by the monodromy, but the appropriate Poisson structure is different to the one emerging here and moreover the one emerging here is not preserved by the map. This is utterly mysterious to me and requires clarification!