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PBW Filtration and Bases for Symplectic Lie Algebras

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We study the PBW filtration on the highest weight representations $V(\lambda)$ of $\mathfrak{sp}_{2n}$. This filtration is induced by the standard degree filtration on $U(\mathfrak{n}^-)$. We give a description of the associated graded $S(\mathfrak{n}^-)$-module $\operatorname{gr} V(\lambda)$ in terms of generators and relations. We also construct a basis of $\operatorname{gr} V(\lambda)$. As an application we derive a graded combinatorial formula for the character of $V(\lambda)$ and obtain a new class of bases of the modules $V(\lambda)$.

1 Introduction

In this paper, we continue the study of the PBW filtration on irreducible representations of simple Lie algebras initiated in [9]. The goal of this paper is to develop the theory of PBW-graded modules for symplectic Lie algebras $\mathfrak{sp}_{2n}$. We start with recalling the definition of the PBW filtration.

Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a Cartan decomposition. For a dominant integral λ, we denote by $V(\lambda)$ the irreducible $\mathfrak{g}$-module with highest weight λ. Fix a highest weight vector $v_\lambda \in V(\lambda)$. Then $V(\lambda) = \mathrm{U}(\mathfrak{n}^-)v_\lambda$, where $\mathrm{U}(\mathfrak{n}^-)$ denotes the universal enveloping algebra of $\mathfrak{n}^-$. The degree filtration $\mathrm{U}(\mathfrak{n}^-)_s$ on $\mathrm{U}(\mathfrak{n}^-)$ is defined by

$$\mathrm{U}(\mathfrak{n}^-)_s = \operatorname{span}\{x_1 \cdots x_l : \ x_i \in \mathfrak{n}^-, l \le s\}.$$

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In particular, $\mathrm{U}(\mathfrak{n}^-)_0 = \mathbb{C}$ and $\mathrm{gr}\,\mathrm{U}(\mathfrak{n}^-) \simeq S(\mathfrak{n}^-)$, where $S(\mathfrak{n}^-)$ denotes the symmetric algebra over $\mathfrak{n}^-$. The filtration of $\mathrm{U}(\mathfrak{n}^-)$ by the subspaces $\mathrm{U}(\mathfrak{n}^-)_s$ induces a filtration of $V(\lambda)$ by the subspaces $V(\lambda)_s$:

$$V(\lambda)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\lambda.$$

We call this filtration the PBW filtration. The central objects of our paper are the associated graded spaces $\mathrm{gr}\,V(\lambda)$ as $S(\mathfrak{n}^-)$-modules for $\mathfrak{g}$ of type $\mathtt{C}_n$.

We note that $\mathrm{gr}\,V(\lambda) = S(\mathfrak{n}^-)v_\lambda$ is a cyclic $S(\mathfrak{n}^-)$-module. So one has

$$\mathrm{gr}\,V(\lambda) \simeq S(\mathfrak{n}^-)/I(\lambda),$$

for some ideal $I(\lambda) \subset S(\mathfrak{n}^-)$. For example, for any simple root α_i the power $f_{\alpha_i}^{(\lambda,\alpha_i)+1}$ of a root vector $f_{\alpha_i} \in \mathfrak{n}^-_{-\alpha_i}$ belongs to $I(\lambda)$ since $f_{\alpha_i}^{(\lambda,\alpha_i)+1} v_\lambda = 0$ in $V(\lambda)$. To describe $I(\lambda)$ explicitly, we prepare some notation. All positive roots of $\mathfrak{sp}_{2n}$ can be divided into two groups:

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n,$$

$$\alpha_{i,\bar{j}} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n.$$

In particular, $\alpha_{1,\bar{1}}$ is the highest root. Consider the action of the opposite subalgebra $\mathfrak{n}^+$ on $V(\lambda)$. It is easy to see that $\mathfrak{n}^+ V(\lambda)_s \hookrightarrow V(\lambda)_s$, so we obtain the structure of an $U(\mathfrak{n}^+)$-module on $\mathrm{gr}\,V(\lambda)$ as well. We show:

Theorem A. The ideal $I(\lambda)$ is generated as $S(\mathfrak{n}^-)$-module by the subspace

$$\mathrm{U}(\mathfrak{n}^+) \circ \mathrm{span}\{ f_{\alpha_{i,j}}^{(\lambda,\alpha_{i,j})+1}, 1 \le i \le j \le n-1, \ f_{\alpha_{i,\bar{i}}}^{(\lambda,\alpha_{i,n})+1}, \ 1 \le i \le n\}. \qquad \square$$

Theorem A should be understood as a commutative analog of the well-known description of $V(\lambda)$ as the quotient

$$V(\lambda) \simeq \mathrm{U}(\mathfrak{n}^-)/\langle f_{\alpha_i}^{(\lambda,\alpha_i)+1}, 1 \le i \le n\rangle$$

(see, for example, [14]).

Our second problem (closely related to the first one) is to construct a monomial basis of $\mathrm{gr}\,V(\lambda)$. The elements $\prod_{\alpha>0} f_\alpha^{s_\alpha} v_\lambda$ with $s_\alpha \ge 0$ obviously span $\mathrm{gr}\,V(\lambda)$ (recall that the order in $\prod_{\alpha>0} f_\alpha^{s_\alpha}$ is not important since f_α are considered as elements of $S(\mathfrak{n}^-)$).

For each λ, we construct a set $S(\lambda)$ of multi-exponents $\mathbf{s} = \{s_\alpha\}_{\alpha>0}$ such that the elements

$$f^{\mathbf{s}} v_\lambda = \prod_{\alpha>0} f_\alpha^{s_\alpha} v_\lambda, \quad \mathbf{s} \in S(\lambda)$$

form a basis of $\operatorname{gr} V(\lambda)$. To give a definition of $S(\lambda)$, we need the notion of a symplectic version of *Dyck path*, which is precisely defined in Definition 2.2. The definition is similar to the one for usual Dyck paths, see, for example, [9]). In short, a Dyck path $\mathbf{p} = (p(0), \ldots, p(k))$ is a sequence of positive roots starting at a simple root α_i, ending at a root α_j or $\alpha_{j,\bar{j}}$, $j \geq i$ and obeying some recursion rules. We denote by $\mathbb{D}$ the set of all Dyck paths.

For a dominant weight λ, we introduce a polytope $P(\lambda) \subset \mathbb{R}^{n^2}_{\geq 0}$:

$$P(\lambda) := \left\{ (s_\alpha)_{\alpha>0} \mid \forall \mathbf{p} \in \mathbb{D} : \begin{array}{ll} \text{if } p(0) = \alpha_i,\ p(k) = \alpha_j & \text{then } s_{p(0)} + \cdots + s_{p(k)} \leq (\lambda, \alpha_{i,j}), \\ \text{if } p(0) = \alpha_i,\ p(k) = \alpha_{j,\bar{j}} & \text{then } s_{p(0)} + \cdots + s_{p(k)} \leq (\lambda, \alpha_{i,n}) \end{array} \right\}.$$

Let $S(\lambda)$ be the set of integral points in $P(\lambda)$.

We show the following.

Theorem B. The set of elements $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$, forms a basis of $\operatorname{gr} V(\lambda)$. □

For $\mathbf{s} \in S(\lambda)$ define the weight

$$\operatorname{wt}(\mathbf{s}) := \sum_{1 \leq j \leq k \leq n} s_{\alpha_{j,k}} \alpha_{j,k} + \sum_{1 \leq j \leq k < n} s_{\alpha_{j,\bar{k}}} \alpha_{j,\bar{k}}.$$

As an important application we obtain the following corollary.

Corollary 1.1.

(i) For each $\mathbf{s} \in S(\lambda)$ fix an arbitrary order of factors f_α in the product $\prod_{\alpha>0} f_\alpha^{s_\alpha}$. Let $f^{\mathbf{s}} = \prod_{\alpha>0} f_\alpha^{s_\alpha}$ be the ordered product. Then the elements $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$, form a basis of $V(\lambda)$.

(ii) $\dim V(\lambda) = \sharp S(\lambda)$.

(iii) $\operatorname{char} V(\lambda) = \sum_{\mathbf{s} \in S(\lambda)} \mathrm{e}^{\lambda - \operatorname{wt}(\mathbf{s})}$. □

We note that the order in the corollary above is important since we are back to the action of the (in general) not commutative enveloping algebra. We thus obtain a family of bases for irreducible $\mathfrak{sp}_{2n}$-modules. The existence of these bases (with the same indexing set) was proved by Vinberg for $\mathfrak{sp}_4$ (see [17]).

The modules $\operatorname{gr} V(\lambda)$ have one more nice property. Namely, given two dominant integral weights λ and μ, consider the subspace $\operatorname{gr} V(\lambda,\mu) \hookrightarrow \operatorname{gr} V(\lambda) \otimes \operatorname{gr} V(\mu)$ generated from the product of highest weight vectors: $\operatorname{gr} V(\lambda,\mu) = S(\mathfrak{n}^-)(v_\lambda \otimes v_\mu)$. We prove that $\operatorname{gr} V(\lambda,\mu) \simeq \operatorname{gr} V(\lambda+\mu)$ as $S(\mathfrak{n}^-)$-modules. This is an analog of the corresponding classical result. In type A, this statement was proved in [9]. Dualizing the embedding $\operatorname{gr} V(\lambda+\mu) \hookrightarrow \operatorname{gr} V(\lambda) \otimes \operatorname{gr} V(\mu)$, one obtains an algebra structure on the space $\bigoplus_\lambda (\operatorname{gr} V(\lambda))^*$. The projective spectrum of this algebra is a certain degeneration of the symplectic flag variety. In type A it was studied in [6].

Remark 1.2. The data labeling the basis vectors is similar to that for the symplectic Gelfand–Tsetlin patterns (see [2, 13]). However, these bases are very different from the symplectic GT basis. On the combinatorial side the connection with the Gelfand–Tsetlin patterns was recently clarified by Ardila et al. [1]. Generalizing a result of Stanley, they show that for every partition λ there exists a marked poset (P, A, λ) such that the Gelfand–Tsetlin polytope coincides with the corresponding marked order polytope and our polytope $P(\lambda)$ coincides with the corresponding marked chain polytope. Note that both polytopes have the same Ehrhart polynomials [1]. □

We finish the introduction with several remarks. The PBW filtration for highest weight representations was considered in [4–7, 9, 16]. It was shown that it has important applications in algebraic geometry, representation theory of current and affine algebras and in mathematical physics.

There exist special representations $V(\lambda)$ such that the operators $f^{\mathbf{s}}$ consist only of mutually commuting root vectors, even before passing to $\operatorname{gr} V(\lambda)$. These modules can be described via the theory of abelian radicals and turned out to be important in the theory of vertex operator algebras (see [8, 10, 12]).

Finally, we note that $\operatorname{gr} V(\lambda)$ carries an additional grading on each weight space $V(\lambda)^\mu$ of $V(\lambda)$:

$$\operatorname{gr} V(\lambda)^\mu = \bigoplus_{s\geq 0} \operatorname{gr}_s V(\lambda)^\mu = \bigoplus_{s\geq 0} \frac{V(\lambda)^\mu_s}{V(\lambda)^\mu_{s-1}}.$$

The graded character of the weight space is the polynomial

$$p_{\lambda,\mu}(q) := \sum_{s\geq 0} \left(\dim \frac{V(\lambda)^\mu_s}{V(\lambda)^\mu_{s-1}}\right) q^s.$$

Define the degree

$$\deg(\mathbf{s}) := \sum_{1\le j\le k\le n} s_{\alpha_{j,k}} + \sum_{1\le j\le k<n} s_{\alpha_{j,\bar{k}}}$$

for $\mathbf{s} \in S(\lambda)$, and let $S(\lambda)^{\mu}$ be the subset of elements such that $\mu = \lambda - \mathrm{wt}(\mathbf{s})$. Then we have the following corollary.

Corollary. $p_{\lambda,\mu}(q) = \sum_{\mathbf{s}\in S(\lambda)^{\mu}} q^{\deg \mathbf{s}}$. □

We note that our filtration is different from the Brylinski–Kostant filtration (see [3, 15]).

Our paper is organized as follows:

In Section 2, we introduce notation and state the problems. Sections 3 and 4 are devoted to the proof of Theorems A and B. In Section 3, we prove the spanning property of our basis and in Section 4 we finalize the proof.

2 Definitions

Let R^+ be the set of positive roots of $\mathfrak{sp}_{2n}$. For each $\alpha \in R^+$, we fix a non-zero element $f_\alpha \in \mathfrak{n}^-_{-\alpha}$. Let α_i, ω_i $i = 1, \ldots, n$ be the simple roots and the fundamental weights. All positive roots of $\mathfrak{sp}_{2n}$ can be divided into two groups:

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n,$$

$$\alpha_{i,\bar{j}} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n$$

(note that $\alpha_{i,n} = \alpha_{i,\bar{n}}$). We will use the following short versions:

$$\alpha_i = \alpha_i, \quad \alpha_{\bar{i}} = \alpha_{i,\bar{i}}, \quad f_{i,j} = f_{\alpha_{i,j}}, \quad f_{i,\bar{j}} = f_{\alpha_{i,\bar{j}}}.$$

We recall the usual order on the alphabet $J = \{1, \ldots, n, \overline{n-1}, \ldots, \bar{1}\}$

$$1 < 2 < \cdots < n-1 < n < \overline{n-1} < \cdots < \bar{1}.$$

Let $\mathfrak{sp}_{2n} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the Cartan decomposition. Consider the increasing degree filtration on the universal enveloping algebra of $\mathrm{U}(\mathfrak{n}^-)$:

$$\mathrm{U}(\mathfrak{n}^-)_s = \mathrm{span}\{x_1 \cdots x_l : x_i \in \mathfrak{n}^-, l \le s\}, \tag{2.1}$$

for example, $\mathrm{U}(\mathfrak{n}^-)_0 = \mathbb{C}\cdot 1$.

For a dominant integral weight $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$ let $V(\lambda)$ be the corresponding irreducible highest weight $\mathfrak{sp}_{2n}$-module with a highest weight vector v_λ. Since $V(\lambda) = \mathrm{U}(\mathfrak{n}^-)v_\lambda$, the filtration (2.1) induces an increasing filtration $V(\lambda)_s$ on $V(\lambda)$:

$$V(\lambda)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\lambda.$$

We call this filtration the PBW filtration and study the associated graded space gr $V(\lambda)$. In the following lemma, we describe some operators acting on gr $V(\lambda)$. Let $S(\mathfrak{n}^-)$ denote the symmetric algebra of $\mathfrak{n}^-$.

Lemma 2.1. The action of $\mathrm{U}(\mathfrak{n}^-)$ on $V(\lambda)$ induces the structure of a $S(\mathfrak{n}^-)$-module on gr $V(\lambda)$ and

$$\mathrm{gr}(V(\lambda)) = S(\mathfrak{n}^-)v_\lambda.$$

The action of $\mathrm{U}(\mathfrak{n}^+)$ on $V(\lambda)$ induces the structure of a $U(\mathfrak{n}^+)$-module on gr $V(\lambda)$. □

Our aims are:

- to describe gr $V(\lambda)$ as an $S(\mathfrak{n}^-)$-module, that is, describe the ideal $I(\lambda) \hookrightarrow S(\mathfrak{n}^-)$ such that gr $V(\lambda) \simeq S(\mathfrak{n}^-)/I(\lambda)$;
- to find a basis of gr $V(\lambda)$.

The description of the ideal is given in the introduction (see Theorem A). To describe the basis, we introduce the notion of the symplectic Dyck paths.

Definition 2.2. A symplectic Dyck path (or simply a path) is a sequence

$$\mathbf{p} = (p(0), p(1), \ldots, p(k)), \quad k \geq 0$$

of positive roots satisfying the following conditions:

(a) the first root is simple, $p(0) = \alpha_i$ for some $1 \leq i \leq n$;
(b) the last root is either simple or the highest root of a symplectic subalgebra, more precisely $p(k) = \alpha_j$ or $p(k) = \alpha_{\overline{j}}$ for some $i \leq j \leq n$;
(c) the elements in between obey the following recursion rule: If $p(s) = \alpha_{r,q}$ with $r, q \in J$, then the next element in the sequence is of the form either $p(s+1) = \alpha_{r,q+1}$ or $p(s+1) = \alpha_{r+1,q}$, where $x+1$ denotes the smallest element in J which is bigger than x. □

To give a visual interpretation of the notion of a Dyck-path for $\mathfrak{sp}_8$, arrange the positive roots in the form of a triangle. In this picture, a Dyck path is a path in the directed graph, starting at a simple root and ending at one of the edges.

$$\begin{array}{ccccccccccccc}
\alpha_{1,1} & \to & \alpha_{1,2} & \to & \alpha_{1,3} & \to & \alpha_{1,4} & \to & \alpha_{1,\bar{3}} & \to & \alpha_{1,\bar{2}} & \to & \alpha_{1,\bar{1}} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \alpha_{2,2} & \to & \alpha_{2,3} & \to & \alpha_{2,4} & \to & \alpha_{2,\bar{3}} & \to & \alpha_{2,\bar{2}} & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & \alpha_{3,3} & \to & \alpha_{3,4} & \to & \alpha_{3,\bar{3}} & & & & \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & \alpha_{4,4} & & & & & &
\end{array}$$

Denote by $\mathbb{D}$ the set of all Dyck-paths. For a dominant weight $\lambda = \sum_{i=1}^{n} m_i \omega_i$ let $P(\lambda) \subset \mathbb{R}_{\geq 0}^{n^2}$ be the polytope

$$P(\lambda) := \left\{ (s_\alpha)_{\alpha>0} \mid \forall \mathbf{p} \in \mathbb{D} : \begin{array}{ll} \text{if } p(0) = \alpha_i,\ p(k) = \alpha_j & \text{then } s_{p(0)} + \cdots + s_{p(k)} \\ & \quad \leq m_i + \cdots + m_j, \\ \text{if } p(0) = \alpha_i,\ p(k) = \alpha_{\bar{j}} & \text{then } s_{p(0)} + \cdots + s_{p(k)} \\ & \quad \leq m_i + \cdots + m_n \end{array} \right\}, \tag{2.2}$$

and let $S(\lambda)$ be the set of integral points in $P(\lambda)$.

For a multi-exponent $\mathbf{s} = \{s_\beta\}_{\beta>0}$, $s_\beta \in \mathbb{Z}_{\geq 0}$, let $f^{\mathbf{s}}$ be the element

$$f^{\mathbf{s}} = \prod_{\beta \in R^+} f_\beta^{s_\beta} \in S(\mathfrak{n}^-).$$

In the next two sections, we prove the following theorem (Theorem B from the introduction), which immediately implies Corollary 1.1.

Theorem 2.3. The set $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$, forms a basis of $\operatorname{gr} V(\lambda)$. □

Proof. In Section 3, we show that the elements $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$, span $\operatorname{gr} V(\lambda)$, see Theorem 3.4. In Section 4, we show that the elements are linear independent in $\operatorname{gr} V(\lambda)$ (see Theorem 4.6), which finishes the proof. ■

3 The Spanning Property

We start with writing down the powers of certain positive roots annihilating a highest weight vector in an irreducible $\mathfrak{sp}_{2n}$-module.

Lemma 3.1. Let $\lambda=\sum_{i=1}^n m_i\omega_i$ be the $\mathfrak{sp}_{2n}$-weight and let $V(\lambda)$ be the corresponding highest weight module with highest weight vector v_λ. Then

$$f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1}v_\lambda=0,\quad 1\le i\le j\le n-1, \tag{3.1}$$

$$f_{\alpha_{i,\bar{i}}}^{m_i+\cdots+m_n+1}v_\lambda=0,\quad 1\le i\le n. \tag{3.2}$$

□

Proof. For each positive root α, we have the corresponding $\mathfrak{sl}_2$-triple $\{e_\alpha, h_\alpha, f_\alpha\}$. Now the lemma follows immediately from the $\mathfrak{sl}_2$-theory. ■

In the following, we use the differential operators ∂_α defined by

$$\partial_\alpha f_\beta=\begin{cases} f_{\beta-\alpha} & \text{if } \beta-\alpha\in\triangle^+,\\ 0 & \text{otherwise.}\end{cases}$$

As in the $\mathtt{A}_n$-case (see [9]), we have a natural action of $U(\mathfrak{n}^+)$ on $S(\mathfrak{n}^-)$ coming from the natural action of $U(\mathfrak{n}^+)$ on $S(\mathfrak{g})$ and the identification $S(\mathfrak{n}^-)\simeq S(\mathfrak{g})/S(\mathfrak{n}^-)S_+(\mathfrak{h}\oplus\mathfrak{n}^+)$. The operators ∂_α satisfy the property

$$\partial_\alpha f_\beta=c_{\alpha,\beta}(\operatorname{ad} e_\alpha)(f_\beta),$$

where $c_{\alpha,\beta}$ are some nonzero constants. In what follows, we sometimes use the equality $\alpha_{i,\bar{n}}=\alpha_{i,n}$. We also use the notation

$$\partial_{i,j}=\partial_{\alpha_{i,j}},\quad \partial_{i,\bar{j}}=\partial_{\alpha_{i,\bar{j}}}.$$

Lemma 3.2. The only nontrivial vectors of the form $\partial_\beta f_\alpha$, $\alpha,\beta>0$ are as follows: for $\alpha=\alpha_{i,j}$, $1\le i\le j\le n$

$$\partial_{i,s}f_{i,j}=f_{s+1,j},\quad i\le s<j,\qquad \partial_{s,j}f_{i,j}=f_{i,s-1},\quad i<s\le j, \tag{3.3}$$

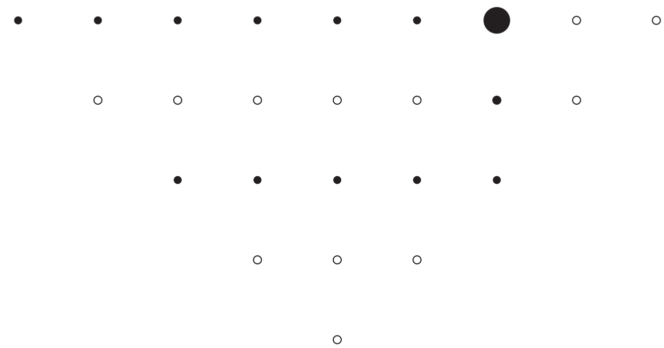
and for $\alpha=\alpha_{i,\bar{j}}$, $1\le i\le j\le n$

$$\partial_{i,s}f_{i,\bar{j}}=f_{s+1,\bar{j}},\quad i\le s<j,\qquad \partial_{i,s}f_{i,\bar{j}}=f_{j,\overline{s+1}},\quad j\le s,\qquad \partial_{i,\bar{s}}f_{i,\bar{j}}=f_{j,s-1},\quad j<s, \tag{3.4}$$

$$\partial_{s+1,\bar{j}}f_{i,\bar{j}}=f_{i,s},\quad i\le s<j,\qquad \partial_{j,\overline{s+1}}f_{i,\bar{j}}=f_{i,s},\quad j\le s,\qquad \partial_{j,s-1}f_{i,\bar{j}}=f_{i,\bar{s}},\quad j<s. \tag{3.5}$$

□

Let us illustrate this lemma by the following picture in type C_5.



Here all circles correspond to the positive roots of the root system of type C_5 in the following way: in the upper row, we have from left to right $\alpha_{1,1}, \ldots, \alpha_{1,5}, \alpha_{1,\bar{4}}, \ldots, \alpha_{1,\bar{1}}$, in the second row, we have from left to right $\alpha_{2,2}, \ldots, \alpha_{2,5}, \alpha_{2,\bar{4}}, \ldots, \alpha_{2,\bar{2}}$, and the last line corresponds to the root $\alpha_{5,5}$. Now let us take the root $\alpha_{1,\bar{3}}$ (which corresponds to the fat circle). Then all roots which can be obtained by applying the operators ∂_β are depicted as filled small circles.

The following remark will be important to us.

Remark 3.3. Formula (3.3) reproduces the picture in type A_n. Formulas (3.3)–(3.5) resemble the situation in type A_{2n-1}. The difference is that in the symplectic case the roots $\partial_\beta f_\alpha$ with fixed α do not form two segments (as in type A), but three segments. □

Our goal is to prove the following theorem.

Theorem 3.4.

(i) The vectors $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$ span $\mathrm{gr}\, V(\lambda)$.

(ii) Let $I(\lambda)$ be the ideal $I(\lambda) = S(\mathfrak{n}^-)(\mathrm{U}(\mathfrak{n}^+) \circ R)$, where

$$R = \mathrm{span}\{ f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1},\ 1 \le i \le j \le n-1,\ \ f_{\alpha_{i,\bar{i}}}^{m_i+\cdots+m_n+1},\ 1 \le i \le n\}.$$

There exists a monomial order on $S(\mathfrak{n}^-) = \mathbb{C}[f_\alpha \mid \alpha > 0]$, denoted by "$\succ$", such that for any $\mathbf{s} \notin S(\lambda)$ there exists a homogeneous expression (a straightening law) of the form

$$f^{\mathbf{s}} - \sum_{\mathbf{s} \succ \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}} \in I(\lambda). \tag{3.6}$$

□

Remark 3.5. In the following, we refer to (3.6) as a *straightening law* for the polynomial ring $S(\mathfrak{n}^-) = \mathbb{C}[f_\alpha \mid \alpha > 0]$ with respect to the ideal $I(\lambda)$. Such a straightening law implies that in the quotient ring $S(\mathfrak{n}^-)/I(\lambda)$ we can express $f^{\mathbf{s}}$ for $\mathbf{s} \notin S(\lambda)$ as a linear combination of monomials which are smaller in the monomial order than $f^{\mathbf{s}}$, but of the same total degree since the expression in (3.6) is homogeneous. □

We show first that (ii) implies (i).

Proof. [(ii) $\Rightarrow$ (i)] The elements in R obviously annihilate $v_\lambda \in \operatorname{gr} V(\lambda)$, and so do the elements of $\mathrm{U}(\mathfrak{n}^+) \circ R$, and hence so do the elements of the ideal $I(\lambda)$. As a consequence, we get a surjective map $S(\mathfrak{n}^-)/I(\lambda) \to \operatorname{gr} V(\lambda)$.

Suppose $\mathbf{s} \notin S(\lambda)$. We know by (ii) that $f^{\mathbf{s}} = \sum_{\mathbf{s} \succ \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}}$ in $S(\mathfrak{n}^-)/I(\lambda)$. If some $\mathbf{t}$ with nonzero coefficient $c_{\mathbf{t}}$ is not an element of $S(\lambda)$, then we can again apply a straightening law and replace $f^{\mathbf{t}}$ by a linear combination of smaller monomials. Since there are only finite number of monomials of the same total degree, by repeating the procedure if necessary, after a finite number of steps we obtain an expression of $f^{\mathbf{s}}$ in $S(\mathfrak{n}^-)/I(\lambda)$ as a linear combination of elements $f^{\mathbf{t}}$, $\mathbf{t} \in S(\lambda)$. It follows that the set $\{f^{\mathbf{t}} \mid \mathbf{t} \in S(\lambda)\}$ is a spanning set for $S(\mathfrak{n}^-)/I(\lambda)$, and hence, by the surjection above, we get a spanning set $\{f^{\mathbf{t}} v_\lambda \mid \mathbf{t} \in S(\lambda)\}$ for $\operatorname{gr} V(\lambda)$. ■

To prove the second part, we need to define the total order. We start by defining a total order on the variables:

$$\begin{array}{rcl} & f_{n,n} & > \\ f_{n-1,\overline{n-1}} > & f_{n-1,n} & > f_{n-1,n-1} > \\ f_{n-2,\overline{n-2}} > f_{n-2,\overline{n-1}} > & f_{n-2,n} & > f_{n-2,n-1} > f_{n-2,n-2} > \\ \ldots > & \ldots & > \ldots > \\ f_{1,\overline{1}} > f_{1,\overline{2}} > \cdots > f_{1,\overline{n-1}} > & f_{1,n} & > f_{1,n-1} > \cdots > f_{1,2} > f_{1,1}. \end{array} \tag{3.7}$$

We use the same notation for the induced homogeneous lexicographic ordering on the monomials. Note that this monomial order $>$ is not the order $\succ$. To define the latter, we need some more notation. Let

$$s_{\bullet,j} = \sum_{i=1}^{j} s_{i,j}, \quad s_{\bullet,\overline{j}} = \sum_{i=1}^{j} s_{i,\overline{j}},$$

$$s_{i,\bullet} = \sum_{j=i}^{n} s_{i,j} + \sum_{j=i}^{n-1} s_{i,\overline{j}}.$$

Define a map d from the set of multi-exponents $\mathbf{s}$ to $\mathbb{Z}^n_{\geq 0}$:

$$d(\mathbf{s}) = (s_{n,\bullet}, s_{n-1,\bullet}, \ldots, s_{1,\bullet}).$$

So, $d(\mathbf{s})_i = s_{n-i+1,\bullet}$. We say $d(\mathbf{s}) > d(\mathbf{t})$, if there exists an i such that

$$d(\mathbf{s})_1 = d(\mathbf{t})_1, \ldots, d(\mathbf{s})_i = d(\mathbf{t})_i, d(\mathbf{s})_{i+1} > d(\mathbf{t})_{i+1}.$$

Definition 3.6. For two monomials $f^{\mathbf{s}}$ and $f^{\mathbf{t}}$, we say $f^{\mathbf{s}} \succ f^{\mathbf{t}}$ if either

(a) the total degree of $f^{\mathbf{s}}$ is greater than the total degree of $f^{\mathbf{t}}$; or
(b) both have the same total degree, but $d(\mathbf{s}) < d(\mathbf{t})$; or
(c) both have the same total degree, $d(\mathbf{s}) = d(\mathbf{t})$, but $f^{\mathbf{s}} > f^{\mathbf{t}}$. □

In words, if both have the same total degree, this definition says that $f^{\mathbf{s}}$ is greater than $f^{\mathbf{t}}$ if $d(\mathbf{s})$ is smaller than $d(\mathbf{t})$, or $d(\mathbf{s}) = d(\mathbf{t})$ but $f^{\mathbf{s}} > f^{\mathbf{t}}$ with respect to the homogeneous lexicographic ordering on $\mathbb{C}[f_\alpha \mid \alpha > 0]$.

Remark 3.7. It is easy to check that "$\succ$" defines a *monomial ordering*, that is, if $f^{\mathbf{s}} \succ f^{\mathbf{t}}$ and $f^{\mathbf{m}} \neq 1$, then

$$f^{\mathbf{s}} f^{\mathbf{m}} = f^{\mathbf{s}+\mathbf{m}} \succ f^{\mathbf{t}} f^{\mathbf{m}} = f^{\mathbf{t}+\mathbf{m}} \succ f^{\mathbf{t}}.$$ □

Slightly abusing notation, we use the same symbol $\succ$ also for the multi-exponents: we write $\mathbf{s} \succ \mathbf{t}$ if and only if $f^{\mathbf{s}} \succ f^{\mathbf{t}}$.

Proof of Theorem 3.4(ii). We discuss first some reduction steps. Let $\mathbf{s}$ be a multi-exponent violating some of the Dyck paths condition from the definition of $S(\lambda)$ and let $\mathbf{p}$ be a corresponding Dyck path. We write $\mathbf{s}$ as a sum $\mathbf{s} = \mathbf{s}^1 + \mathbf{s}^2$, where $\mathbf{s}^1$ is defined as follows: $s^1_\alpha = s_\alpha$ if $\alpha \in \mathbf{p}$ and $s^1_\alpha = 0$ if $\alpha \notin \mathbf{p}$, so s^1_α has support (i.e. nonzero entries) only on $\mathbf{p}$. Now obviously we still have $\mathbf{s}^1 \notin S(\lambda)$. If we have a straightening law for $f^{\mathbf{s}^1}$:

$$f^{\mathbf{s}^1} - \sum_{\mathbf{s}^1 \succ \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}} \in I(\lambda)$$

then multiplication by $f^{\mathbf{s}^2}$ gives a straightening law for $f^{\mathbf{s}} = f^{\mathbf{s}^1} f^{\mathbf{s}^2}$, because $\succ$ is a monomial order.

So it suffices to find a straightening law for those $\mathbf{s} \notin S(\lambda)$ which are supported on a Dyck path $\mathbf{p}$ and $\mathbf{s}$ violates the Dyck path condition for $S(\lambda)$ for the path $\mathbf{p}$.

Suppose first that the Dyck path $\mathbf{p}$ is such that $p(0) = \alpha_i$, $p(k) = \alpha_j$ for some $1 \leq i \leq j < n$. We are going to show that in this case we get a straightening law by the corresponding result for the Lie algebra $\mathfrak{sl}_n$ from [9]. In fact, consider the Lie subalgebra $M \subset \mathfrak{sp}_{2n}$ generated by the elements $e_{\alpha_{i,i}}, f_{\alpha_{i,i}}, h_{\alpha_{i,i}}$, $1 \leq i < n$. This subalgebra is isomorphic to $\mathfrak{sl}_n$. Let $M = \mathfrak{n}_M^+ \oplus \mathfrak{h}_M \oplus \mathfrak{n}_M^-$ be the Cartan decomposition obtained by setting $\mathfrak{n}_M^+ = \mathfrak{n}^+ \cap M$, $\mathfrak{n}_M^- = \mathfrak{n}^- \cap M$ and $\mathfrak{h}_M = \mathfrak{h} \cap M$. Let

$$R_M = \mathrm{span}\{f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1}, 1 \leq i \leq j \leq n-1\} \subset S(\mathfrak{n}_M^-) \subset S(\mathfrak{n}^-).$$

Then $R_M \subset R$ and $U(\mathfrak{n}_M^+) \circ R_M \subset U(\mathfrak{n}^+) \circ R$. Set $\lambda_M = \sum_{i=1}^{n-1} m_i \omega_i$ and let $I_M(\lambda_M)$ be the ideal $I_M(\lambda) := S(\mathfrak{n}_M^-)(U(\mathfrak{n}_M^+) \circ R_M) \subset S(\mathfrak{n}_M^-)$.

We have an obvious inclusion $I_M(\lambda_M) \subset I(\lambda)$. But note that the ideal $I_M(\lambda_M)$ is considered in [9] for the $\mathfrak{sl}_n$-case (recall, $M \simeq \mathfrak{sl}_n$). Also the Dyck path considered here is an $\mathfrak{sl}_n$ Dyck path, because all roots occurring in the path are roots in the subroot-system associated to the Lie-subalgebra M. It follows by [9] that we have a straightening law

$$f^{\mathbf{s}} - \sum_{\mathbf{s} > \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}} \in I_M(\lambda_M) \subset I(\lambda). \tag{3.8}$$

It remains to show that in the sum above we can replace "$>$" by "$\succ$". For this, we need to recall the proof in the type $\mathtt{A}$-case. Recall that we work now with the subalgebra $M \simeq \mathfrak{sl}_n \subset \mathfrak{sp}_{2n}$. To get the straightening law above, one starts with the element $f_{1,i}^{s_{p(0)}+\cdots+s_{p(k)}} \in R_M$. Applying the ∂-operators (see [9]) one shows that

$$B = f_{1,1}^{s_{\bullet,1}} f_{1,2}^{s_{\bullet,2}} \cdots f_{1,i}^{s_{\bullet,i}} \in R_M.$$

One applies then the following ∂-operators to B to get

$$A = \partial_{1,1}^{s_{2,\bullet}} \partial_{1,2}^{s_{3,\bullet}} \cdots \partial_{1,i-1}^{s_{i,\bullet}} B \in R_M \tag{3.9}$$

(since $\mathbf{s}$ is supported on $\mathbf{p}$ and $p(k) = \alpha_j$, $j < n$, we have $s_{l,\bullet} = \sum_{j=l}^{n-1} s_{l,j}$). We show in [9] that

$$A = \sum_{\mathbf{t} \leq \mathbf{s}} c_{\mathbf{t}} F^{\mathbf{t}} \tag{3.10}$$

for some $c_{\mathbf{s}} \neq 0$, which gives rise to the straightening law in (3.8). Now in this special case Lemma 3.2 implies that the application of the ∂-operators in (3.9) produces only summands such that $d(\mathbf{s}) = d(\mathbf{t})$ for any $\mathbf{t}$ occurring in the sum with a nonzero coefficient. Hence we can replace "$>$" by "$\succ$" in (3.8), which finishes the proof of the theorem in this case.

Now assume $p(0) = \alpha_{i,i}$ and $p(k) = \alpha_{j,\bar{j}}$ for some $j \geq i$. We include the case $j = n$ by writing $\alpha_{n,n} = \alpha_{n,\bar{n}}$. We proceed by induction on n. For $n = 1$ we have $\mathfrak{sp}_2 = \mathfrak{sl}_2$, so we can refer to [9]. Now assume that we have proved the existence of a straightening law for all symplectic algebras of rank strictly smaller than n. If $i > 1$, then the Dyck path is also a Dyck path for the symplectic subalgebra $L \simeq \mathfrak{sp}_{2n-2(i-1)}$ generated by $e_{\alpha_{k,k}}, f_{\alpha_{k,k}}, h_{\alpha_{k,k}}$, $i \leq k \leq n$. Let $\mathfrak{n}_L^+, \mathfrak{n}_L^-$, etc. be defined by the intersection of $\mathfrak{n}^+, \mathfrak{n}^-$, etc. with L and set $\lambda_L = \sum_{k=i}^n m_k \omega_k$. It is now easy to see that the straightening law for $f^{\mathbf{s}}$ viewed as an element in $S(\mathfrak{n}_L^-)$ with respect to $I_L(\lambda_L)$ defines also a straightening law for $f^{\mathbf{s}}$ viewed as an element in $S(\mathfrak{n}^-)$ with respect to $I(\lambda)$.

So from now on we fix $p(0) = \alpha_1$ and $p(k) = \alpha_{i,\bar{i}}$ for some $i \in \{1, \ldots, n\}$. For a multi-exponent $\mathbf{s}$ supported on $\mathbf{p}$, set

$$\Sigma = \sum_{l=0}^{k} s_{p(l)} > m_1 + \cdots + m_n.$$

We have obviously $f_{1,\bar{1}}^{\Sigma} \in I(\lambda)$. We consider now two operators

$$\Delta_1 := \partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}} \underbrace{\partial_{i+1,\overline{i+1}}^{s_{\bullet,i}} \cdots \partial_{n,\bar{n}}^{s_{\bullet,n-1}}}_{\delta_3} \underbrace{\partial_{1,n-1}^{s_{\bullet,n-1}+s_{\bullet,\bar{n}}} \cdots \partial_{1,i}^{s_{\bullet,i}+s_{\bullet,\overline{i+1}}}}_{\delta_2} \underbrace{\partial_{1,\bar{i}}^{s_{\bullet,i-1}} \cdots \partial_{1,\bar{3}}^{s_{\bullet,2}} \partial_{1,\bar{2}}^{s_{\bullet,1}}}_{\delta_1}$$

(so $\Delta_1 := \partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}} \delta_3 \delta_2 \delta_1$) and

$$\Delta_2 := \partial_{1,1}^{s_{2,\bullet}} \partial_{1,2}^{s_{3,\bullet}} \cdots \partial_{1,i-2}^{s_{i-1,\bullet}}.$$

We will show that

$$\Delta_2 \Delta_1 f_{1,\bar{1}}^{\Sigma} = c_{\mathbf{s}} f^{\mathbf{s}} + \sum_{\mathbf{s} \succ \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}} \tag{3.11}$$

with complex coefficients $c_{\mathbf{s}}$ and $c_{\mathbf{t}}$, where $c_{\mathbf{s}} \neq 0$. Since $\Delta_2 \Delta_1 f_{1,\bar{1}}^{\Sigma} \in I(\lambda)$, the proof of (3.11) finishes the proof of the theorem. A first step in the proof of (3.11) is the following lemma.

Recall the alphabet $J=\{1,\dots,n,\overline{n-1},\dots,\overline{1}\}$. Let $q_1,\dots,q_i\in J$ be a sequence of increasing elements defined by

$$q_k=\max\{l\in J:\ \alpha_{k,l}\in\mathbf{p}\}.$$

For example, $q_i=\bar{i}$. The roots of $\mathbf{p}$ are then of the form

$$\alpha_{1,1},\dots,\alpha_{1,q_1},\alpha_{2,q_1},\dots,\alpha_{2,q_2},\dots,\alpha_{i,q_{i-1}},\dots,\alpha_{i,q_i}.$$

Lemma 3.8. Set $f^{\mathbf{s}'}=f_{1,1}^{s_{\bullet,1}}f_{1,2}^{s_{\bullet,2}}\cdots f_{1,q_{i-1}}^{s_{\bullet,q_{i-1}}-s_{i,q_{i-1}}}f_{i,q_{i-1}}^{s_{i,q_{i-1}}}\cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}$. Then $\Delta_1 f_{1,\bar{1}}^{\Sigma}$ is of the form

$$\Delta_1 f_{1,\bar{1}}^{\Sigma}=c_{\mathbf{s}'}f^{\mathbf{s}'}+\sum_{\mathbf{s}'\succ\mathbf{t}}c_{\mathbf{t}}f^{\mathbf{t}} \tag{3.12}$$

such that $c_{\mathbf{s}'}\neq 0$. In addition, if $f^{\mathbf{t}}$, $\mathbf{t}\neq\mathbf{s}'$, is a monomial occurring in this sum, then one of the following statements holds:

- there exists an index j such that $d(\mathbf{t})_j>0$ for some $j\in\{1,2,\dots,n-i\}$,
- $d(\mathbf{t})_j=0$ for all $j\in\{1,2,\dots,n-i\}$ and $d(\mathbf{t})_{n-i+1}>s_{i,\bullet}$,
- $d(\mathbf{t})=d(\mathbf{s}')$ and $f_{i,i}^{t_{i,i}}f_{i,i+1}^{t_{i,i+1}}\cdots f_{i,\bar{i}}^{t_{i,\bar{i}}}<f_{i,i}^{s_{i,i}}f_{i,i+1}^{s_{i,i+1}}\cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}$. □

Before proving the lemma, we explain in the following corollary the reason why we need the lemma. The corollary is proved after the proof of the lemma.

Corollary 3.9. If $f^{\mathbf{t}}\neq f^{\mathbf{s}'}$ is a monomial occurring in (3.12), then $\Delta_2 f^{\mathbf{t}}$ is a sum of monomials $f^{\mathbf{k}}$ such that $f^{\mathbf{s}}\succ f^{\mathbf{k}}$. □

Proof of the lemma. One sees easily by induction that

$$\delta_1(f_{1,\bar{1}}^{\Sigma})=f_{1,1}^{s_{\bullet,1}}f_{1,2}^{s_{\bullet,2}}\cdots f_{1,i-1}^{s_{\bullet,i-1}}f_{1,\bar{1}}^{\Sigma-s_{\bullet,1}-s_{\bullet,2}-\cdots-s_{\bullet,i-1}}.$$

Since $\alpha_{1,j}-\alpha_{1,\ell}$, $1\le j<i$, $i<\ell\le n$, and $\alpha_{1,j}-\alpha_{\ell,\bar{\ell}}$, $1\le j<i$, $i<\ell\le n$, and $\alpha_{1,j}-\alpha_{1,i-1}$, $1\le j<i$, are never positive roots, all factors of δ_2 and δ_3, as well as $\partial_{1,i-1}$, annihilate the vector

$$f^{\mathbf{x}}=f_{1,1}^{s_{\bullet,1}}f_{1,2}^{s_{\bullet,2}}\cdots f_{1,i-1}^{s_{\bullet,i-1}}.$$

Therefore

$$\Delta_1(f_{1,\bar{1}}^{\Sigma})=f^{\mathbf{x}}\partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}}\delta_3\delta_2(f_{1,\bar{1}}^{\Sigma-s_{\bullet,1}-s_{\bullet,2}-\cdots-s_{\bullet,i-1}}).$$

To visualize the following procedure, one should think of the variables $f_{i,j}$ as being arranged in a triangle as in the picture after Lemma 3.2, or in the following example (type $\mathtt{C}_4$):

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
f_{11} & f_{12} & f_{13} & f_{14} & f_{1\bar{3}} & f_{1\bar{2}} & f_{1\bar{1}} \\
\hline
 & f_{22} & f_{23} & f_{24} & f_{2\bar{3}} & f_{2\bar{2}} & \\
\cline{2-6}
 & & f_{33} & f_{34} & f_{3\bar{3}} & & \\
\cline{3-5}
 & & & f_{44} & & & \\
\cline{4-4}
\end{array} \tag{3.13}$$

With respect to the ordering ">", the largest element is in the bottom row and the smallest element is in the top row on the left side. We enumerate the rows and columns as the indices of the variables, so the top row is the 1st row, the bottom row the nth row, the columns are enumerated from left to right, so we have the 1st column on the left side and the most right one is the $\bar{1}$st column.

The operator $\partial_{1,q}$, $1 \le q \le n-1$, kills all $f_{1,j}$ for $1 \le j \le q$, $\partial_{1,q}(f_{1,j}) = f_{q+1,j}$ for $j = q+1, \ldots, \overline{q+1}$, $\partial_{1,q}(f_{1,\bar{j}}) = f_{j,\overline{q+1}}$ for $j = 1, \ldots, q$, and $\partial_{1,q}$ kills all $f_{k,\ell}$ for $k \ge 2$. Because of the set of indices of the operators occurring in δ_2, the operator applied to $f_{1,\bar{1}}^{\Sigma - s_{\bullet,1} - s_{\bullet,2} - \cdots - s_{\bullet,i-1}}$ never increases the zero entries in the first row, column $\bar{i}$ up to column $\bar{2}$. As a consequence, the application of δ_2 produces the monomial

$$f^{\mathbf{x}} f_{1,\overline{i+1}}^{s_{\bullet,i}+s_{\bullet,\overline{i+1}}} \cdots f_{1,\overline{n-1}}^{s_{\bullet,n-2}+s_{\bullet,\overline{n-1}}} f_{1,n}^{s_{\bullet,n-1}+s_{\bullet,n}} f_{1,\bar{1}}^{s_{\bullet,\bar{i}}} + \sum c_{\mathbf{k}} f^{\mathbf{k}},$$

where the monomials $f^{\mathbf{k}}$ occurring in the sum are such that the corresponding triangle (see (3.13)) has at least one non-zero entry in one of the rows between the $(i+1)$th row and the nth row (counted from top to bottom). This implies $d(\mathbf{k})_j > 0$ for some $j = 1, \ldots, n-i$. The operators δ_3 and $\partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}}$ do not change this property, because (in the language of the scheme (3.13) above) the operators $\partial_{j,\bar{j}}$ used to compose δ_3 either kill a monomial or, in the language of the scheme (3.13), they subtract from an entry in the $\bar{j}$th column, kth row and add to the entry in the same row, but $(j-1)$th column. The operator $\partial_{1,i-1}$ subtracts from the entries in the top row. Since the entries in the top row, column $\overline{i-1}$ up to $\bar{2}$, are zero, it adds to the entries in the ith row. The only exception is $\partial_{1,i-1}$ applied to $f_{1,\bar{1}}$, the result is $f_{1,\bar{i}}$. It follows that the monomials $f^{\mathbf{k}'}$ occurring in $\partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}} \delta_3 f^{\mathbf{k}}$ already have the desired properties, because we have just seen that $d(\mathbf{k}')_j > 0$ for some $j = 1, \ldots, n-i$.

So to finish the proof of the lemma, it suffices to look at

$$f^{\mathbf{x}}\partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}}\delta_3 f_{1,\overline{i+1}}^{s_{\bullet,i}+s_{\bullet,\overline{i+1}}}\cdots f_{1,\overline{n-1}}^{s_{\bullet,n-2}+s_{\bullet,\overline{n-1}}} f_{1,n}^{s_{\bullet,n-1}+s_{\bullet,n}} f_{1,\bar{1}}^{s_{\bullet,\bar{i}}} \tag{3.14}$$

$$= f^{\mathbf{x}}\partial_{1,i-1}^{s_{\bullet,\bar{i}}+s_{i,\bullet}} f_{1,i}^{s_{\bullet,i}} f_{1,i+1}^{s_{\bullet,i+1}}\cdots f_{1,n}^{s_{\bullet,n}} f_{1,\overline{n-1}}^{s_{\bullet,\overline{n-1}}}\cdots f_{1,\overline{i+1}}^{s_{\bullet,\overline{i+1}}} f_{1,\bar{1}}^{s_{\bullet,\bar{i}}} \tag{3.15}$$

Note that the operator $\partial_{1,i-1}$ being applied to any variable in (3.15) but to $f_{1,\bar{1}}$, increases the degree with respect to the variables $f_{i,*}$ or gives zero. We note also that $\partial_{1,i-1} f_{1,\bar{1}} = f_{1,\bar{i}}$. So (3.14) written as a linear combination $\sum c_{\mathbf{k}} f^{\mathbf{k}}$ of monomials such that $d(\mathbf{k})_j = 0$ for $j = 1, \ldots, n-i$ and $d(\mathbf{k})_{n-i+1} \geq s_{i,\bullet}$.

It remains to consider the case where $d(\mathbf{k})_{n-i+1} = s_{i,\bullet}$. This is only possible if $\partial_{1,i-1}$ is applied $s_{\bullet,\bar{i}} + s_{i,\bullet}$-times to $f_{1,\bar{1}}^{s_{\bullet,\bar{i}}}$, in which case $d(\mathbf{k})$ has only two nonzero entries: $d(\mathbf{k})_n = \Sigma - s_{i,\bullet}$ and $d(\mathbf{k})_{n-i+1} = s_{i,\bullet}$, so $d(\mathbf{k}) = d(\mathbf{s}')$. If $\mathbf{k} \neq \mathbf{s}'$, then necessarily $f_{i,i}^{t_{i,i}} f_{i,i+1}^{t_{i,i+1}} \cdots f_{i,\bar{i}}^{t_{i,\bar{i}}} < f_{i,i}^{s_{i,i}} f_{i,i+1}^{s_{i,i+1}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}$. ■

Proof of the corollary. The operators used to compose Δ_2 do not change anymore the entries of $d(\mathbf{t})$ for the first $n-i+1$ indices.

Suppose first $\mathbf{t}$ is such that there exists an index j such that $d(\mathbf{t})_j > 0$ for some $j \in \{1, 2, \ldots, n-i\}$ or $d(\mathbf{t})_{n-i+1} > s_{i,\bullet}$. By the description of the operators occurring in Δ_2, every monomial $f^{\mathbf{k}}$ occurring with a nonzero coefficient in $\Delta_2 f^{\mathbf{t}}$ has this property too and hence $f^{\mathbf{s}} \succ f^{\mathbf{k}}$.

Next assume $d(\mathbf{t}) = d(\mathbf{s}')$ and $f_{i,i}^{t_{i,i}} f_{i,i+1}^{t_{i,i+1}} \cdots f_{i,\bar{i}}^{t_{i,\bar{i}}} < f_{i,i}^{s_{i,i}} f_{i,i+1}^{s_{i,i+1}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}$. Recall that $\mathbf{t}_{1,\overline{i-1}} = \cdots = \mathbf{t}_{1,\bar{1}} = 0$. It follows that the operators occurring in Δ_2 always only subtract from one of the entries in the top row and add to the entry in the same column and a corresponding row (of index strictly smaller than i). It follows that all monomials $f^{\mathbf{k}}$ occurring in $\Delta_2(f^{\mathbf{t}})$ have the property: $d(\mathbf{k}) = d(\mathbf{s})$. Since $f_{i,i}^{t_{i,i}} f_{i,i+1}^{t_{i,i+1}} \cdots f_{i,\bar{i}}^{t_{i,\bar{i}}} < f_{i,i}^{s_{i,i}} f_{i,i+1}^{s_{i,i+1}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}$, it follows that $f^{\mathbf{s}} > f^{\mathbf{k}}$ and hence $f^{\mathbf{s}} \succ f^{\mathbf{k}}$. ■

Continuation of the proof of Theorem 3.4(ii). We have seen that to prove Theorem 3.4(ii), it suffices to prove (3.11). By Lemma 3.8 and Corollary 3.9, it remains to prove for $f^{\mathbf{s}'}$ that $\Delta_2 f^{\mathbf{s}'}$ is a linear combination of $f^{\mathbf{s}}$ with a non trivial coefficient and monomials strictly smaller than $f^{\mathbf{s}}$. The following lemma proves this claim and hence finishes the proof of the theorem.

The following lemma completes the proof of part (ii) of Theorem 3.4.

Lemma 3.10. The operator $\Delta_2 := \partial_{1,1}^{s_{2,\bullet}} \partial_{1,2}^{s_{3,\bullet}} \cdots \partial_{1,i-2}^{s_{i-1,\bullet}}$ applied to the monomial $f^{\mathbf{s}'}$ is a linear combination of $f^{\mathbf{s}}$ and smaller monomials:

$$\Delta_2 f^{\mathbf{s}'} = cf^{\mathbf{s}} + \sum_{\mathbf{s} \succ \mathbf{t}} c_{\mathbf{t}} f^{\mathbf{t}} \quad \text{where } c \neq 0. \tag{3.16}$$

□

■

Proof. First note that all monomials $f^{\mathbf{k}}$ occurring in $\Delta_2 f^{\mathbf{s}'}$ have the same total degree. Recall that $\mathbf{s}'_{1,\overline{i-1}} = \cdots = \mathbf{s}'_{1,\bar{1}} = 0$. It follows that the operators occurring in Δ_2 always only subtract from one of the entries in the top row and add to the entry in the same column and a corresponding row (of index strictly smaller than i and strictly greater than 1). Thus, all monomials $f^{\mathbf{k}}$ occurring in $\Delta_2(f^{\mathbf{s}'})$ have the same multi-degree. In fact, we will see below that $f^{\mathbf{s}}$ is a summand and hence $d(\mathbf{k}) = d(\mathbf{s})$.

So in the following, we can replace the ordering $\succ$ by $>$ since, in this special case, the latter implies the first.

The elements $f_{i,j}$ and $f_{i,\bar{j}}$, $2 \leq i \leq j \leq n$ are in the kernel of the operators $\partial_{1,k}$ for all $1 \leq k \leq n$, and so are the variables $f_{1,j}$, $j \leq k$ in the first k columns.

The operator $\partial_{1,k}$, $1 \leq k \leq n$, "moves" the variables $f_{1,j}$, $k+1 \leq j \leq n$ from the first row to the variable $f_{k+1,j}$ in the same column.

The operator $\partial_{1,k}$, $1 \leq k \leq n$ "moves" the variables $f_{1,\bar{j}}$, $k+1 \leq j \leq n$ from the first row to the variable $f_{k+1,\bar{j}}$ in the same column. For $j \leq k$, the operator makes the variables switch also the column, it moves the variable $f_{1,\bar{j}}$ to the variable $f_{j,\overline{k+1}}$ in the jth row and $(\overline{k+1})$th column.

If $i = 1, 2$, then Δ_2 is the identity operator, $f^{\mathbf{s}} = f^{\mathbf{s}'}$ and hence the lemma is trivially true. Now assume $i \geq 3$. We note that the monomial

$$f_{1,1}^{s_{1,1}} \cdots f_{1,q_1}^{s_{1,q_1}} \cdot \left(\partial_{1,1}^{s_{2,q_1}} f_{1,q_1}^{s_{2,q_1}} \cdots \partial_{1,1}^{s_{2,q_2}} f_{1,q_2}^{s_{2,q_2}}\right) \cdot \ldots$$
$$\cdot \left(\partial_{1,i-2}^{s_{i-1,q_{i-2}}} f_{1,q_{i-2}}^{s_{i-1,q_{i-2}}} \cdots \partial_{1,i-2}^{s_{i-1,q_{i-1}}} f_{1,q_{i-1}}^{s_{i-1,q_{i-1}}}\right) \left(f_{i,q_{i-1}}^{s_{i,q_{i-1}}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}\right)$$

is proportional to $f^{\mathbf{s}}$ and appears as a summand in $\Delta_2 f^{\mathbf{s}'}$. Our goal is to show that all other monomials in $\Delta_2 f^{\mathbf{s}'}$ are less than $f^{\mathbf{s}}$.

All monomials share the common factor $(f_{i,q_{i-1}}^{s_{i,q_{i-1}}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}})$. The maximal variable smaller than those occurring in this factor is the variable $f_{i-1,q_{i-1}}$. Note that if $j < i-1$ then for any $q \in J$ the variable $\partial_{1,j} f_{1,q}$ lies in the $(j+1)$th row and $j+1 < i$. The operator $\partial_{1,i-2}$ is applied $s_{i-1,\bullet}$ times and the unique maximal monomial in the sum expression of $\partial_{1,i-2}^{s_{i-1,\bullet}} f^{\mathbf{s}'}$ is

$$f_{1,1}^{s_{\bullet,1}} f_{1,2}^{s_{\bullet,2}} \cdots f_{1,q_{i-2}}^{s_{\bullet,q_{i-2}}-s_{i-1,q_{i-2}}} \left(f_{i-1,q_{i-2}}^{s_{i-1,q_{i-2}}} \cdots f_{i-1,q_{i-1}}^{s_{i-1,q_{i-1}}}\right)\left(f_{i,q_{i-1}}^{s_{i,q_{i-1}}} \cdots f_{i,\bar{i}}^{s_{i,\bar{i}}}\right).$$

In fact, applying the operator $\partial_{1,i-2}$ to any of the variables $f_{1,j}$ such that $j \neq q_{i-2}, \ldots, q_{i-1}$, one gets a monomial smaller in the order $>$. The exponents $s_{i-1,j}$, $j = q_{i-2}, \ldots, q_{i-1}$, are the maximal powers such that $\partial_{1,i-2}$ can be applied to $f_{1,j}^{y}$ because either $q_{i-2} < j < q_{i-1}$, and then $y = s_{\bullet,j} = s_{i-1,j}$, or $j = q_{i-1}$, then $s_{i-1,q_{i-1}}$ is the power with which the variable occurs in $f^{\mathbf{s}'}$, or $j = q_{i-2}$, then only the power $s_{i-1,q_{i-2}}$ of the operator is left.

Repeating the arguments for the operators $\partial_{1,i-3}$, etc. we complete the proof of the lemma. ■

4 Main Theorem

Recall that in [1] the equality $\#S(\lambda) = \dim V(\lambda)$ is proved using purely combinatorial tools. Combining this result with Theorem 3.4 we obtain Theorems A and B from the introduction. However, in this section, we present a representation theoretical proof of the equality $\#S(\lambda) = \dim V(\lambda)$ by showing that the vectors $f^{\mathbf{s}}$, $\mathbf{s} \in S(\lambda)$, are linearly independent in $\mathrm{gr}\, V(\lambda)$. The advantage of our proof is that in the course of the proof we obtain the following important statement: the subspace of $\mathrm{gr}\, V(\lambda) \otimes \mathrm{gr}\, V(\mu)$ generated from the product of highest weight vectors is isomorphic to $\mathrm{gr}\, V(\lambda+\mu)$ (see Theorem 4.6(iii)).

4.1 Fundamental weights and minimal sets

In this subsection, we study the case $\lambda = \omega_i$. The following lemma follows from the definition of $S(\omega_i)$.

Lemma 4.1. $S(\omega_i)$ consists of all $\mathbf{s}$ such that $s_\alpha \leq 1$ and the support of $\mathbf{s}$ is given by the set

$$M^{\mathbf{s}} = \{\alpha_{j_l,\overline{k_l}} \mid l = 1, \ldots, p\} \cup \{\alpha_{t_l,r_l} \mid l = 1, \ldots, q\}$$

with the following conditions:

$$1 \leq j_1 < j_2 < \cdots < j_p \leq i\,;\ 1 \leq k_1 < k_2 < \cdots < k_p,$$

$$j_p < t_1 < t_2 \cdots < t_q \leq i \leq r_1 < \cdots < r_q \leq n. \qquad \square$$

Remark 4.2. We note that $\sharp M^{\mathbf{s}} \leq i$ and every path contains at most one element from $M^{\mathbf{s}}$, since the roots on a path are ordered with respect to the order $>$. □

Lemma 4.3. For every fundamental weight ω_i, we have

$$\sharp S(\omega_i) = \dim V(\omega_i). \qquad \square$$

Proof. Follows from [1] or by establishing a bijection with Kashiwara–Nakashima tableaux or by showing that

$$\sharp S(\omega_i) + \sharp S(\omega_{i-2}) + \cdots = \binom{2n}{i}$$

(compare with $\Lambda^i V(\omega_1) = V(\omega_i) \oplus V(\omega_{i-2}) \oplus \cdots$, see [11]). ■

We set

$$R_i = \{\beta \in R^+ \mid (\omega_i, \beta) \neq 0\}.$$

Let $\lambda = \sum m_j \omega_j \in P^+$ and $\mathbf{s} \in S(\lambda)$. We set

$$R_i^{\mathbf{s}} = \{\beta \in R_i \mid s_\beta \neq 0\}.$$

From now on let i be the minimal index, s.t. $m_i \neq 0$.

Definition 4.4. For $\mathbf{s} \in S(\lambda)$ denote by $M_i^{\mathbf{s}}$ the set of minimal elements in $R_i^{\mathbf{s}}$ with respect to the order $>$ (see (3.7)). Denote by $\mathbf{m}_i^{\mathbf{s}}$ the tuple $m_\beta = 1$, if $\beta \in M_i^{\mathbf{s}}$ and 0 otherwise. □

Lemma 4.5. Let $\lambda = \sum m_i \omega_i$ and i minimal with $m_i \neq 0$. If $\mathbf{s} \in S(\lambda)$, then $\mathbf{m}_i^{\mathbf{s}} \in S(\omega_i)$ and $\mathbf{s} - \mathbf{m}_i^{\mathbf{s}} \in S(\lambda - \omega_i)$. □

Proof. We first note that the statement $\mathbf{m}_i^{\mathbf{s}} \in S(\omega_i)$ follows from Remark 4.2. Let us prove that $\mathbf{s} - \mathbf{m}_i^{\mathbf{s}} \in S(\lambda - \omega_i)$. For this, we need to show that the conditions from (2.2) for $\mathbf{s} - \mathbf{m}_i^{\mathbf{s}}$ are satisfied for all paths $\mathbf{p}$. Let $\mathbf{p} = (p(0), \ldots, p(k))$. Let $p(0) = \alpha_a$. Then we know that $\sum_{l=0}^k s_{p(l)} \leq m_a + \cdots + m_b$, where $p(k) = \alpha_b$ if $b < n$ and $p(k) = \alpha_{j,\bar{j}}$ if $b = n$ $(j \geq a)$. The cases $b < i$ or $a > i$ are trivial. So we assume $a \leq i$ and $b \geq i$. If $M_i^{\mathbf{s}} \cap \mathbf{p} \neq \emptyset$, then

$$\sum_{l=0}^{k} (s - \mathbf{m}_i^{\mathbf{s}})_{p(l)} \leq m_a + \cdots + m_b - 1.$$

Now assume that $M_i^{\mathbf{s}} \cap \mathbf{p} = \emptyset$. Let l be the minimal number such that $s_{p(l)} > 0$. Then there exists $\alpha \in M_i^{\mathbf{s}}$ such that $\alpha < p(l)$. Therefore, there exists a path $\mathbf{p}'$ containing α, $p(l), \ldots, p(k)$. We note that

$$m_i + \cdots + m_b \geq \sum_{l \geq 0} s_{p'(l)} > \sum_{l \geq 0} s_{p(l)}.$$

Therefore, $\sum_{l \geq 0} (s - \mathbf{m}_i^{\mathbf{s}})_{p(l)} \leq m_i + \cdots + m_b$. ■

4.2 Proof of the main theorem

In the following, we write $V^a(\lambda)$ for the associated graded module $\operatorname{gr} V(\lambda)$. Denote by $V^a(\lambda, \mu) \hookrightarrow V^a(\lambda) \otimes V^a(\mu)$ the $S(\mathfrak{n}^-)$-submodule generated by the tensor product $v_\lambda \otimes v_\mu$ of the highest weight vectors.

Theorem 4.6.

(i) The vectors $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$ form a basis of $V^a(\lambda)$,

(ii) Let $V^a(\lambda) = S(\mathfrak{n}^-)/I(\lambda)$. Then $I(\lambda) = S(\mathfrak{n}^-)(\mathrm{U}(\mathfrak{n}^+) \circ R)$, where

$$R = \operatorname{span}\{ f_{\alpha_{i,j}}^{m_i + \cdots + m_j + 1}, 1 \leq i \leq j \leq n-1, \ f_{\alpha_{i,\bar{i}}}^{m_i + \cdots + m_n + 1}, 1 \leq i \leq n \}.$$

(iii) The $S(\mathfrak{n}^-)$ modules $V^a(\lambda, \mu)$ and $V^a(\lambda + \mu)$ are isomorphic. □

The proof of the theorem is by an inductive procedure. We know that part (i) of the theorem holds for all fundamental weights. For a dominant weight $\lambda = \sum_i a_i \omega_i$ denote by $|\lambda| = \sum a_i$ the sum of the coefficients. A first step in the proof is the following proposition:

Let λ be a dominant weight, and let i be the minimal number such that $(\lambda, \alpha_i) \neq 0$.

Proposition 4.7. The vectors $f^{\mathbf{s}}(v_{\lambda - \omega_i} \otimes v_{\omega_i})$, $\mathbf{s} \in S(\lambda)$, are linearly independent in $V^a(\lambda - \omega_i) \otimes V^a(\omega_i)$, and the vectors $f^{\mathbf{s}}(v_\lambda)$, $\mathbf{s} \in S(\lambda)$, form a basis for $V^a(\lambda)$. □

Proof. The proof is by induction on $|\lambda|$. If λ is a fundamental weight, then the first part of the claim makes no sense and the second part is true.

So assume now $|\lambda| \geq 2$ and assume that the second part of the proposition holds for all dominant weights μ such that $|\mu| < |\lambda|$. We prove now the first part of the proposition for λ.

Assume that there exists some vanishing linear combination

$$\sum_{\mathbf{s}\in S(\lambda)} c_{\mathbf{s}} f^{\mathbf{s}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})=0. \tag{4.1}$$

We will prove that $c_{\mathbf{s}}=0$ for all $\mathbf{s}$.

Recall first that we have an order $\succ$ on the set of $\mathtt{C}_n$-multi-exponents (see Definition 3.6) such that if $\mathbf{t}\notin S(\lambda)$ then

$$f^{\mathbf{t}}v_\lambda=\sum_{\substack{\mathbf{t}\succ\mathbf{s}\\ \mathbf{s}\in S(\lambda)}} d_{\mathbf{s}} f^{\mathbf{s}}v_\lambda.$$

Another important ingredient will be the elements $\mathbf{m}_i^{\mathbf{s}}$ (Definition 4.4). Recall that i is minimal such that $(\lambda,\alpha_i)\neq 0$, $\mathbf{m}_i^{\mathbf{s}}\in S(\omega_i)$ and $\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}\in S(\lambda-\omega_i)$.

The proof is by contradiction. Assume that $c_{\mathbf{s}}\neq 0$ for some $\mathbf{s}$. In the following, we fix such an element $\mathbf{s}\in S(\lambda)$ and we assume without loss of generality that $c_{\mathbf{t}}=0$ for all $\mathbf{t}\succ\mathbf{s}$.

The vector space $V^a(\lambda-\omega_i)\otimes V^a(\omega_i)$ has a basis given by the elements $f^{\mathbf{a}}v_{\lambda-\omega_i}\otimes f^{\mathbf{b}}v_{\omega_i}$, $\mathbf{a}\in S(\lambda-\omega_i)$, $\mathbf{b}\in S(\omega_i)$. For all $\mathbf{t}\in S(\lambda)$ such that $c_{\mathbf{t}}\neq 0$ in (4.1) we express $f^{\mathbf{t}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})$ as a linear combination of these basis elements, that is, we will write

$$f^{\mathbf{t}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})=\sum_{\substack{\mathbf{a}\in S(\lambda-\omega_i)\\ \mathbf{b}\in S(\omega_i)}} K^{\mathbf{t}}_{\mathbf{a},\mathbf{b}} f^{\mathbf{a}}v_{\lambda-\omega_i}\otimes f^{\mathbf{b}}v_{\omega_i}.$$

In the next step, we show that $K^{\mathbf{t}}_{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}},\mathbf{m}_i^{\mathbf{s}}}=0$ for all $\mathbf{t}\neq\mathbf{s}$ and $K^{\mathbf{s}}_{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}},\mathbf{m}_i^{\mathbf{s}}}\neq 0$.

Using the rules for the action on a tensor product we see:

$$f^{\mathbf{s}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})=C f^{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}}v_{\lambda-\omega_i}\otimes f^{\mathbf{m}_i^{\mathbf{s}}}v_{\omega_i}+\sum_{\mathbf{r}_1+\mathbf{r}_2=\mathbf{s}} p_{\mathbf{r}_1,\mathbf{r}_2} f^{\mathbf{r}_1}v_{\lambda-\omega_i}\otimes f^{\mathbf{r}_2}v_{\omega_i}, \tag{4.2}$$

where C is a nontrivial constant (a product of binomial coefficients) and $\mathbf{r}_1\neq\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}$, $\mathbf{r}_2\neq\mathbf{m}_i^{\mathbf{s}}$. The elements $f^{\mathbf{r}_1}v_{\lambda-\omega_i}$, $f^{\mathbf{r}_2}v_{\omega_i}$ need not be basis elements, we discuss the several possible cases separately. First assume that $\mathbf{r}_2\in S(\omega_i)\setminus\{\mathbf{m}_i^{\mathbf{s}}\}$. Then $f^{\mathbf{r}_1}v_{\lambda-\omega_i}\otimes f^{\mathbf{r}_2}v_{\omega_i}$ is a sum of basis elements of the form $f^{\mathbf{a}}v_{\lambda-\omega_i}\otimes f^{\mathbf{r}_2}v_{\omega_i}$, where $(\mathbf{a},\mathbf{r}_2)\neq(\mathbf{s}-\mathbf{m}_i^{\mathbf{s}},\mathbf{m}_i^{\mathbf{s}})$. For the same reason, if $\mathbf{r}_1\in S(\lambda-\omega_i)\setminus\{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}\}$, then $f^{\mathbf{r}_1}v_{\lambda-\omega_i}\otimes f^{\mathbf{r}_2}v_{\omega_i}$ is a sum of basis elements of the form $f^{\mathbf{r}_1}v_{\lambda-\omega_i}\otimes f^{\mathbf{b}}v_{\omega_i}$ where $(\mathbf{r}_1,\mathbf{b})\neq(\mathbf{s}-\mathbf{m}_i^{\mathbf{s}},\mathbf{m}_i^{\mathbf{s}})$. If $\mathbf{r}_1\notin S(\lambda-\omega_i)$ and

$\mathbf{r}_2 \notin S(\omega_i)$, then

$$f^{\mathbf{r}_1} v_{\lambda-\omega_i} = \sum_{\substack{\mathbf{r}_1 \succ \mathbf{a} \\ \mathbf{a} \in S(\lambda-\omega_i)}} e_{\mathbf{a}} f^{\mathbf{a}} v_{\lambda-\omega_i} \quad \text{and} \quad f^{\mathbf{r}_2} v_{\omega_i} = \sum_{\substack{\mathbf{r}_2 \succ \mathbf{b} \\ \mathbf{b} \in S(\omega_i)}} d_{\mathbf{b}} f^{\mathbf{b}} v_{\omega_i}$$

with some constants $e_{\mathbf{a}}$ and $d_{\mathbf{b}}$. But among the pairs $(\mathbf{a}, \mathbf{b})$ the pair $(\mathbf{s} - \mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}})$ cannot appear, because

$$(\mathbf{s} - \mathbf{m}_i^{\mathbf{s}}) + \mathbf{m}_i^{\mathbf{s}} = \mathbf{s} = \mathbf{r}_1 + \mathbf{r}_2 \succ \mathbf{a} + \mathbf{b}.$$

Therefore, the expression of $f^{\mathbf{s}}(v_{\lambda-\omega_i} \otimes v_{\omega_i})$ as a sum of the basis elements is of the form

$$C f^{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}} v_{\lambda-\omega_i} \otimes f^{\mathbf{m}_i^{\mathbf{s}}} v_{\omega_i} + \sum_{\substack{\mathbf{a} \in S(\lambda-\omega_i), \mathbf{b} \in S(\omega_i) \\ (\mathbf{a},\mathbf{b}) \neq (\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}})}} p_{\mathbf{a},\mathbf{b}} f^{\mathbf{a}} v_{\lambda-\omega_i} \otimes f^{\mathbf{b}} v_{\omega_i}$$

and hence $K^{\mathbf{s}}_{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}}} \neq 0$.

Now let us consider a term $f^{\mathbf{t}}(v_{\lambda-\omega_i} \otimes v_{\omega_i})$, $\mathbf{t} \neq \mathbf{s}$, such that $c_{\mathbf{t}} \neq 0$ in (4.1). We write again

$$f^{\mathbf{t}}(v_{\lambda-\omega_i} \otimes v_{\omega_i}) = \sum_{\mathbf{r}_1+\mathbf{r}_2=\mathbf{t}} p_{\mathbf{r}_1,\mathbf{r}_2} f^{\mathbf{r}_1} v_{\lambda-\omega_i} \otimes f^{\mathbf{r}_2} v_{\omega_i}, \tag{4.3}$$

and express each of the terms $f^{\mathbf{r}_1} v_{\lambda-\omega_i} \otimes f^{\mathbf{r}_2} v_{\omega_i}$ as a sum of the basis elements

$$f^{\mathbf{r}_1} v_{\lambda-\omega_i} \otimes f^{\mathbf{r}_2} v_{\omega_i} = \sum_{\mathbf{a} \in S(\lambda-\omega_i), \mathbf{b} \in S(\omega_i)} q_{\mathbf{a},\mathbf{b}} f^{\mathbf{a}} v_{\lambda-\omega_i} \otimes f^{\mathbf{b}} v_{\omega_i}.$$

Recall that $\mathbf{a}$ is less than or equal to $\mathbf{r}_1$, and $\mathbf{b}$ is less than or equal to $\mathbf{r}_2$. We claim that none of the couples $(\mathbf{a}, \mathbf{b})$ occurring with a nonzero coefficient $q_{\mathbf{a},\mathbf{b}}$ is equal to $(\mathbf{s} - \mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}})$. The proof is by contradiction: If $(\mathbf{a}, \mathbf{b}) = (\mathbf{s} - \mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}})$, then $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{t}$ is greater than or equal to $\mathbf{a} + \mathbf{b} = \mathbf{s} - \mathbf{m}_i^{\mathbf{s}} + \mathbf{m}_i^{\mathbf{s}} = \mathbf{s}$, which is not possible, because $c_{\mathbf{t}} = 0$ if $\mathbf{t} \succ \mathbf{s}$. Hence $K^{\mathbf{t}}_{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}, \mathbf{m}_i^{\mathbf{s}}} = 0$ for all $\mathbf{t} \neq \mathbf{s}$.

It follows that if we express each of the summands in (4.1) as a linear combination of the basis elements $f^{\mathbf{a}} v_{\lambda-\omega_i} \otimes f^{\mathbf{b}} v_{\omega_i}$, $\mathbf{a} \in S(\lambda - \omega_i)$, $\mathbf{b} \in S(\omega_i)$, then the term $f^{\mathbf{s}-\mathbf{m}_i^{\mathbf{s}}} v_{\lambda-\omega_i} \otimes f^{\mathbf{m}_i^{\mathbf{s}}} v_{\omega_i}$ occurs only once with a nonzero coefficient, which is not possible unless $c_{\mathbf{s}} = 0$ in (4.1). Hence all coefficients vanish in the expression in (4.1), proving the linear independence.

To prove the second part of the proposition recall the degree filtration $\mathrm{U}(\mathfrak{n}^-)_s$ on $\mathrm{U}(\mathfrak{n}^-)$:

$$\mathrm{U}(\mathfrak{n}^-)_s = \mathrm{span}\{x_1 \cdots x_l : \ x_i \in \mathfrak{n}^-, l \le s\},$$

and recall that for a dominant weight μ we set

$$V(\mu)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\mu.$$

Then $V^a(\mu)$ is the associated graded $S(\mathfrak{n}^-)$-module. The tensor product $V^a(\lambda-\omega_i)\otimes V^a(\omega_i)$ of the graded modules with grading

$$V^a(\lambda-\omega_i)\otimes V^a(\omega_i) = \bigoplus_{k\ge 0}(\oplus_{k=\ell+m}(V^a(\lambda-\omega_i))_\ell \otimes (V^a(\omega_i))_m)$$

is the associated graded module for the filtration

$$(V(\lambda-\omega_i)\otimes V(\omega_i))_k = \sum_{k=\ell+m} V(\lambda-\omega_i)_\ell \otimes V(\omega_i)_m.$$

Recall the total order on the set of positive roots. We write $f^{\mathbf{s}} \in U(\mathfrak{n}^-)$ for $\mathbf{s}\in S(\lambda)$ for the ordered product of the corresponding root vectors. The linear independence of the vectors $f^{\mathbf{s}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})$, $\mathbf{s}\in S(\lambda)$, in $V^a(\lambda-\omega_i)\otimes V^a(\omega_i)$ implies the linear independence of the vectors $f^{\mathbf{s}}(v_{\lambda-\omega_i}\otimes v_{\omega_i})$, $\mathbf{s}\in S(\lambda)$, in $V(\lambda-\omega_i)\otimes V(\omega_i)$. Since these vectors are all contained in the Cartan component $V(\lambda)\hookrightarrow V(\lambda-\omega_i)\otimes V(\omega_i)$, we obtain the inequality $|S(\lambda)|\le \dim V(\lambda)$. We know already that the vectors $f^{\mathbf{s}}(v_\lambda)$, $\mathbf{s}\in S(\lambda)$, span $V^a(\lambda)$ (Section 3, the straightening law), so $|S(\lambda)|\ge \dim V^a(\lambda)=\dim V(\lambda)$ and hence:

$$|S(\lambda)| = \dim V^a(\lambda).$$

It follows that the vectors $f^{\mathbf{s}}(v_\lambda)$, $\mathbf{s}\in S(\lambda)$, are in fact a basis for $V^a(\lambda)$. ■

Using the straightening law in Section 3 we get as an immediate consequence.

Corollary 4.8. Let $V^a(\lambda) = S(\mathfrak{n}^-)/I(\lambda)$. Then $I(\lambda) = S(\mathfrak{n}^-)(\mathrm{U}(\mathfrak{n}^+)\circ R)$, where

$$R = \mathrm{span}\{f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1}, 1\le i\le j\le n-1, \ f_{\alpha_{i,\bar{i}}}^{m_i+\cdots+m_n+1}, 1\le i\le n\}. \qquad \square$$

Using the defining relations for $V^a(\lambda)$, it is easy to see that we have a canonical surjective map $V^a(\lambda) \to V^a(\lambda - \omega_i, \omega_i)$ sending v_λ to $v_{\lambda-\omega_i} \otimes v_{\omega_i}$. By Proposition 4.7, we know that the image of basis $\{f^{\mathbf{s}}(v_\lambda), \mathbf{s} \in S(\lambda)\} \subset V^a(\lambda)$ remains linearly independent and hence:

Corollary 4.9. The $S(\mathfrak{n}^-)$ modules $V^a(\lambda - \omega_i, \omega_i)$ and $V^a(\lambda)$ are isomorphic. □

Proof of Theorem 4.6. The first and second parts of the theorem follow from Proposition 4.7 and Corollary 4.8. It remains to prove the third part. As above, it is easy to see that we have a canonical surjective map $V^a(\lambda + \mu) \to V^a(\lambda, \mu)$ sending $v_{\lambda+\mu}$ to $v_\lambda \otimes v_\mu$.

The corollary above says that our theorem holds if $\mu = \omega_i$. Iterating, we obtain that both $V^a(\lambda, \mu)$ and $V^a(\lambda + \mu)$ sit inside the tensor product

$$V^a(\omega_1)^{\otimes(\lambda+\mu,\alpha_1)} \otimes \cdots \otimes V^a(\omega_n)^{\otimes(\lambda+\mu,\alpha_n)}$$

as highest components (generated from the tensor product of highest weight vectors). This proves the theorem. ■

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