1. Organizational and methodological issues

- *The aim of the course.* The course "Basic Representation Theory" is aimed at mastering basic concepts and tools of modern Representation Theory, understanding of the role Representation Theory plays in Mathematics and Physical Sciences (in particular, in Quantum Mechanics).
- *Objectives of the course.* Upon successful completion of the course, the students will be able to understand mathematical papers, in which Representation Theory is used (except perhaps the works that deal with the development of special branches of Representation Theory per se). They will understand the concepts of Representation Theory that are hidden behind quantum mechanical formalism. A student will *know* the following concepts: the notion of a representation, morphisms of representations, invariant subspaces, irreducible and completely reducible representations, basic operations on representations and modules (direct sums, tensor products, exterior tensor products, symmetric and exterior powers), complexification of real representations, subrepresentations and quotient representations, the Maschke theorem, the Schur lemma, characters of representations, group algebras, Young diagrams and tableau, Lie groups and Lie algebras, the exponential mapping, the highest weight modules, angular momentum operator in quantum mechanics, spin of a particle. A student will be *able to*: find the decomposition of a tensor product, symmetric power or exterior power of representations into irreducible representations; find character tables of the simplest groups; find the highest weight vectors in representations of the simplest Lie algebras; decide wether a given representation is irreducible; decide wether a given representation is completely reducible; find how the spin-state of a particle transforms under an orthogonal change of variables; expand functions into spherical harmonics.
- *Original methodological approaches used in the course.* Representation theory is the most applied part of algebra. Methods of Representation Theory play an important role in Combinatorics, Dynamical Systems, Analysis. Representation Theory provides a very useful language for quantum mechanics. In contrast to many other expositions of Representation Theory, we pay a lot of attention to applications. The discussion of fundamental concepts is preceded by a careful motivation derived from the needs of different branches of mathematics and mathematical physics, not only algebra. The exposition is arranged as the way from particular examples to general theorems. We consistently consider particular examples as being more important than general results. Every technique we use is first explained in the simplest partial case, which makes the presentation clearer at the cost of being less "efficient". We rely on the principle that it is better to know different proofs of the same theorem rather than the same proof of different theorems. We emphasize the computational aspect of Representation Theory. Many particular computations are discussed that are important for developing an intuition. Computations may be rather involved and may require the use of computer algebra systems.
- *The place of the course in the system of innovative qualifications that are formed in the course of study.* The course is offered to the first year Master of Science students in Mathematics. This is an international M.Sc. programme conducted by

the department of Mathematics in English. We expect to have students with very different background in Mathematics. The topic of the course shows the unity of Mathematics. The main qualification that the students are suppose to receive is the ability to find connections between different mathematical subjects.

2. The content of the course

What makes this course unique (description of scientific and methodological features, comparison with similar courses offered by the NRU HSE and other universities in Russia and worldwide). The course “Basic Representation Theory” is among the traditional courses offered by the Math in Moscow programme (a joint student exchange program of the Independent University of Moscow and the NRU HSE). Last year, E.Yu. Smirnov taught this course as a joint course of the Master of Science programme in Mathematics and the Math in Moscow programme. In this year, our department has converted the M.Sc. program into an international programme, conducted in English, and this course is offered as a course of students’ choice in the cycle “Topics in algebra”, which is one the two major cycles. On one hand, the organization of this course follows the main highlights developed by the Math in Moscow programme. On the other hand, the scientific content and the method of presentation correspond to the author’s teaching philosophy and are different from those employed by different instructors of this course in recent years. Representation Theory is an important part of the mathematical culture. Several educational centers in Mathematics (notably the Mechanics and Mathematics Department of the MSU, the Scientific-Educational Center of the RAS Steklov Institute, the Independent University of Moscow) sometimes offer special topics courses in Representation Theory. However, this is never done on a regular basis, and the content of these courses may very essentially differ depending on mathematical taste of their authors. Since the subject of Representation Theory is not viewed as basic, the presentation of it in a special topics course is not subject to common requirements to the structure and the content. In contrast, the curriculum of the M.Sc. programme in Mathematics and of the Math in Moscow programme shows this course as one of the basic courses. Therefore, some minimal requirements to the content have been developed. They take into account that the students attending this course do not necessarily major in algebra. A more specific feature of this programme is that it is based on an “interdisciplinary” approach. Representation Theory is viewed not only as a self-contained section of algebra but also and more importantly as the language that is useful in the study of many other mathematical subjects. We often start with applications, motivate a certain concept of a tool, then introduce it in the simplest possible set-up, after which we either proceed with a general discussion or just hint at how a general theory may develop. A traditional for this course methodological trick is the following: many particular examples are worked out; sometimes the same theorem is proved in different ways; sometimes a particular case is proved before the general statement, etc.

Plan of the course. Allocation of classroom hours among the topics and types of assignments.

No	Topic	Total hours	Lectures	Recitation sessions
1	Introduction. Basic concepts of Representation Theory.	20	4	4
2	Invariant subspaces and complete reducibility.	12	2	2
3	Basic operations on representations.	20	4	4
4	Properties of complex irreducible representations.	30	6	6
5	The character theory.	30	6	6
6	Representations of SU_2 and SO_3.	30	6	6
7	Lie groups and Lie algebras.	38	4	4

The structure of the syllabus.

Section 1: Introduction. Basic concepts of Representation Theory. Reminder: transformation groups and group actions. Orbits and stabilizers. Symmetry groups of regular polyhedra. Linear transformation groups and matrix groups. Group representations. The exponential function on matrices and linear operators. One-parameter subgroups of the general linear group. Permutation representations of finite groups, regular representations. Examples: representations of the groups S_3 and Q_8.

References: [1, 5, 8]

Section 2: Invariant subspaces and complete reducibility. Invariant subspaces. Subrepresentations, quotient representations, morphisms of representations. Irreducible representations. Completely reducible representations. Examples of irreducible and completely reducible representations. Finite-dimensional representations of compact groups are completely reducible. Examples: representations of the groups A_4 and S_4.

References: [1, 2, 3, 4, 6]

Section 3: Basic operations on representations. Dual representation. Direct sum of representations. Tensor product of representations. Examples: functions of many variables, many-particle states. The Hom functor. Exterior tensor product. Exterior and symmetric powers of representations. Complexification of real representations.

References: [1, 2, 3, 6]

Section 4: Properties of complex irreducible representations. The Schur lemma. The exterior tensor product of irreducible representations is irreducible. Spaces of matrix elements. Matrix elements span the space of functions on a finite group. Orthogonality relations for matrix elements. The Peter–Weyl theorem.

References: [1, 2, 3, 4, 6]

Section 5: The character theory. Characters of complex representations. Orthogonality relations for characters. Projection formulas. Decomposition of the regular representation. Examples: representations of A_5 and S_5. Representations of symmetric groups. Young diagrams and Young tableau. Young symmetrizers and Specht modules.

References: [1, 2, 4, 7, 9]

Section 6: Representations of SU_2 and SO_3. The group SU_2 as the group of norm 1 quaternions. Two-sheeted covering of the three-dimensional special orthogonal group by the group SU_2. Irreducible representations of the group SU_2. Spherical harmonics and irreducible representations of SO_3. Angular momentum in quantum mechanics. Spin states of quantum particles.

References: [1, 7, 10, 11, 12]

Section 7: Lie groups and Lie algebras. Topological groups and Lie groups. Examples of Lie groups. Lie algebras. Examples: cross products, commutators, Poisson brackets. The exponential mapping. The asymptotic expansion of the commutator of exponentials. Description of low-dimensional Lie algebras. Representations of the algebra $\mathfrak{sl}_2$.

References: [1, 2, 3, 7, 12]

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Sample topics of course projects.

Topic 1: The Hurwitz problem on the products of sums of squares and Clifford modules.
Topic 2: Gelfand–Zetlin bases for representations of symmetric groups and general linear groups.
Topic 3: Geometry and combinatorics of Gelfand–Zetlin polytopes.
Topic 4: Young–Jucys–Murphy elements.
Topic 5: Statistical properties of Young diagrams. The Vershik–Kerov theorem.
Topic 6: Flag varieties and Schubert cells.

3. Grading policy

We will use the following means of evaluation: homework assignments, quizzes, the written midterm test, the written final exam.

Quiz: 1–2 questions for 5 minutes,
Homework: 3–6 problems for one week,
Midterm and Final: 8 problems for 3 hours.

The intermediate grade is computed at the end of the first module by the formula ¡¡0.5 times the average homework grade in the first module plus 0.5 times the midterm grade¿¿.

The final grade is computed at the end of the term by the formula ¡¡0.3 times the cumulated grade plus 0.3 times the intermediate grade plus 0.4 times the final exam grade¿¿, where the cumulated grade is the average homework grade in the second module. Quiz results and classroom participation are taken into account as bonus points when computing the cumulated grade.

4. Problems for quizzes, tests, homework assignments

The following is a pull of problems, from which homework assignments and questions for quizzes may be taken. This is just a sample. It can also be used by students as a list of practice problems while preparing for a midterm or the final exam.

4.1. **Questions for quizzes.**

- Give a definition of an abstract group.
- Give a definition of a group representation.
- How many elements are there in the group A_4?
- How many elements are there in the group D_n?
- Find the number of all orientation-preserving symmetries (rotations) of a regular cube.
- Find the number of all orientation-preserving symmetries (rotations) of a regular tetrahedron.
- Find the number of all orientation-preserving symmetries (rotations) of a regular icosahedron.
- Define an irreducible representation.
- Define a completely reducible representation.
- Define an invariant subspace of a representation.
- Define a morphism of representation.
- The group $\mathbb{R}$ acts on the space $\mathbb{R}[x]$ of polynomials in x by the formula $S(t)(f(x)) = f(tx)$. Is S a linear representation of $\mathbb{R}$?
- The group $\mathbb{R}$ acts on the space $\mathbb{R}[x]$ of polynomials in x by the formula $S(t)(f(x)) = f(e^t x)$. Is S a linear representation of $\mathbb{R}$?
- The group $\mathbb{R}$ acts on the space $\mathbb{R}[x]$ of polynomials in x by the formula $S(t)(f(x)) = e^t f(x)$. Is S a linear representation of $\mathbb{R}$?
- The group $\mathbb{R}$ acts on the space $\mathbb{R}[x]$ of polynomials in x by the formula $S(t)(f(x)) = f(x) + t$. Is S a linear representation of $\mathbb{R}$?
- The group $\mathbb{R}$ acts on the space $\mathbb{R}[x]$ of polynomials in x by the formula $S(t)(f(x)) = e^t f(x+t)$. Is S a linear representation of $\mathbb{R}$?
- Which of the following topological groups are compact: $\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{T}^1$, $SL_2(\mathbb{R})$, $SO_3(\mathbb{R})$?
- Let G be the symmetry group of a regular cube, and χ the character of the corresponding permutation representation. Find the value of χ at the rotation of the cube around its major diagonal.

4.2. **Representations of commutative groups.**

Problem 1. Is the map $f : \mathbb{R} \to M_2(\mathbb{R})$ given by the formula

$$f(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

a representation of $\mathbb{R}$? Rigorously justify your answer.

Problem 2. Every linear representation of $\mathbb{R}$ in a real finite-dimensional vector space V has the form $t \mapsto e^{tA}$, where $A : V \to V$ is some linear operator.

Problem 3. Is it true that any complex finite-dimensional representation of $\mathbb{Z}$ is a restriction of some complex representation of $\mathbb{R}$? The same question for real representations.

Problem 4. Find all irreducible real representations of the groups $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_m$.

Problem 5. Find all representations of the Klein 4-group.

Problem 6. Let $B : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map with the property

$$|B(x, y)| = |x| \cdot |y|,$$

where $|z|$ denotes the standard Euclidean norm of a vector z (the square root of the sum of squares of the coordinates). Prove that n is even.

Hint: we have $B(x, y) = (x_1A_1 + x_2A_2)y$ for some linear operators $A_1,\ A_2 : \mathbb{R}^n \to \mathbb{R}^n$. Prove that these operators are orthogonal and that $(A_1y, A_2y) = 0$ for all $y \in \mathbb{R}^n$. Deduce that the operator $I = A_2A_1^{-1}$ satisfies the relation $I^2 = -1$, hence gives rise to a representation of $\mathbb{C}$ in $\mathbb{R}^n$.

Problem 7. Find all irreducible representations of $\mathbb{Z}_m$ (the additive group of residues modulo m).

4.3. **General properties of representations.**

Problem 8. Suppose that a group G has no non-trivial one-dimensional representations over $\mathbb{R}$. Prove that the image of every homomorphism $\phi : G \to \mathrm{GL}_n(\mathbb{R})$ lies in $\mathrm{SL}_n(\mathbb{R})$.

Problem 9. Show that $e^{\mathrm{tr} A} = \det A$ for every real square matrix A.

Problem 10. For the following maps f, prove that f is a matrix representation of the group $\mathbb{R}$, and find a matrix A, for which $f(t) = e^{tA}$:

$$(a) \quad f(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

$$(b) \quad f(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Problem 11. Prove that the complexification of every irreducible real representation is either an irreducible complex representation, or the sum of two irreducible complex representations of the same dimension.

Problem 12. Prove that the complexification of an odd-dimensional irreducible real representation is irreducible.

Let T be a representation of a group G, and S a representation of a group H.

Problem 13. If both T and S are irreducible, then the representation $T \boxtimes S$ of $G \times H$ is also irreducible.

Problem 14. Let χ_V denote the character of a representation V. Verify that

$$\chi_{U\oplus V} = \chi_U + \chi_V, \quad \chi_{U\otimes V} = \chi_U \cdot \chi_V,$$
$$\chi_{Sym^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}, \quad \chi_{\Lambda^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}.$$

Problem 15. Let G be a finite group acting on a finite set X. Then we can extend this action to a linear representation of G in $\mathbb{C}[X]$ (the space of formal linear combinations of the elements of X). This representation is called a *permutation representation.* Find its character.

Problem 16. Suppose that a group G acts on a set X, and let $R : G \to \mathbb{C}[X]$ be the corresponding permutation representation. Show that $\mathrm{tr}(R(g))$ equals to the number of fixed points of $g : X \to X$.

Problem 17. Show that if U is an irreducible representation, and V a one-dimensional representation, then $U \otimes V$ is also an irreducible representation.

Let G be a finite group, and $\chi : G \to \mathbb{C}$ a function on G. Set

$$E_\chi(g,t) = \exp\left(\sum_{n=1}^{\infty} \frac{\chi(g^n)t^n}{n}\right).$$

Problem 18. Let $\chi : G \to \mathbb{C}$ be a character of a G-module V, and χ_n the character of the G-module $Sym^n(V)$. Show that $\chi_n(g)$ coincides with the coefficient with t^n in $E_\chi(g,t)$.

4.4. **Complete reducibility.**

Problem 19. Let T be an orthogonal or unitary representation of a group G. Show that, for every $g \in G$, all complex eigenvalues of $T(g)$ have modulus one.

Problem 20. Give an example of a complex representation of the group $\mathbb{Z}$ that is not unitary.

Problem 21. Prove that any quotient representation of a completely reducible representation is completely reducible.

Problem 22. Prove that the identity representation of $GL_n(\mathbb{C})$ is irreducible for every n.

Problem 23. Prove that the identity representation of $O_n(\mathbb{R})$ is irreducible for every n.

Problem 24. For which linear operators $A : \mathbb{C}^n \to \mathbb{C}^n$ is the representation $t \mapsto e^{tA}$ of $\mathbb{R}$ in $\mathbb{C}^n$ completely reducible?

Problem 25. For which pairs of linear operators A, $B : \mathbb{C}^n \to \mathbb{C}^n$ is the map $(x,y) \mapsto e^{xA+yB}$ a completely reducible linear representation of $\mathbb{R}^2$?

4.5. **Representations of S_3.**

Problem 26. Describe all irreducible complex representations of S_3.

Answer: there are 3 such representations: T (trivial), T^* (sign), and U (standard). The representation U admits a basis e_1, e_2 of eigenvectors for $\tau = (123)$, the corresponding eigenvalues being the two nontrivial cubic roots of unity. The transposition $\sigma = (12)$ interchanges the basis elements.

Problem 27. Find the character table for S_3.

Answer:

	1	σ	τ
T	1	1	1
T^*	1	-1	1
U	2	0	-1

Problem 28. Decompose the tensor power $U^{\otimes n}$ into irreducibles.

Answer:

$$U^{\otimes n} = \left(\frac{2^{n-1}+(-1)^n}{3}\right)(T \oplus T^*) \oplus \left(\frac{2^n+(-1)^{n-1}}{3}\right) U.$$

Problem 29. Find the number of partitions

$$\{1, 2, \dots, n\} = A \sqcup B$$

of the set $\{1, 2, \dots, n\}$ into two subsets A and B such that $|A| \equiv |B| \pmod 3$. The partitions $A \sqcup B$ and $B \sqcup A$ are regarded the same.

Answer: $\frac{2^{n-1}+(-1)^n}{3}$.

Problem 30. Show that

$$Sym^2(U) = U \oplus T, \quad Sym^3(U) = U \oplus T \oplus T^*,$$

$$Sym^4(U) = 2U \oplus T, \quad Sym^5(U) = 2U \oplus (T \oplus T^*),$$

$$Sym^6(U) = 2U \oplus T \oplus (T \oplus T^*)$$

In fact, for every odd n, we have

$$Sym^n(U) = \left(\left[\frac{n+1}{6}\right]+1\right)(T \oplus T^*) \oplus \left(\frac{n+1}{2} - \left[\frac{n+1}{6}\right] - 1\right) U,$$

and, for every even n, we have

$$Sym^n(U) = T \oplus \left(\left[\frac{n}{6}\right]+1\right)(T \oplus T^*) \oplus \left(\frac{n}{2} - \left[\frac{n}{6}\right] - 1\right) U,$$

Problem 31. Prove that

$$Sym^3(Sym^2 U) = Sym^2(Sym^3 U)$$

(meaning that the two representations are isomorphic). Is it true that $Sym^m(Sym^n U)$ and $Sym^n(Sym^m U)$ are always isomorphic?

Problem 32. Let χ_n be the character of the representation $Sym^n(U)$. Then

$$\chi_n(id) = n+1, \quad \chi_n(\sigma) = \frac{1+(-1)^n}{2}, \quad \chi_n(\tau) = \frac{\zeta^{n+1} - \zeta^{-n-1}}{\sqrt{3}i},$$

where $\zeta = \exp(\frac{2\pi i}{3})$.

4.6. **Representations of Q_8.** We let Q_8 denote the group consisting of the 8 quaternions ± 1, $\pm i$, $\pm j$, $\pm k$ with the usual quaternionic multiplication.

Problem 33. Show that the representation of Q_8 in $\mathbb{H}$ given by quaternionic multiplication is an irreducible real representation; however, its complexification is not irreducible.

Problem 34. Find the character table for Q_8.

Answer:

	1	-1	$\pm i$	$\pm j$	$\pm k$
T_0	1	1	1	1	1
T_1	1	1	1	-1	-1
T_2	1	1	-1	1	-1
T_3	1	1	-1	-1	1
U	2	-2	0	0	0

Problem 35. Find the abelianization of Q_8.

Answer: the Klein 4-group.

Problem 36. Show that $\mathbb{H}_{\mathbb{C}} = U \oplus U$ in the category of Q_8-modules.

Problem 37. Find the decomposition of $U^{\otimes n}$ into a direct sum of irreducible complex representations.

Answer: if n is even, then

$$U^{\otimes n} = 2^{n-2}(T_0 \oplus T_1 \oplus T_2 \oplus T_3);$$

if n is odd, then

$$U^{\otimes n} = 2^{n-1}U.$$

Problem 38. Let $B : \mathbb{R}^3 \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map with the property

$$|B(x, y)| = |x| \cdot |y|,$$

where $|z|$ denotes the standard Euclidean norm of a vector z (the square root of the sum of squares of the coordinates). Prove that n is divisible by 4.

Hint: we have $B(x, y) = (x_1A_1 + x_2A_2 + x_3A_3)y$ for some linear operators A_1, A_2, A_3. By an orthogonal coordinate change in the y-space, we can arrange that $A_1 = id$. Prove that in this case id, A_2 and A_3 generate a linear representation of Q_8, in which A_2 is the image of i, and A_3 is the image of j.

Problem 39. Decompose the representations Sym^2U, Sym^3U and Sym^4U into a sum of irreducible representations.

Answer:

$$Sym^2U = T_1 \oplus T_2 \oplus T_3,$$
$$Sym^3U = U \oplus U,$$
$$Sym^4U = 2T_0 \oplus (T_1 \oplus T_2 \oplus T_3).$$

Problem 40. Let χ_n denote the character of the representation Sym^nU. We have

$$\chi_n(1) = n + 1, \quad \chi_n(-1) = (-1)^n(n + 1), \quad \chi_n(\pm i) = \chi_n(\pm j) = \chi_n(\pm k) = \Re(i^n).$$

Problem 41. Show that, if n is odd, then Sym^nU is isomorphic to $\frac{n+1}{2}U$. If n is even, then Sym^nU is isomorphic to

$$\left(\frac{n+1+3(-1)^{\frac{n}{2}}}{4}\right) T_0 \oplus \left(\frac{n+1-(-1)^{\frac{n}{2}}}{4}\right)(T_1 \oplus T_2 \oplus T_3).$$

4.7. **Representations of A_4.**

Problem 42. Find the character table for A_4.

Answer:

	1	4	4	3
	1	ρ_+	ρ_-	σ
T_0	1	1	1	1
T_1	1	ζ	ζ^2	1
T_2	1	ζ^2	ζ	1
V	3	0	0	-1

Here $\zeta = e^{2\pi i/3}$. In the upper row, we have indicated the cardinalities of the corresponding conjugacy classes.

Problem 43. Show that

$$V^{\otimes 2} = T_0 \oplus T_1 \oplus T_2 \oplus 2V.$$

Problem 44. Show that

$$V^{\otimes n} = \left(\frac{3^n - (-1)^n}{4}\right) V \oplus \left(\frac{3^{n-1} - (-1)^{n-1}}{4}\right)(T_0 \oplus T_1 \oplus T_2).$$

Problem 45. Let χ_n be the character of the representation Sym^nV, and $E_\chi(g,t)$ be as in Problem 18. Show that

$$E_{\chi_n}(1,t) = \frac{1}{(1-t)^3}, \quad E_{\chi_n}(\rho_\pm,t) = \frac{1}{(1-t^3)^{\frac{1}{3}}}, \quad E_{\chi_n}(\sigma,t) = \frac{1}{(1+t)^2(1-t)}.$$

Problem 46. Find the character χ_n of the representation Sym^nV.

Answer:

$$\chi(1) = \frac{(n+1)(n+2)}{2},$$

$$\chi(\rho_\pm) = \begin{cases} \frac{(-\frac{2}{3}+\frac{n}{3})!}{(-\frac{2}{3})!\frac{n}{3}!}, & n \equiv 0 \pmod 3, \\ 0 & n \not\equiv 0 \pmod 3 \end{cases}$$

$$\chi(\sigma) = \frac{(1+3(-1)^n + 2(-1)^n n)}{4}.$$

Problem 47. Show that

$$Sym^2V = V \oplus T_0 \oplus T_1 \oplus T_2.$$

4.8. **Representations of S_4.**

Problem 48. Find the character of the monomial representation of S_4.

Answer:

$$\chi(1) = 4, \quad \chi(2) = 2,$$

Problem 49. Find the character table for S_4.

Answer:

	1	6	8	3	6
	1	2	3	2^2	4
T_0	1	1	1	1	1
T_1	1	-1	1	1	-1
U	2	0	-1	2	0
V_0	3	1	0	-1	-1
V_1	3	-1	0	-1	1

Problem 50. Prove that V_0 is isomorphic to the representation of S_4 by symmetries of a regular tetrahedron.

Problem 51. Prove that V_1 is isomorphic to the representation of S_4 by rotations of a regular cube.

Problem 52. Verify that $V_1 = V_0 \otimes T_1$

Problem 53. Prove that U is obtained as the composition of the quotient map $S_4 \to S_3$ and the representation of S_3 by symmetries of a regular triangle.

Problem 54. Find $V_0^{\otimes n}$.

Answer:

$$V_0^{\otimes n} = \left(\frac{2+3(-1)^n+3^{n-1}}{8}\right) T_0 \oplus \left(\frac{-2+(-1)^{n-1}+3^{n-1}}{8}\right) T_1 \oplus \left(\frac{(-1)^n+3^{n-1}}{4}\right) U \oplus$$
$$\oplus \left(\frac{2-3(-1)^n+3^n}{8}\right) V_0 \oplus \left(\frac{-2+(-1)^n+3^n}{8}\right) V_1.$$

Problem 55. The group S_4 acts on the faces of a regular cube. Find the corresponding permutation representation.

Answer: $T_0 \oplus U \oplus V_1$.

Problem 56. The group S_4 acts on the edges of a regular cube. Find the corresponding permutation representation.

Answer: $T_0 \oplus U \oplus 2V_0 \oplus V_1$.

Problem 57. The group S_4 acts on the vertices of a regular cube. Find the corresponding permutation representation.

Answer: $T_0 \oplus T_1 \oplus V_0 \oplus V_1$.

Problem 58. Find the restrictions of all irreducible representations of S_4 to A_4.

Answer: $T_0|_{A_4} = T_0$, $T_1|_{A_4} = T_0$, $U|_{A_4} = T_1 \oplus T_2$, $V_0|_{A_4} = V$, $V_1|_{A_4} = V$.

Problem 59. Find the restrictions of all irreducible representations of S_4 to S_3.

Answer: $T_0|_{S_3} = T_0$, $T_1|_{S_3} = T_1$, $U|_{S_3} = U$, $V_0|_{S_3} = U \oplus T_0$, $V_1|_{S_3} = U \oplus T_1$.

4.9. **Representations of A_5.**

Problem 60. Describe all conjugacy classes in the rotation group of a regular icosahedron.

Answer: 1 = the identity, $\frac{1}{5}$ = rotations by angle $\pm\frac{1}{5}(2\pi)$ around the lines connecting pairs of opposite vertices, $\frac{2}{5}$ = rotations by angle $\pm\frac{2}{5}(2\pi)$ around the lines connecting pairs of opposite vertices, $\frac{1}{3}$ = rotations by angle $\pm\frac{1}{3}(2\pi)$ around the lines connecting the centers of opposite faces, $\frac{1}{2}(2\pi)$ around the lines connecting the centers of opposite edges.

Problem 61. Let I be the representation of A_5 by rotations of a regular icosahedron. Find the character of the representation I.

Problem 62. Find the character table for A_5.

Answer:

	1	12	12	20	15
	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{2}$
T	1	1	1	1	1
I	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1
I	3	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1
U	4	-1	-1	1	0
V	5	0	0	-1	1

4.10. **Representations of S_5.**

Problem 63. Find the character table for S_5.

Answer:

	1	10	15	20	20	30	24
	1	2	2^2	3	$2\cdot 3$	4	5
T_0	1	1	1	1	1	1	1
T_1	1	-1	1	1	-1	-1	1
TV_0	4	2	0	1	-1	0	-1
TV_1	4	-2	0	1	1	0	-1
UV_0	5	1	1	-1	1	-1	0
UV_1	5	-1	1	-1	-1	1	0
W	6	0	-2	0	0	0	1

Problem 64. Find the restrictions of all irreducible representations of S_5 to S_4.

Answer:

$$T_0|_{S_4} = T_0, \quad T_1|_{S_4} = T_1, \quad TV_0|_{S_4} = T_0 \oplus V_0, \quad TV_1|_{S_4} = T_1 \oplus V_1,$$

$$UV_0|_{S_4} = U \oplus V_0, \quad UV_1|_{S_4} = U \oplus V_1, \quad W|_{S_4} = V_0 \oplus V_1.$$

4.11. **Gelfand–Zetlin bases.**

Problem 65. The following is a two-dimensional representation of S_3:

	$(1,0)$	$(0,1)$
s_1	$(1,0)$	$(-1,-1)$
s_2	$(0,1)$	$(1,0)$

4.12. **Lie groups and algebras.**

Problem 66. Show that the following groups have natural Lie group structures: $\mathbb{Z}$, $\mathbb{R}$, $GL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $O_n(\mathbb{R})$.

Problem 67. Find the real dimension of the following Lie groups: $SL_n(\mathbb{C})$, $SO_n(\mathbb{R})$, U_n.

Problem 68. Prove that the Euclidean space $\mathbb{R}^3$ equipped with the vector product is a Lie algebra.

Problem 69. Deduce from the Jacobi identity for the vector product in $\mathbb{R}^3$ that the three altitudes of any triangle meet at a common point.

Problem 70. Let D be the vector space of the first order differential operators on $\mathbb{R}$, i.e.

$$D = \operatorname{span}\left\langle p(x)\frac{\partial}{\partial x}, q(x)\right\rangle,$$

where $p(x), q(x)$ are arbitrary smooth functions on $\mathbb{R}$ ($q(x)$ is the operator of multiplication by a given function $q(x)$). Check that D is a Lie algebra with respect to the standard bracket $[A, B] = AB - BA$.

A subspace $\mathfrak{h} \subset \mathfrak{g}$ is said to be a *Lie subalgebra of* $\mathfrak{g}$ if $[h_1, h_2] \in \mathfrak{h}$ for all $h_1, h_2 \in \mathfrak{h}$.

Problem 71. Show that

$$\operatorname{span}\left\langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right\rangle$$

is a Lie subalgebra of D. Prove that this subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

A Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Problem 72. Show that the following algebras are semisimple: $\mathfrak{sl}_n(\mathbb{C})$; $\mathfrak{so}_n(\mathbb{C})$.

4.13. **Representations of SU_2 and SO_3.**

Problem 73. Show that SU_2 is isomorphic (as a topological group) to the group of unit quaternions.

Problem 74. Unit quaternions q act on the space $\mathbb{R}^3 = \{x \in \mathbb{H} \mid \Re(x) = 0\}$ by the formula $x \mapsto qxq^{-1}$. This defines a homomorphism of SU_2 to SO_3. Prove that this homomorphism is surjective. Find its kernel.

The representation Φ_n of the group SU_2 in the space $Sym^n(\mathbb{C}^2)$ is defined as the symmetric power of the standard representation.

Problem 75. Find the character of the representation Φ_n restricted to the subgroup of diagonal matrices.

Answer: Set

$$A(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

Then

$$\chi_n(A(z)) = \sum_{k=0}^{n} z^{n-2k} = \frac{z^{-n} - z^{n+2}}{1 - z^2},$$

where χ_n denotes the character of Φ_n.

Problem 76. Decompose $\Phi_a \otimes \Phi_b$ into irreducible representations.

Answer:

$$\Phi_a \otimes \Phi_b = \Phi_{a+b} \oplus \Phi_{a+b-2} \oplus \cdots \oplus \Phi_{a-b+2} \oplus \Phi_{a-b}.$$

Problem 77. Decompose $Sym^2\Phi_n$, $n = 2, 3, 4, 5$ into the sum of irreducible representations.

Answer:

$$Sym^2(\Phi_2) = \Phi_0 \oplus \Phi_4, \quad Sym^2(\Phi_3) = \Phi_2 \oplus \Phi_6,$$
$$Sym^2(\Phi_4) = \Phi_0 \oplus \Phi_4 \oplus \Phi_8, \quad Sym^2(\Phi_5) = \Phi_2 \oplus \Phi_6 \oplus \Phi_{10}.$$

Problem 78. Show that

$$Sym^2(\Phi_n) = \Phi_{2n} \oplus \Phi_{2n-4} \oplus \Phi_{2n-8} \oplus \ldots.$$

The group SU_2 can be regarded as the unit sphere in the Euclidean space $\mathbb{R}^4$.

Problem 79. The 3-dimensional volume of SU_2 is equal to $2\pi^2$.

Problem 80. Let $f : SU_2 \to \mathbb{C}$ be a continuous class function. Then

$$\int_{SU_2} f(X)\,\omega(X) = 4\pi \int_0^{\pi} f(A(e^{i\phi}))\sin^2(\phi)\,d\phi.$$

Here ω is the standard volume form on SU_2.

Problem 81. We have

$$\chi_n(A(e^{i\phi})) = \frac{\cos n\phi - \cos(n+2)\phi}{1 - \cos 2\phi}.$$

Problem 82. Prove that

$$\int_{SU_2} \chi_n(X)\chi_m(X)\,\omega(X) = \begin{cases} 2\pi^2, & n = m \\ 0, & n \neq m \end{cases}$$

Deduce that

$$\frac{1}{\pi}\int_0^{\pi} \frac{(\cos n\phi - \cos(n+2)\phi)(\cos m\phi - \cos(m+2)\phi)}{1 - \cos 2\phi} d\phi = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Problem 83. Prove that

$$Sym^3(\Phi_2) = Sym^2(\Phi_3), \quad Sym^4(\Phi_2) = Sym^2(\Phi_4).$$

$$Sym^3(\Phi_5) = \Phi_3 \oplus \Phi_5 \oplus \Phi_7 \oplus \Phi_9 \oplus \Phi_{11} \oplus \Phi_{15}.$$

Let $P_{m,k}(n)$ denote the number of partitions

$$n = a_1 + \cdots + a_k,$$

in which $0 \leq a_i \leq m$ for all $i = 1, \ldots, k$.

Problem 84. The character of $Sym^k(\Phi_m)$ has the form

$$\sum_{n=0}^{mk} P_{k,m}(n) z^{mk-2n}.$$

Problem 85. Prove the following properties of the numbers $P_{m,k}(n)$:

$$P_{m,k}(n) = P_{k,m}(n),$$
$$\sum_{n=0}^{mk} P_{m,k}(n) = \binom{m+k}{m},$$
$$P_{m,k}(n) = P_{m,k}(mk-n).$$

Problem 86. Prove that, for any given $m > 0$ and $k > 0$, the sequence $P_{m,k}(n)$ is unimodular, i.e. the only local maximum of this sequence is at $n = mk/2$ or $n = (mk \pm 1)/2$.

Problem 87. The generating function for the numbers $P_{m,k}(n)$ with fixed m has the form

$$\sum_{k,n\geq 0} P_{k,m}(n) x^n y^k = \prod_{i=1}^{m} \frac{1}{1-yx^i}.$$

4.14. **Angular momentum in quantum mechanics.** We will use the units, in which $\hbar = 1$. Recall that the components of the angular momentum operator are defined by the formulas

$$\hat{l}_x = y\hat{p}_z - z\hat{p}_y, \quad \hat{l}_y = z\hat{p}_x - x\hat{p}_z, \quad \hat{l}_z = x\hat{p}_y - y\hat{p}_x,$$

where

$$\hat{p}_x = -i\frac{\partial}{\partial x}, \quad \hat{p}_y = -i\frac{\partial}{\partial y}, \quad \hat{p}_z = -i\frac{\partial}{\partial z}$$

are the components of the momentum operator.

Problem 88. Find the commutators $[\hat{l}_i, \hat{p}_k]$.

Problem 89. Find the commutators $[\hat{l}_i, \hat{l}_j]$.

Problem 90. Express the operator $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$ in spherical coordinates.

Problem 91. Prove that all eigenvalues of $\hat{l}_z$ are integer.

Problem 92. Prove that the eigenvalues of $\hat{l}^2$ have the form $l(l+1)$, where l is an integer. Moreover, the the eigenvalues of the operator $\hat{l}_z$ restricted to the eigenspace of $\hat{l}^2$ corresponding to the eigenvalue $l(l+1)$ are exactly all integers in the interval $[-l, l]$.

4.15. **Representations of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$.** We fix the following basis in the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It satisfies the following commutation relations:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Problem 93. For any positive integer n, there is a unique $(n+1)$-dimensional irreducible representation $V^{(n)}$ of $\mathfrak{sl}_2(\mathbb{C})$, with H having eigenvalues n, $n-2$, $\ldots$, $-n+2$, $-n$.

Problem 94. Let V be the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ (by infinitesimal special linear coordinate changes in the space of linear forms in 2 variables). Prove that

$$V^{(n)} = Sym^n(V).$$

Problem 95. The representation $V^{(n)}$ of $\mathfrak{sl}_2(\mathbb{C})$ generates a representation of $SL_2(\mathbb{C})$, whose restriction to $SU_2 \subset SL_2(\mathbb{C})$ coincides with Φ_n.