Basic representation theory: practice problem list

Recall that if $\sigma : G \to S(X)$ is an action of a group G on a set X, then the corresponding *permutation representation* ρ of G on $\mathbb{C}[X]$ (the space of all complex-valued functions on X) is given by the formula

$$\rho(g)(\phi) = \phi \circ \sigma(g^{-1}).$$

Here $\phi \in \mathbb{C}[X]$, and $\circ$ denotes the composition.

Problem 1. Let G be the isometry group of a regular cube acting on the set of vertices, and χ the character of the corresponding permutation representation. Find the value of χ at the rotation of the cube around its major diagonal.

Problem 2. Give an example of a representation of $\mathbb{Z}$ that is neither irreducible nor representable as a direct sum of irreducible representations.

Problem 3. Prove that any representation of $\mathbb{Z} \oplus \mathbb{Z}$ has a one-dimensional invariant subspace.

Problem 4. Find all two-dimensional irreducible representations of $\mathbb{Z}_4$. Rigorously justify your answer.

Problem 5. Suppose that a group G has no non-trivial one-dimensional representations over $\mathbb{R}$. Prove that the image of every homomorphism $\phi : G \to \mathrm{GL}_n(\mathbb{R})$ lies in

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}.$$

Problem 6. Suppose that a group G acts on a set X, and let $R : G \to \mathbb{C}[X]$ be the corresponding permutation representation. Show that $\mathrm{tr}(R(g))$ equals to the number of fixed points of $g : X \to X$.

Problem 7. Is there a matrix representation of $\mathbb{Z}_5$ such that

$$\rho(\overline{1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}?$$

Problem 8. Let ρ be a unitary representation of a group G. Show that, for every $g \in G$, all complex eigenvalues of $\rho(g)$ have modulus one.

Problem 9. Are the following maps linear representations of the group $\mathbb{Z} \oplus \mathbb{Z}$?

(*a*) $\quad \rho : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Hom}(\mathbb{R}, \mathbb{R}), \quad \rho(m, n)(x) = e^{m+2n}x.$

(*b*) $\quad \rho : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Hom}(\mathbb{R}, \mathbb{R}), \quad \rho(m, n)(x) = (m + n)x.$

Rigorously justify your answer.

Problem 10. Give an example of a complex representation of the group $\mathbb{Z}$ that is not unitary.

Problem 11. Prove that the identity representation of $GL_n(\mathbb{C})$ is irreducible for every n.

Problem 12. Prove that the identity representation of $O_n(\mathbb{R})$ is irreducible for every n.

Representations of S_3. The following is the character table for S_3.

	1	σ	τ
T	1	1	1
S	1	-1	1
U	2	0	-1

Problem 13. Find a decomposition of $Sym^6(U)$ into a direct sum of irreducible representations.

Representations of Q_8. We let Q_8 denote the group consisting of the 8 quaternions ± 1, $\pm i$, $\pm j$, $\pm k$ with the usual quaternionic multiplication, i.e.

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The *algebra of quaternions* is defined as the linear span of 1, i, j, k (regarded as linearly independent vectors), to which the multiplication is extended by bilinearity.

Problem 14. Show that the representation of Q_8 in $\mathbb{H}$ given by quaternionic multiplication is an irreducible real representation.

Problem 15. Prove that the quotient group $Q_8/\{\pm 1\}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Problem 16. Find the character table for Q_8.

Answer:

	1	-1	$\pm i$	$\pm j$	$\pm k$
T_0	1	1	1	1	1
T_1	1	1	1	-1	-1
T_2	1	1	-1	1	-1
T_3	1	1	-1	-1	1
U	2	-2	0	0	0

Problem 17. Find the decomposition of $U^{\otimes n}$ into a direct sum of irreducible complex representations.

Problem 18. Decompose the representations $Sym^2 U$, $Sym^3 U$ and $Sym^4 U$ into a sum of irreducible representations.

Representations of A_4. The group A_4 is the group of all even permutations of 4 elements. It is isomorphic to the rotation group of a regular tetrahedron.

Problem 19. Find the character table for A_4.

Answer:

	1	4	4	3
	1	ρ_+	ρ_-	σ
T_0	1	1	1	1
T_1	1	ζ	ζ^2	1
T_2	1	ζ^2	ζ	1
V	3	0	0	-1

Here $\zeta = e^{2\pi i/3}$. In the upper row, we have indicated the cardinalities of the corresponding conjugacy classes.

Problem 20. Decompose $V^{\otimes 2}$ into a direct sum of irreducible representations.

Problem 21. The group A_4 acts by rotations of a regular tetrahedron. Consider the corresponding action on the set of edges. Find the character of the corresponding permutation representation and decompose this representation into a direct sum of irreducible representations.

Problem 22. Find the character of the representation $Sym^3 V$.

Problem 23. Show that

$$Sym^2 V = V \oplus T_0 \oplus T_1 \oplus T_2.$$

Representations of S_4. The group S_4 is the group of all permutations of $\{1, 2, 3, 4\}$. It is isomorphic to the rotation group of a regular cube and of a regular octahedron.

Problem 24. Consider the action of S_4 on the set $\{1, 2, 3, 4\}$. Let W be the corresponding permutation representation of S_4. Find its character. Show that W has a 1-dimensional invariant subspace generated by constants. Find an invariant complement of this subspace, and the character of the corresponding representation.

Problem 25. Find the character table for S_4.

Answer:

	1	6	8	3	6
	1	2	3	2^2	4
T_0	1	1	1	1	1
T_1	1	-1	1	1	-1
U	2	0	-1	2	0
V_0	3	1	0	-1	-1
V_1	3	-1	0	-1	1

Problem 26. Prove that V_0 is isomorphic to the representation of S_4 by symmetries of a regular tetrahedron.

Problem 27. Prove that V_1 is isomorphic to the representation of S_4 by rotations of a regular cube.

Problem 28. Verify that $V_1 = V_0 \otimes T_1$.

Problem 29. Prove that U is obtained as the composition of the quotient map $S_4 \to S_3$ and the representation of S_3 by symmetries of a regular triangle.

In the following problems, "find" a representation of S_4 means decompose it into a direct sum of irreducible representations.

Problem 30. Find $V_0^{\otimes 3}$.

Problem 31. The group S_4 acts on the faces of a regular cube. Find the corresponding permutation representation.

Problem 32. The group S_4 acts on the edges of a regular cube. Find the corresponding permutation representation.

Problem 33. The group S_4 acts on the vertices of a regular cube. Find the corresponding permutation representation.

Problem 34. Find the restrictions of all irreducible representations of S_4 to A_4.

Problem 35. Find the restrictions of all irreducible representations of S_4 to S_3. Here the group S_4 consists of permutations of 1, 2, 3, 4, and the group S_3 consists of permutations of 1, 2, 3.

Representations of A_5.

Problem 36. Describe all conjugacy classes in the rotation group of a regular icosahedron.

Answer: $1 =$ the identity, $\frac{1}{5} =$ rotations by angle $\pm\frac{1}{5}(2\pi)$ around the lines connecting pairs of opposite vertices, $\frac{2}{5} =$ rotations by angle $\pm\frac{2}{5}(2\pi)$ around the lines connecting pairs of opposite vertices, $\frac{1}{3} =$ rotations by angle $\pm\frac{1}{3}(2\pi)$ around the lines connecting the centers of opposite faces, $\frac{1}{2}(2\pi)$ around the lines connecting the centers of opposite edges.

Problem 37. Let I be the representation of A_5 by rotations of a regular icosahedron. Find the character of the representation I.

Problem 38. Find the character table for A_5.

Answer:

	1	12	12	20	15
	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{2}$
T	1	1	1	1	1
I_0	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1
I_1	3	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1
U	4	-1	-1	1	0
V	5	0	0	-1	1

Problem 39. The group A_5 acts on the edges of a regular icosahedron. Find the corresponding permutation representation.

Representations of S_5.

Problem 40. Find all conjugacy classes of S_5.

Problem 41. Compute the dimension of the space of all class functions on S_5.

Problem 42. Find the character of the regular representation of S_5.

Problem 43. Find the character of the permutation representation of S_5 associated with the standard action of S_5 on the set $\{1, 2, 3, 4, 5\}$.

Problem 44. Find the character table for S_5.

Answer:

	1	10	15	20	20	30	24
	1	2	2^2	3	$2\cdot 3$	4	5
T_0	1	1	1	1	1	1	1
T_1	1	-1	1	1	-1	-1	1
TV_0	4	2	0	1	-1	0	-1
TV_1	4	-2	0	1	1	0	-1
UV_0	5	1	1	-1	1	-1	0
UV_1	5	-1	1	-1	-1	1	0
W	6	0	-2	0	0	0	1

Problem 45. Find the restrictions of all irreducible representations of S_5 to S_4.

Problem 46. Find the restrictions of all irreducible representations of S_5 to A_5.