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TULLOCK RENT-SEEKING CONTEST AND ITS SOLUTION IN SECURE STRATEGIES

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It is well known that a pure-strategy Nash equilibrium does not exist for a two-player rent-seeking contest when the contest success function parameter α is greater than two. We analyze the contest using the concept of an equilibrium in secure strategies, which is a generalization of the Nash equilibrium. It is defined by two conditions: no one can get profit from worsening the situation of other players and no one can get profit without creating a threat to lose more than he gains. We show that such an equilibrium always exists. Moreover, for $\alpha > 2$ it is unique up to a permutation, and has a lower rent dissipation than in a mixed-startegy Nash equilibrium.

Key Words: Rent Seeking, Tullock Contest, Equilibrium in Secure Strategies, Noncooperative Games.

JEL classification: D72, D03, L12, C72.

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Content

1. Introduction

Many economic and political interactions can be modelled as contests. A contest arises when several players claim the ownership of some resource, and when the probability of one player obtaining the resource (or her share of the resource) is an increasing function of her irrecoverable effort (and a decreasing function of the effort of other players). A canonical example of a contest is a rent-seeking contest (Tullock 1968,1980, Posner 1975, Nitzan 1994), when firms make lobbying outlays in order to obtain a monopoly status, either as a producer, or as an importer of some good. Other examples of contests include resource allocation problems (Burkov and Kondratyev, 1981), sports (Szymanski, 2003), advertisement (Schmalensee, 1972), wars (Garfinkel and Skaperdas, 2006), litigation (Warneryd, 2000, Baye, Kovenock and de Vries, 2005, Osborne, 2002), economic growth (Polterovich, 2001), R&D contests, electoral competition, and marketing, among others (Konrad, 2007). Redistribution through contests consumes a significant fraction of income worldwide: estimates in the range of 7-15% of GDP have been obtained (Krueger, 1974, Posner, 1975, Cowling and Mueller, 1978, Laband and Sophocleus, 1988).

Modeling rent-seeking contests involves defining the contest success function that translates the effort of the players into the probabilities that each player will obtain the resource[1]. Skaperdas (1996) provides an axiomatization of such functions. Several axioms that he considered were monotonicity of one's probability of success in one's own effort, independence of irrelevant alterantives (that the ratio of probabilities for two

[1]An alterantive approach is to model a contest through an all-pay auction, where the valuations of players are private information (Baye, Kovenock and de Vries, 1996, Hillman and Riley, 1989).

players should not depend on the efforts of other players), anonymity (that probabilities of players should not depend on their identity) and zero-degree homogeneity (that multiplying the effort of every player by a constant will not change the outcome). It was shown that the only function that satisfies these four axioms was the following:

$$p_i = \frac{x_i^\alpha}{\sum_{j=1}^{n} x_j^\alpha}, \quad (x \neq 0) \tag{1}$$

where p_i is the success probability of player i, and x_j is the effort of player $j = 1, \ldots, n$. For convenience we extend the contest success function as $p_i = \frac{1}{n}$ when all $x_i = 0$. The parameter α specifies the returns to rent-seeking technology. As α increases, larger effort than one's opponents produces a bigger advantage; for $\alpha = 0$, all players share the prize equally regardless of their efforts; for $\alpha = \infty$, the player with the highest x receives the prize with probability one. This functional form was widely used in literature since Tullock (1980). The rent-seeking game itself involves payoffs

$$U_i = R_i p_i - x_i \tag{2}$$

to each player, where R_i is that player's idyosyncratic valuation of the resource. This setting is background to most other work in contest theory; it has been extended to include non-simultaneous order of moves (Baye and Shin 1999), group benefits (Nitzan 1991, Baik and Lee, 1997), budget and other constraints (Che and Gail, 1997, Schoonbeek and Kooperman, 1997), endogenous or stochastic number of players (Epstein and Nitzan, 2002, Lim and Matros, 2009), endogenous size of the prize (Leitzel and Alexeev, 1995). The central question in analyzing a rent-seeking contest is that of rent dissipation: are the combined efforts of

players less than or equal to the value of the prize that is being contested? In games with payoffs (2) dissipation is usually less than complete if an equilibrium exists.

The existence of a pure-strategy equilibrium in this game is, however, not guaranteed for $\alpha > 1$. For instance, in a symmetric case (with $R_i = R$) a pure-strategy equilibrium exists if and only if $\alpha \leqslant \frac{n}{n-1}$. Indeed, the utility function is concave in x_i for all effort levels of other players only if $\alpha \leqslant 1$.

For a symmetric, two-player contest Baye, Kovenock and de Vries (1994) showed that for $\alpha > 2$, rent dissipation in a mixed strategy equilibrium is complete; however, the equilibrium was not characterized analytically. For asymmetric contests or contests with more than two players, as far as we know there are no works that attempt to solve for the mixed-strategy equilibrium.

2. Equilibria in secure strategies

We now proceed to define the solution concept that we are going to use to analyze the rent-seeking game defined by (2). We are going to use the concept of an equilibrium in secure strategies, first proposed in Iskakov (2005). This is a generalization of Nash equilibria which introduces an additional criteria of security. The classic equilibrium concept is stable against individual deviations of every player. We require an equilibrium profile to be stable only against those individual deviations that cannot be subsequently exploited by other players. Below we provide definitions of Equilibrium in Secure Strategies (EinSS) from Iskakov and Iskakov (2012).

Consider n-person non-cooperative game in the normal form $G = \langle I, S, u \rangle$. The concept of equilibria is based on the notion

of threat and on the notion of secure strategy.

Definition 1. ***A threat*** *of player j to player i at strategy profile s is a pair of strategy profiles $\{s, (s'_j, s_{-j})\}$ such that $u_j(s'_j, s_{-j}) > u_j(s)$ and $u_i(s'_j, s_{-j}) < u_i(s)$. The strategy profile s is said* ***to pose a threat*** *from player j to player i.*

Definition 2. *A strategy s_i of player i is* ***a secure strategy*** *for player i at given strategies s_{-i} of all other players if profile s poses no threats to player i. A strategy profile s is* ***a secure profile*** *if all strategies are secure.*

A threat is when a player can deviate, making herself better off and some other player worse off. A secure profile is when no player can improve her payoff by making another player worse off.

Definition 3. ***A secure deviation*** *of player i with respect to s is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ for any threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ of player $j \neq i$ to player i.*

A secure deviation of a player must satisfy two conditions. First, it must make the player better off. Second, any subsequent threat by another player must not make the player's payoff less than he had in his original position. Note that a secure deviation does not necessarily mean a deviation into a secure profile. After a secure deviation the profile (s'_i, s_{-i}) can pose threats to player i. However these threats can not make his or her profit less than in the initial profile s. We assume that the player with incentive to maximize his or her profit securely will look for secure deviations.

Definition 4. *A secure strategy profile is* ***an Equilibrium in Secure Strategies (EinSS)*** *if no player has a secure deviation.*

There are two conditions in the definition of EinSS. There are no threats in the profile and there are no profitable secure deviations. Any Nash equilibrium poses no threats so it is a secure profile. And no player in Nash equilibrium can improve his or her profit using whatever deviation. Both conditions of the EinSS are fulfilled. Therefore we get:

Property 1. *Any Nash equilibrium is an Equilibrium in Secure Strategies.*

Consider the following two-player game:

	L	C	R
U	0,0	0,4	0,3
C	4,0	2,2	-1,-1
D	3,0	-1,-1	-2,-2

The only Nash equilibrium in this game is (C, C) when players get equal payoffs $(2, 2)$. However, there are also two other equilibria in secure strategies — namely, (D, L) and (U, R) in which one player gains 3 and the other player has to be content with zero payoff to avoid losses. Take the first of these profiles. If the row player deviates and chooses strategy C, then the column player can also choose strategy C. That will improve the column player's utility relative to (C, L) and reduce the row player's utility relative to (D, L). Hence, C is not a secure deviation for the row player. Similarly, U is not a secure deviation, as it does not increase the row player's utility. As for the column player, choosing either C or R will reduce her utility relative to (D, L).

There are no other secure strategy equilibria. If we take (C, L), then strategy C is a secure deviation by the column player, as (C, C) poses no threats for either player. Similarly, for (U, L), strategy D is a secure deviation for the row player (and R is such a deviation for the column player as well). Strategy profiles (D, C) and (D, R) are, obviously, not equilibria as well.

This example shows that even if there is a unique Nash equilibrium (which seems to complete the study of the game) there may be additional equilibria in secure strategies which significantly alter the overall picture. In the given case there are three stable profiles which have different values for players. Which of them will be realized in the game is not predetermined and each player is interested in the profile favorable to him (like in the game of battle of the sexes).

The Nash equilibrium is the profile in which the strategy of each player is the best response to strategies of other players. In a similar way, the strategy of each player in the EinSS turns out to be the best secure response.

Definition 5. *A startegy s_i of player i is **a best secure response** to strategies s_{-i} of all other players if player i has no more profitable secure strategy at s_{-i}. A profile s^* is **the Best Secure Response profile** (BSR-profile) if strategies of all players are best secure responses.*

Denote by $BS_i(s_{-i})$ the set of all best secure responses of player i to s_{-i}. Customarily, we will call $BS_i(\cdot)$ the best secure response function of player i and $BS(s) = \{s^* | s_i^* \in BS_i(s_{-i})\}$ the best secure response function for all players.

The EinSS is a secure profile by definition. And it must be the best secure response for each player since otherwise there

is a player who can increase the payoff by secure deviation. Therefore we get:

Property 2. *Any Equilibrium in Secure Strategies is a BSR-profile.*

This property provides a practical method for finding EinSS. First, all BSR-profiles are to be found. Then those BSR-profiles which are not satisfy the definition of EinSS are to be excluded.

3. The rent-seeking game

In this section we will first characterize the Nash equilibrium in the rent-seeking game, and analyze its existence. Consider a game of n players with the same valuation of the resource (i.e. $R_i = 1$). Then the utility function of player i takes the form:

$$U_i = x_i^\alpha / \sum_{j=1}^{n} x_j^\alpha - x_i, \quad (x \neq 0) \tag{3}$$

Without loss of generality we assume below that the strategy space of each player is $0 \leqslant x_i \leqslant 1$. We also assume that if the strategies of all players are zero, all get the same prize $1/n$.

When $0 < \alpha \leqslant 1$ we have $\frac{\partial^2 U_i}{\partial x_i^2} < 0$. The payoff functions are concave and single-peaked. Best response of player i is defined by the first order condition $\frac{\partial U_i}{\partial x_i} = 0$. In order to resolve it let us introduce the following invertible functions:

$$\xi^+(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2} \left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i} \right) \right)^{1/\alpha},$$

$$\max\left\{0, \frac{\alpha^2-1}{4\alpha}\right\} \leqslant x_i \leqslant \alpha/4 \tag{4}$$

$$\xi^-(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2} \left(\alpha - 2x_i - \sqrt{\alpha^2 - 4\alpha x_i} \right) \right)^{1/\alpha}, \ 0 \leqslant x_i \leqslant \alpha/4$$

Then the position of the maximum $\hat{x}_i$ of the payoff function U_i is defined by the equation $\tilde{x}_{-i} = \xi^{\pm}_{-i}(\hat{x}_i)$ or

$$\hat{x}_i = \xi^{-1}(\tilde{x}_{-i}) \equiv \begin{cases} (\xi^+)^{-1}(\tilde{x}_{-i}), \ \tilde{x}_{-i} > \frac{\alpha}{4} \\ (\xi^-)^{-1}(\tilde{x}_{-i}), \ \tilde{x}_{-i} \leqslant \frac{\alpha}{4} \end{cases} \tag{5}$$

where $\tilde{x}_{-i} \equiv \left(\sum\limits_{j \neq i} x_j^{\alpha} \right)^{1/\alpha}$.

When $\alpha > 1$ there are three cases of behavior of payoff function U_i shown in Fig.1 depending on the value of $\tilde{x}_{-i} \equiv \left(\sum\limits_{j \neq i} x_j^{\alpha} \right)^{1/\alpha}$. In general case the payoff functions U_i can be double peak in x_i. Left peak arises at $x_i = 0$ and the right peak is defined by the conditions: $\frac{\partial U_i}{\partial x_i} = 0$, $\frac{\partial^2 U_i}{\partial x_i^2} < 0$. The position of the right peak $\hat{x}_i$ is defined by the equation (5). When $\tilde{x}_{-i} < \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ the right peak is higher (see Fig.1a). When $\tilde{x}_{-i} = \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ the payoff function reaches its maximum $U_i = 0$ at both peaks (see Fig.1b). When $\tilde{x}_{-i} > \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ the best response of player i is $x_i = 0$. (see Fig.1c)

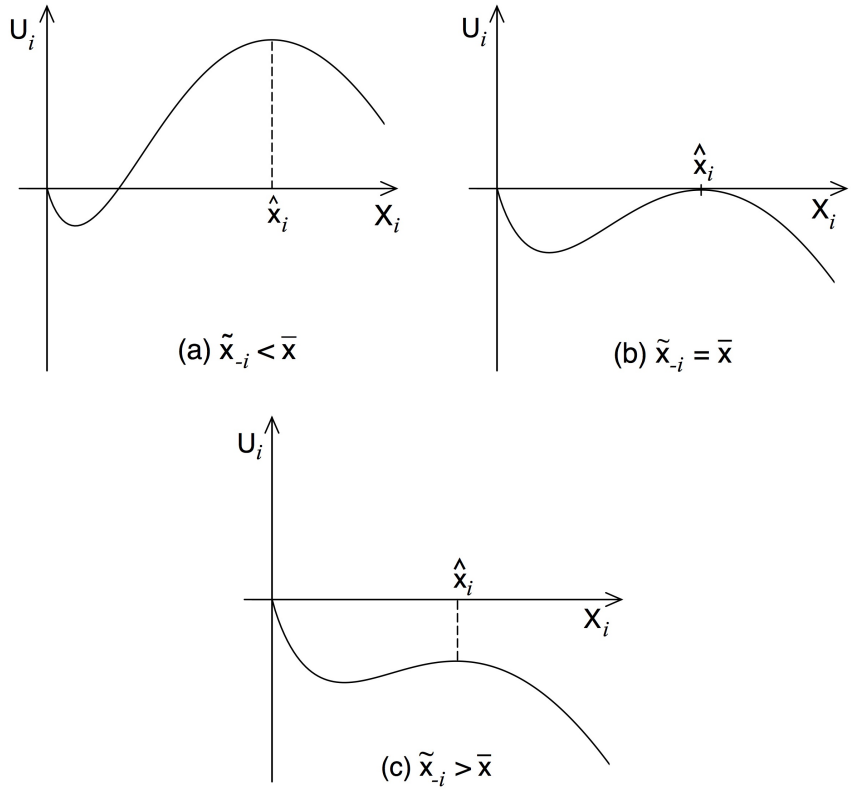

Fig. 1. Three cases of behavior of payoff function U_i when $\alpha > 1$ depending on $\tilde{x}_{-i}$. $\bar{x} \equiv \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}$.

Using notation (4-5) the best response function of player i in the rent-seeking game (3) can be written as:

$$(6) \qquad BR_i(\tilde{x}_{-i}) = \begin{cases} \xi^{-1}(\tilde{x}_{-i}), & 0 < \alpha \leqslant 1 \\ \xi^{-1}(\tilde{x}_{-i}), & \alpha > 1, \quad \tilde{x}_{-i} \leqslant \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \\ 0, & \alpha > 1, \quad \tilde{x}_{-i} \geqslant \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \end{cases},$$

where $\tilde{x}_{-i} \equiv \Big(\sum\limits_{j \neq i} x_j^{\alpha} \Big)^{1/\alpha}$.

When $\alpha \leqslant \frac{n}{n-1}$ this system has a symmetric Nash equilibrium solution $x_i^* = \alpha \frac{n-1}{n^2}$ obtained by Tullock (1980). It is not difficult to show that there are no other Nash equilibria. When $\alpha > \frac{n}{n-1}$ the symmetric Nash equilibrium no longer exists.

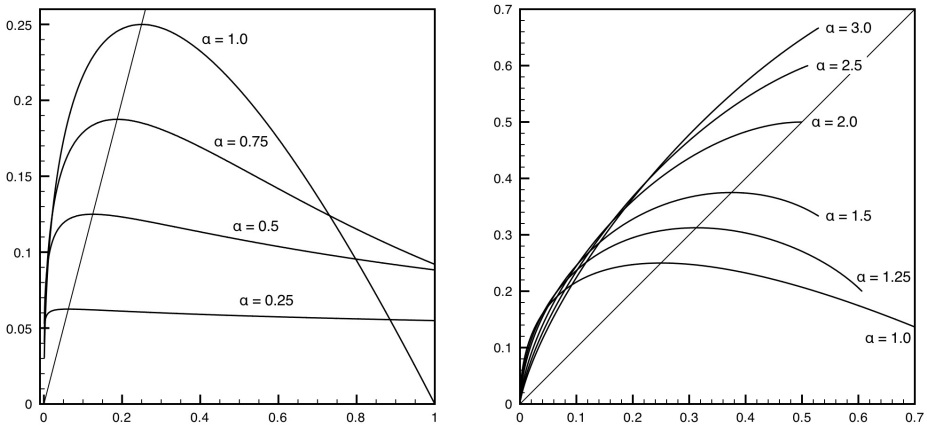

Fig. 2. Best response of player i as a function of $\tilde{x}_{-i}$ and α.

The left part of the Fig.2 shows the best response of player i (along axis Y) as a function of $\tilde{x}_{-i}$ (along axis X) for $\alpha = 0.25, 0.5, 0.75, 1.0$. The right part of the Fig.2 shows the best

secure response for $\alpha = 1.0, 1.25, 1.5, 2.0, 2.5, 3.0$. In case of $n = 2$ one can see that the best response functions of players no longer intersect in the point with $x_1 = x_2$ for $\alpha > 2$ and the symmetric Nash equilibrium no longer exists.

4. EinSS in the game of two equal players

In this section we will characterize the equilibrium in secure strategies in the rent-seeking game of two equal players. The general algorithm of finding solution in secure strategies is following. First the set of secure profiles will be found in the Theorem 1. Then for all profiles on the boundary of this set the conditions of the EinSS will be checked in the Theorem 2.

Theorem 1. *When $0 < \alpha \leqslant 1$ the set of secure profiles (x_1, x_2) in the rent seeking game (3) of two equal players is given by:*

$$\{\xi^{-1}(x_2) \leqslant x_1,\ \xi^{-1}(x_1) \leqslant x_2\}. \tag{7}$$

When $\alpha > 1$ it is given by:

$$\begin{gathered} \{\xi^{-1}(x_2) \leqslant x_1 \leqslant c,\ \xi^{-1}(x_1) \leqslant x_2 \leqslant c\} \cup \\ \cup \{\max(x_1, x_2) \geqslant c\} \cup \\ \cup \{0 \leqslant x_1 \leqslant \eta^{-1}(x_2),\ b \leqslant x_2 \leqslant c\} \cup \\ \cup \{0 \leqslant x_2 \leqslant \eta^{-1}(x_1),\ b \leqslant x_1 \leqslant c\}, \end{gathered} \tag{8}$$

where $\xi^{-1}(x)$ is defined by (4-5), $b = \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}$, $c = \frac{1}{4\alpha}(\alpha + 1)^{\frac{\alpha+1}{\alpha}}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}$ and an auxiliary invertible function η is defined for $\alpha > 1$ on the interval $0 \leqslant x \leqslant \frac{\alpha^2-1}{4\alpha}$ as

$$\eta(x) \equiv x_2 : U_1(x, x_2) = U_1\left(\xi^{-1}(x_2), x_2\right) \tag{9}$$

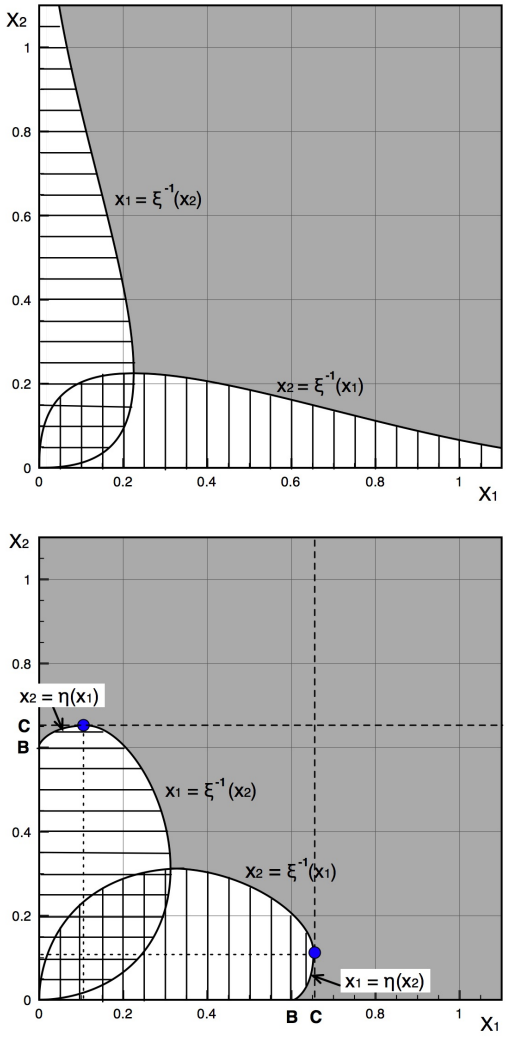

Fig. 3. Secure profiles (gray area) and insecure profiles (vertical and horizontal bars) for two players in the Tullock Contest depend upon α: $\alpha \leqslant 1$ (top) and $\alpha > 1$ (bottom). $B \equiv \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$, $C \equiv \frac{1}{4\alpha}(\alpha+1)^{\frac{\alpha+1}{\alpha}}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$.

In Fig.3 the sets of secure profiles in the rent seeking game of two equal players are shaded by gray color in the plane of strategies (x_1, x_2) for $\alpha \leqslant 1$ (top picture) and for $\alpha > 1$ (bottom picture).

Proof. We will use the following criteria. A pair of strategies is secure in the rent-seeking contest if and only if no player can be made better off by increasing his or her effort (which would always reduce the payoff of another player).

Let us first consider the case of $0 < \alpha \leqslant 1$. The payoff functions of players U_i are concave and single peaked in their strategies x_i. Therefore no one can increase his profit by increasing his effort if and only if $x_1 \geqslant BR_1(x_2), x_2 \geqslant BR_2(x_1)$. According to (6) at $0 < \alpha \leqslant 1$ this condition can be written as $x_1 \geqslant \xi^{-1}(x_2), x_2 \geqslant \xi^{-1}(x_1)$. Therefore the set of secure strategies in the game can be written as $\{(x_1, x_2) : x_1 \geqslant \xi^{-1}(x_2), x_2 \geqslant \xi^{-1}(x_1)\}$. It is shaded by gray in the plane of strategies (x_1, x_2) in the top part of Fig.3. The profiles which pose threat to player 2 are shaded by horizontal bars and profiles which pose a threat to player 1 are shaded by vertical bars.

Let us now consider the case of $\alpha > 1$. Consider for example the set of profiles which pose a threat to player 2. The profile with $x_1 = x_2 = 0$ always poses a threat for player 2 since his competitor can arbitrarily increase his strategy and decrease the payoff of player 2 from 1/2 to 0. Other profiles pose a threat to player 2 if and only if player 1 can increase his payoff U_1 by increasing his effort x_1. As we know the payoff function $U_1(x_1, x_2)$ can be two peak in x_1 depending on x_2 (see Fig.4). The left peak arises at $x_1 = 0$ and the right one at $x_1 = \xi^{-1}(x_2)$.

When $0 \leqslant x_2 < b$ the right peak is higher than the left one and profile poses a threat to player 2 when $x_1 \leqslant \xi^{-1}(x_2)$ (upper plot on Fig.4). When $b < x_2 < c$ the left peak is higher than the right one and non-secure profiles for player 2 lie in the interval $\eta^{-1}(x_2) < x_1 < \xi^{-1}(x_2)$ (lower plot on Fig.4). Finally when $x_2 \geqslant c$ the function U_1 decrease monotonically in x_1 and all profiles are secure for player 2.

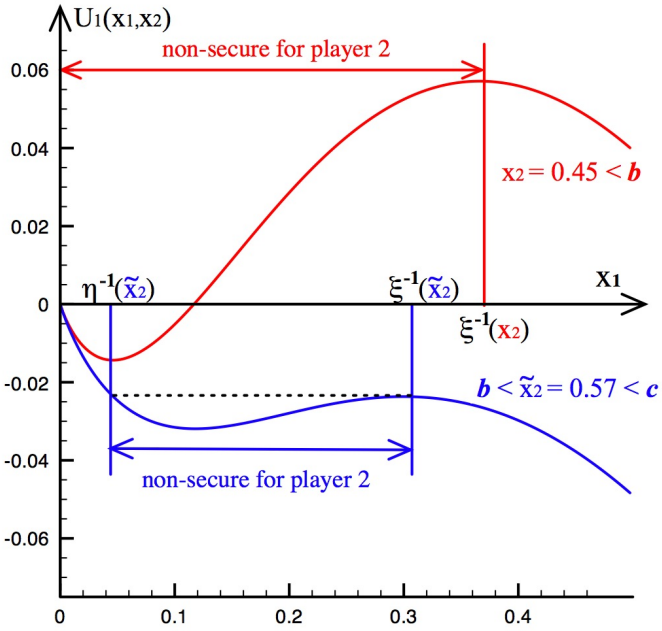

Fig. 4. The strategies of player 1 which pose a threat to player 2 at $\alpha > 1$ when $x_2 < b$ (upper plot) and when $b < x_2 < c$ (lower plot) ($\alpha = 1.5$, $b \approx 0.529$, $c \approx 0.609$, $x_2 = 0.45 > b$ и $b < \tilde{x}_2 = 0.57 < c$).

The set of profiles which pose a threat to player 2 are shaded by horizontal bars in the bottom part of Fig.3. Those profiles which pose a threat to player 1 are shaded by vertical bars respectively. The set of profiles (x_1, x_2) secure for player 2 can be formally written as:

$$\{x_2 < b, x_1 \geqslant \xi^{-1}(x_2)\} \cup$$
$$\cup \{b \leqslant x_2 < c, x_1 \leqslant \eta^{-1}(x_2) \text{ or } x_1 \geqslant \xi^{-1}(x_2)\} \cup \{x_2 \geqslant c\}$$

Consequently the set of profiles secure for player 1 can be written symmetrically:

$$\{x_1 < b, x_2 \geqslant \xi^{-1}(x_1)\} \cup$$
$$\cup \{b \leqslant x_1 < c, x_2 \leqslant \eta^{-1}(x_1) \text{ or } x_2 \geqslant \xi^{-1}(x_1)\} \cup \{x_1 \geqslant c\}$$

The intersection of these sets is the set of secure profiles in the game. It is shaded by gray in the right part of Fig.3. One can easily verify that it can be written as (4). □

Now we are ready to formulate the basic result.

Theorem 2. *If $0 < \alpha < 1$ the Tullock Contest (3) of two players reaches the following unique equilibrium in secure strategies (which is also Nash equilibrium):*

$$\{(\alpha/4, \alpha/4)\}. \tag{10}$$

If $1 \leqslant \alpha \leqslant 2$ there are following equilibria in secure strategies in the Tullock Contest (3):

$$\{(\alpha/4, \alpha/4)\} \quad \textit{and} \tag{11}$$
$$\{(0, \bar{x}), (\bar{x}, 0)\}, \quad \bar{x} = \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}, \quad \alpha > 1$$

and all other equilibria in secure strategies lie on the curve:

$$\left\{(x_1,\ \xi^+(x_1)):\ \frac{\alpha-1}{\alpha}\leqslant x_1<\frac{\alpha}{4}\right\}\ \cup \tag{12}$$
$$\cup\ \left\{(\xi^+(x_2),\ x_2):\ \frac{\alpha-1}{\alpha}\leqslant x_2<\frac{\alpha}{4}\right\},$$

where $\xi^+(x_i)\equiv\left(\frac{x_i^{\alpha-1}}{2}\left(\alpha-2x_i+\sqrt{\alpha^2-4\alpha x_i}\right)\right)^{1/\alpha}$,
$\max\left\{0,\frac{\alpha^2-1}{4\alpha}\right\}\leqslant x_i\leqslant\alpha/4$.
If $\alpha>2$ *the Tullock Contest (3) reaches only two equilibria in secure strategies:*

$$\{(0,\bar{x}),\ (\bar{x},0)\}. \tag{13}$$

Remark. Our numerical computations showed that all points on the curve (12) are in fact multiple equilibria in secure strategies. The detailed description of this verification is provided in the Appendix A.

Secure profiles and EinSS in the Tullock contest of two players are shown in the plane of strategies (x_1,x_2) in Fig.5. The shaded (gray) area corresponds to secure profiles. The solid points and curves represent EinSS.

Proof. **(1)**. We will need the following estimations for the payoff functions U_i of players given by (3) and functions $\xi^{\pm}$ given by (4) $(i=1,2)$:

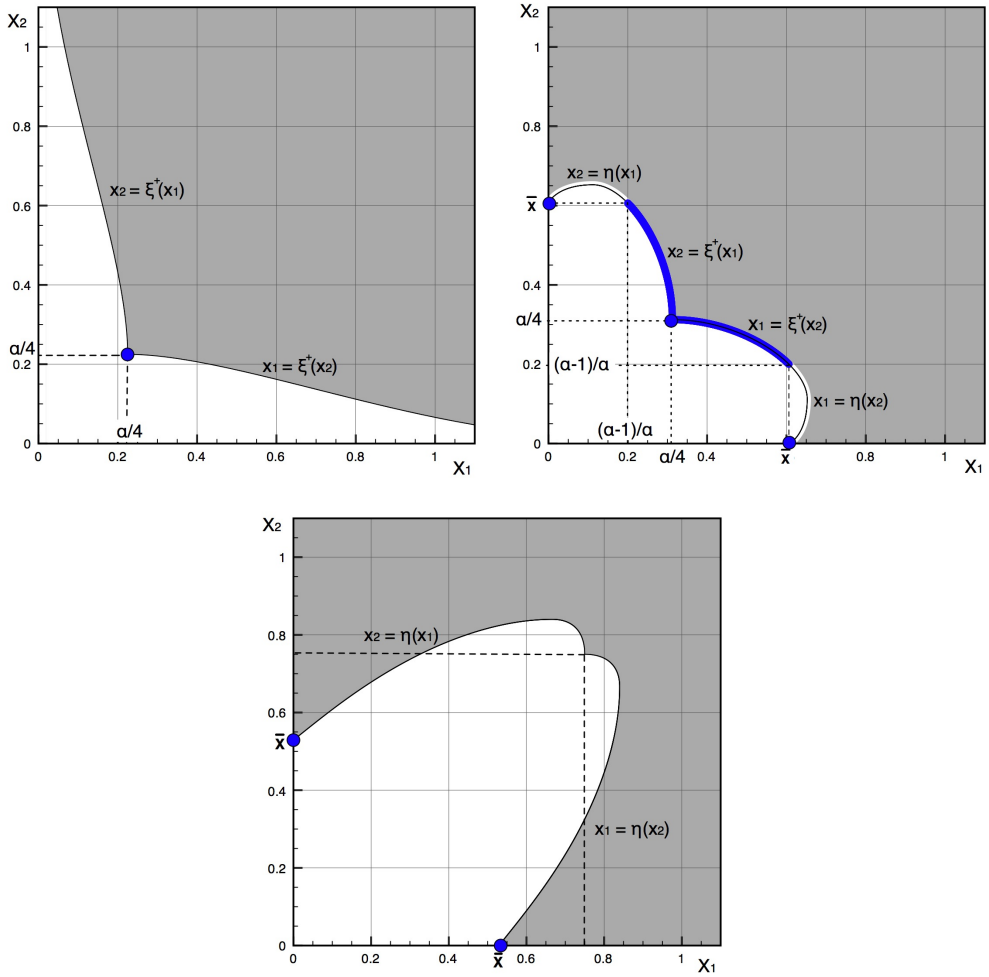

Fig. 5. Secure profiles (gray area) and EinSS (solid points and curves) in the Tullock contest of two players depending on the parameter α: $\alpha < 1$ (left), $1 \leqslant \alpha \leqslant 2$ (right) and $\alpha > 2$ (center). $\bar{x} \equiv \frac{1}{\alpha}\left(\alpha - 1\right)^{\frac{\alpha-1}{\alpha}}$, $\alpha > 1$.

(14) $$\text{When } \alpha < 1; \quad 0 < x_i < \alpha/4 :$$
$$U_{-i}(x_i, \xi^-(x_i)) > U_{-i}(x_i, \xi^+(x_i)), \quad i = 1, 2$$

(15) $$\text{When } \alpha > 1; \quad \frac{\alpha - 1}{\alpha} < x_i < \alpha/4 :$$
$$U_{-i}(x_i, \xi^-(x_i)) < U_{-i}(x_i, \xi^+(x_i)), \quad i = 1, 2$$

The proof is given in the Appendix B.

(2). The profile $(\alpha/4, \alpha/4)$ is an EinSS at $\alpha \leqslant 2$ according to Property 1 because it is a Nash equilibrium in the game (Tullock,1980).

(3). Let us prove that profiles $(0,\ \bar{x})$ and $(\bar{x},\ 0)$ are EinSS at $\alpha > 1$. Consider for example the first one. Player 1 can not increase his payoff in it by whatever deviation. Therefore the profile $(0,\ \bar{x})$ satisfies the definition of EinSS for player 1. Consider player 2. Any deviation into $x_2 > \bar{x}$ is not profitable for him. Deviation into $x_2 = 0$ is not secure for player 2 since player 1 can in response decrease his payoff to zero by arbitrarily small deviation. Let us prove that deviation of player 2 into $x_2 :\ 0 < x_2 < \bar{x} \equiv \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}} < 1$ is not a secure deviation either. Indeed, player 1 in response can deviate into the position arbitrarily close to $x_1 > \frac{\alpha-1}{\alpha} :\ U_1(x_1, x_2) = 0$. Expressing x_2 through x_1 one gets $x_2 = x_1\big(\frac{1-x_1}{x_1}\big)^{1/\alpha},\ x_1 > \frac{\alpha-1}{\alpha}$. Let us prove that in this case $U_2(x_1, x_2) - U_2(0, \bar{x}) = \bar{x} - x_1 - x_2 < 0$ for all

$x_2 \in (0, \bar{x})$, or in terms of x_1:

$$\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} < x_1\left(1+\left(\frac{1-x_1}{x_1}\right)^{1/\alpha}\right) \equiv f(x_1)$$

$$\text{for all} \quad \frac{\alpha-1}{\alpha} < x_1 < 1 \qquad (*)$$

One can easily check that $f''(x_1) = \frac{(1-\alpha)}{\alpha^2 x_1(1-x_1)^2}\left(\frac{1-x_1}{x_1}\right)^{1/\alpha} < 0$ at $\alpha > 1$. Therefore $\min\limits_{\frac{\alpha-1}{\alpha}<x_1<1} f(x_1) = \min\left\{f\left(\frac{\alpha-1}{\alpha}\right), f(1)\right\} = \min\left\{\frac{\alpha-1}{\alpha} + \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}},\ 1\right\}$ and estimation $(*)$ is true. The deviation of player 2 into $0 < x_2 < \bar{x}$ is not a secure deviation. Thus no player can make secure deviation in the profile $\left(0,\ \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}\right)$ if $\alpha > 1$ and it is an EinSS by definition. By symmetry the profile $\left(\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}},\ 0\right)$ is either an EinSS.

(4). If $\alpha < 1$ all EinSS profiles must lie on the boundary of the set of secure profiles (7) found in Theorem 1 (otherwise any player can securely increase his payoff by arbitrarily small deviation). Let us choose any profile on this boundary other than $(\alpha/4, \alpha/4)$. According to Theorem 1 it must be either $(\xi^{-1}(x_2), x_2),\ x_2 > \alpha/4$ or $(x_1, \xi^{-1}(x_1)),\ x_1 > \alpha/4$. Consider for example the first case. Since $x_2 > \alpha/4$ then x_2 according to (5) can be expressed as $x_2 = \xi^+(x_1)$. The profile $(\xi^{-1}(x_2), x_2)$ can be written in the form $(x_1, \xi^+(x_1))$. Then there is a secure deviation of player 1 into the profile $(x_1, \xi^-(x_1))$. Indeed according to (14) we have $U_1(x_1, \xi^-(x_1)) > U_1(x_1, \xi^+(x_1))$ at $\alpha < 1$ and profile $(x_1, \xi^-(x_1))$ is secure for player 1 (since player 2 get in this profile his maximum payoff and pose no threat to player 1). Therefore the profile $(\xi^{-1}(x_2), x_2),\ x_2 > \alpha/4$ is not an EinSS. By symmetry the profile $(x_1, \xi^{-1}(x_1)),\ x_1 > \alpha/4$ is not an

EinSS either. There are no EinSS profiles at $\alpha < 1$ other than $(\alpha/4, \alpha/4)$.

(5). If $\alpha \geqslant 1$ all EinSS profiles must lie on the boundary of the set of secure profiles (4) found in Theorem 1 (otherwise any player can securely increase his payoff by arbitrarily small deviation). From the other hand all EinSS must lie in the set $\{(x_1, x_2) : \max(x_1, x_2) \leqslant \bar{x}\}$ (otherwise at least one player gets negative payoff and can make a secure deviation into zero strategy). From these two conditions it follows that at $\alpha \geqslant 1$ all EinSS other than $(0,\ \bar{x})$ and $(\bar{x},\ 0)$ must lie on the curve (12). In particular it implies that there are no EinSS other than $(0,\ \bar{x})$ and $(\bar{x},\ 0)$ at $\alpha > 2$. $\square$

Like Nash equilibria, equilibria in secure strategies can be Pareto-ranked. By calculating and comparing payoffs of players in the EinSS found in the Theorem 2 one can establish the corresponding result.

Corollary.

- *For* $1 \leqslant \alpha \leqslant \alpha^*$ *all EinSS are Pareto dominated by the Nash equilibrium* $(\alpha/4, \alpha/4)$*, where* $\alpha^* \approx 1.08$ *is found from the condition* $U_1(\bar{x}(\alpha^*), 0) = U_1(\alpha^*/4, \alpha^*/4)$.

- *For* $\alpha^* < \alpha \leqslant \alpha^{**}$ *all EinSS on the curve (12) are Pareto dominated by the Nash equilibrium* $(\alpha/4, \alpha/4)$*, where* $\alpha^{**} \approx 1.22$ *is found from the condition* $U_1(\bar{x}(\alpha^{**}), \frac{\alpha^{**}-1}{\alpha^{**}}) = U_1(\alpha^{**}/4, \alpha^{**}/4)$.

- *For* $\alpha^{**} < \alpha < \alpha^{***}$ *EinSS lying on some interval of the curve (12) are Pareto dominated by the Nash equilibrium*

$(\alpha/4, \alpha/4)$, *where* $\alpha^{***} = \sqrt{2} \approx 1.41$ *is found from the condition* $\frac{\partial U_1(x,\xi^+(x))}{\partial x}|_{x=\alpha^{***}/4-0} = 0$.

- *For* $1 < \alpha < 2$ *there are two EinSS* $(\bar{x}, (\xi^+)^{-1}(\bar{x}))$ *and* $((\xi^+)^{-1}(\bar{x}), \bar{x})$ *Pareto dominated by the monopolistic EinSS* $(\bar{x}, 0)$ *and* $(0, \bar{x})$ *respectively.*

- *For* $\alpha = 2$ *there is only one equilibrium* $(0.5, 0.5)$ *Pareto-dominated by the "monopolistic" EinSS.*

- *For* $\alpha < 1$ *and* $\alpha > 2$ *there are no Pareto-dominated equilibria.*

When $\alpha \lessapprox 1.08$ all other EinSS are Pareto dominated by the Nash equilibrium $(\alpha/4, \alpha/4)$. When $1.08 \lessapprox \alpha < 2$ the two "monopolistic" EinSS $(0,\ \bar{x})$ and $(\bar{x},\ 0)$ coexist with (but are not dominated by) the symmetric Nash equilibrium in a similar way as they coexist in the matrix game example considered earlier:

	L	C	R
U	0,0	0,4	0,3
C	4,0	2,2	-1,-1
D	3,0	-1,-1	-2,-2

There is Nash equilibrium (C, C) which corresponds to the symmetric Nash Equilibrium found by Tullock (1980). The other two EinSS (U, R) and (D, L) correspond to the monopolistic EinSS in the rent-seeking game. In these equilibria the winning monopolist fixes high enough payment for the rent to create the entrance barrier for the other player making him unprofitable to participate in the competition. The difference however with the matrix game is the intermediate EinSS lying on the curve (12).

When $\alpha \lessapprox 1.22$ all these equilibria are Pareto dominated by the Nash equilibrium $(\alpha/4, \alpha/4)$. However when $1.22 \lessapprox \alpha \leqslant 2$ they can be interpreted as an intermediate type of solutions when players participate in the contest non-symmetricaly. One (the "stronger") player with larger level of effort chooses his strategy x and another (the "weaker") player adjust his strategy by choosing his best response $(\xi^+)^{-1}(x)$ at a given x. The weaker player always gains less than the stronger player and less than he would gain in the symmetric Nash equilibrium. The payoff of weaker player monotonically decrease from his payoff in the Nash equilibrium to zero with the increase of the effort of stronger player. One can show that if $\alpha \geqslant \sqrt{2} \approx 1.41$ the payoff of stronger player monotonically increases along the curve (12) with the increase of his effort. Therefore the intermediate EinSS lying on the curve (12) can be considered as positions which in terms of profitability are in between the Nash equilibrium and the monopolistic EinSS. The stronger player continuously increases his payoff and weak player continuously decreases his payoff up to the point $(\frac{\alpha-1}{\alpha}, \frac{1}{\alpha}(\alpha-1)^{(\alpha-1)/\alpha})$ in which the weak player leaves the contest and the strong player settles himself in the monopolistic EinSS.

What is the degree of rent dissipation in EinSS? Rent dissipation is equal to the ratio of total effort of both players to the value of the prize, which in our case is equal to one. The higher is the degree of rent dissipation, the lower is the efficiency of the equilibrium.

For the symmetric Nash equilibrium rent dissipation $x_1+x_2 = \alpha/2$ increases linearly with α and reaches unity at $\alpha = 2$ (see Fig.6, solid line). For $1 \leqslant \alpha \leqslant 2$ there are multiple EinSS (12)

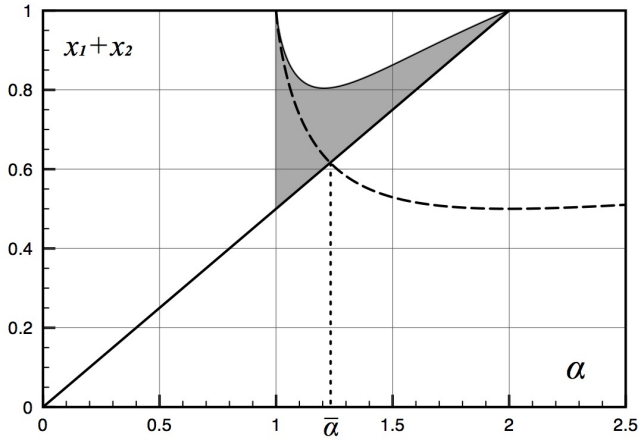

Fig. 6. Minimum and maximum rent dissipation in EinSS depending on α.

with the rent dissipation in the interval $\frac{\alpha}{2} < x_1 + x_2 \leqslant \frac{\alpha-1}{\alpha} + \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ shaded in Fig.6 by gray color. One can see that all these EinSS are less efficient than the Nash equilibrium. The rent dissipation for two monopolistic EinSS is the same and is given by $x_1+x_2 = \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$. It is shown in Fig.6 by dashed line. One can see that for $\bar{\alpha} < \alpha \leqslant 2$ (here, $\bar{\alpha} \approx 1.23$ is the solution to $\frac{\bar{\alpha}}{2} = \frac{1}{\bar{\alpha}}(\bar{\alpha}-1)^{\frac{\bar{\alpha}-1}{\bar{\alpha}}}$) the monopolistic EinSS is more efficient than Nash equilibrium. For $\alpha > 2$, no pure-strategy Nash equilibria exist, and rent dissipation in mixed-startegy equilibria is equal to one, so the rent is completely dissipated. However there are still two monopolistic EinSS with the rent dissipation significantly less than one. Hence for $\alpha > 2$, the concept of EinSS provides more efficient solution than the mixed-strategy Nash equilibrium.

5. Multiple players

In this section we generalize the result of Theorem 2 for the case of several equal players. In this case however we can fully characterize the set of EinSS in the rent-seeking game only for certain values of α.

Theorem 3. *Consider the rent seeking game of n players with payoff functions (3).*

1) If $\alpha \leqslant \frac{n}{n-1}$ there is a symmetric EinSS (which is also Nash equilibrium):

$$x_i^* = \alpha \frac{n-1}{n^2} \quad \text{for all} \quad i = 1, ..., n \tag{16}$$

2) If $\frac{k+1}{k} \leqslant \alpha \leqslant \frac{k}{k-1}$ for some $2 \leqslant k < n$ there are symmetric EinSS of k players (which are also Nash equilibria):

$$x_i^* = \alpha \frac{k-1}{k^2} \quad \text{for } k \text{ players and} \qquad x_i^* = 0 \quad \text{for all other players} \tag{17}$$

3) If $\alpha > 1$ there are monopolistic EinSS:

$$x_i = \bar{x} \equiv \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}} \text{ for some } i \text{ and} \qquad x_j = 0 \text{ for } j \neq i. \tag{18}$$

4) If $\alpha > 2$ there are no EinSS other than monopolistic ones (18).

5) Any other EinSS x^* at $\alpha \leqslant 2$ for all $i = 1, ..., n$ satisfies the following conditions:

$$x_i^* \geqslant \xi^{-1}(\tilde{x}_{-i}^*), \quad \tilde{x}_{-i}^* \equiv \Big(\sum_{j \neq i} x_j^{\alpha} \Big)^{1/\alpha} \tag{19}$$

where $\xi^{-1}(\cdot)$ is given by (5) and at least for one i the inequality (19) is binding.

Proof. **(1)**. The symmetric Nash equilibrium (16) was found by Tullock (1980). It always exists at $\alpha \leqslant \frac{n}{n-1}$ and according to Property 1 it is also an EinSS.

(2). The EinSS (17) correspond to the symmetric Nash equilibria in the game of k players. The condition $\alpha \leqslant \frac{k}{k-1}$ ensures the existence of Nash equilibrium in the game of k players. The condition $\alpha \geqslant \frac{k+1}{k}$ ensures that entering is not profitable for other $(n-k)$ players. One can easily check that (17) are Nash equilibria (and consequently EinSS) in the game of n players.

(3). The definition of EinSS for the profile (18) can be verified straightforwardly in the same way as in the proof of Theorem 2.

(4). Let us prove that there are no EinSS other than monopolistic ones (18) if $\alpha > 2$. Let x^* be an EinSS. According to Property 2 x^* is also a BSR-profile. Assume that

$$\exists\ i,\ j \neq i :\ x_i^* > 0,\ x_j^* > 0 \qquad (*)$$

If $\tilde{x}_{-i}^* > \bar{x}$ then it would be $x_i^* = 0$ (since all other strategies provide negative payoff for player i and therefore x^* could

not be BSR-profile). Consequently $x_j^* \leqslant \tilde{x}_{-i}^* \leqslant \bar{x}$. And by symmetry arguments $x_i^* \leqslant \bar{x}$. If $\alpha < 2$ we have $\bar{x} < \alpha/4$ and therefore $x_i^* \leqslant \bar{x} < \alpha/4$ and $\tilde{x}_{-i}^* \leqslant \bar{x} < \alpha/4$. Under such inequalities the security condition of player j against threats of player i takes the form: $x_i^* \geqslant \xi^{-1}(\tilde{x}_{-i}^*)$ or $\xi^-(x_i^*) \geqslant \tilde{x}_{-i}^*$. Using definition (4) of ξ we can write this condition as: $(\tilde{x}_{-i}^*)^\alpha + (x_i^*)^\alpha \leqslant \frac{\alpha (x_i^*)^\alpha}{2x_i^*}\Big(1 - \sqrt{1 - \frac{4x_i^*}{\alpha}}\Big)$. Since $1 - \sqrt{1-t} < t$ at $0 < t < 1$, then it follows that $(\tilde{x}_{-i}^*)^\alpha + (x_i^*)^\alpha < 2(x_i^*)^\alpha$, i.e. $(x_j^*)^\alpha \leqslant (\tilde{x}_{-i}^*)^\alpha < (x_i^*)^\alpha$. By symmetry arguments $(x_i^*)^\alpha < (x_j^*)^\alpha$. This is a contradiction and an assumption $(*)$ was wrong. Therefore $\forall j \neq i: \;\; x_j^* = 0$. If $x_i^* < \bar{x}$ there is always a threat of other players to choose non-zero strategy and get a positive payoff with simultaneous decreasing the payoff of player i. If $x_i^* > \bar{x}$ player i can always securely increase his payoff by reducing his strategy to $\bar{x}$. Therefore the only possible EinSS is given by (18).

(5). The inequalities (19) are conditions of security of all players $j \neq i$ against the threats of player i. Therefore according to Definition 4 they must be satisfied for any EinSS. If all these inequalities are strict then an arbitrary player can slightly decrease his strategy without violation any of them. This is a secure deviation and initial profile is not an EinSS. Therefore at least for one player inequality (19) must be binding. □

In any Nash equilibrium (17) there are two types of players: those who exert a zero level of effort, and those who exert some other level of effort that is the same for all such players. The monopoly equilibrium (18) is identical to the monopoly equilibrium in the two-player game: one player has a nonzero level of effort, such that other players cannot improve their

utility by entering the contest by choosing a nonzero effort level. When $\alpha < 1$ all players are interested to participate in the game (i.e. $\forall i : x_i^* > 0$). When $\alpha > 2$ only monopolistic equilibria can exist. When $\alpha \in [1, 2]$ there is an intermediate situation. Like in the two-player case monopoly equilibria can coexist with the equilibria for which all players select nonzero effort.

6. Unfair contest of two players

Clark and Riis (1996) extended the axiomatic characterization of the contest success function by Scaperdas (1996) to an unfair contest by relaxing the axiom of anonymity. They showed that the success function which satisfies monotonicity in one's own effort, independence of irrelevant alternatives and zero-degree homogeneity can be uniquely defined. The corresponding payoff function for two players can be written in a similar way to (3):

$$U_1 = \frac{a_1 x_1^\alpha}{a_1 x_1^\alpha + a_2 x_2^\alpha} - x_1, \quad U_2 = \frac{a_2 x_2^\alpha}{a_1 x_1^\alpha + a_2 x_2^\alpha} - x_2, \tag{20}$$

where $\alpha, a_1, a_2 > 0$. Denote $\gamma_1 = a1/a2$, $\gamma_2 = a2/a1$ and introduce notation similar to (4) and (5) ($i \in \{1, 2\}$):

$$\xi^+(x_i) \equiv \left(\gamma_i \frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha},$$

$$\max\left\{0, \frac{\alpha^2 - 1}{4\alpha}\right\} \leqslant x_i \leqslant \alpha/4 \tag{21}$$

$$\xi^-(x_i) \equiv \left(\gamma_i \frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i - \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha}, \ 0 \leqslant x_i \leqslant \alpha/4$$

$$\hat{x}_i = \xi^{-1}(x_{-i}) \equiv \begin{cases} (\xi^+)^{-1}(x_{-i}), & x_{-i} > \gamma_i^{1/\alpha}\frac{\alpha}{4} \\ (\xi^-)^{-1}(x_{-i}), & x_{-i} \leqslant \gamma_i^{1/\alpha}\frac{\alpha}{4} \end{cases} \tag{22}$$

Then the best response function of player i takes the form similar to (6):

$$BR_i(x_{-i}) = \begin{cases} \xi_{-i}^{-1}(x_{-i}), & 0 < \alpha \leqslant 1 \\ \xi_{-i}^{-1}(x_{-i}), & \alpha > 1, \quad x_{-i} \leqslant \gamma_i^{1/\alpha}\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \\ 0, & \alpha > 1, \quad x_{-i} \geqslant \gamma_i^{1/\alpha}\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \end{cases} \tag{23}$$

One can easily find the corresponding symmetric Nash equilibrium similar to (10) and the condition of its existence:

$$x_1 = x_2 = \frac{\alpha a_1 a_2}{(a_1+a_2)^2}, \quad \alpha \leqslant \frac{a1+a2}{\max\{a_1, a_2\}} \tag{24}$$

Therefore in an unfair contest the Nash equilibrium strategies of players are still the same but their payoffs are different.
Monopolistic solutions in secure strategies can be found in the form similar to (13):

$$\begin{gathered} \left\{ \left(0,\ \gamma_1^{1/\alpha}\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}\right) \right\},\ \alpha > 1,\ \frac{(\alpha-1)^{(\alpha-1)}}{\alpha^\alpha} < a_2/a_1 \\ \left\{ \left(\gamma_2^{1/\alpha}\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}},\ 0\right) \right\},\ \alpha > 1,\ \frac{(\alpha-1)^{(\alpha-1)}}{\alpha^\alpha} < a_1/a_2. \end{gathered} \tag{25}$$

In comparison with (13) however there are additional conditions. The "weak" player has to keep monopolistic barrier higher and receive less payoff. Besides, his possibility to settle monopoly occurs at greater α.

The basic result can be formulated by the following Theorem:

Theorem 4. *Consider rent-seeking unfair contest of 2 players with payoff functions (20).*
There is a symmetric EinSS (which coincides with Nash equilibrium):

$$x_1 = x_2 = \frac{\alpha a_1 a_2}{(a_1+a_2)^2}, \quad \alpha \leqslant \frac{a_1+a_2}{\max\{a_1,a_2\}} \tag{26}$$

There are two monopolistic EinSS:

$$\begin{array}{l} \left\{(0,\ \gamma_1^{1/\alpha}\dfrac{1}{\alpha}\,(\alpha-1)^{\frac{\alpha-1}{\alpha}})\right\},\ \alpha>1,\ \dfrac{(\alpha-1)^{(\alpha-1)}}{\alpha^\alpha} < a_2/a_1 \\ \left\{(\gamma_2^{1/\alpha}\dfrac{1}{\alpha}\,(\alpha-1)^{\frac{\alpha-1}{\alpha}},\ 0)\right\},\ \alpha>1,\ \dfrac{(\alpha-1)^{(\alpha-1)}}{\alpha^\alpha} < a_1/a_2. \end{array} \tag{27}$$

Any other EinSS x^* must satisfy the following conditions:

$$\forall i \in \{1,2\}: \quad x_i^* \geqslant \xi^{-1}(x_{-i}^*) > 0 \tag{28}$$

and at least for one player this inequality is binding.
If $\alpha > \frac{a_1+a_2}{\max\{a_1,a_2\}}$ there are no EinSS other than monopolistic ones (27).

Proof. It is similar to the proof of Theorem 2 and given in Appendix C. □

One can employ the different notation: $a_1 = R_1^\alpha,\ a_2 = R_2^\alpha,\ x_1 = \tilde{x}_1/R_1,\ x_2 = \tilde{x}_2/R_2,\ U_1 = \tilde{U}_1/R_1,\ U_2 = \tilde{U}_2/R_2$. Then the payoff functions of players take the form:

$$\tilde{U}_1 = \frac{R_1\tilde{x}_1^\alpha}{\tilde{x}_1^\alpha+\tilde{x}_2^\alpha} - \tilde{x}_1, \quad \tilde{U}_2 = \frac{R_2\tilde{x}_2^\alpha}{\tilde{x}_1^\alpha+\tilde{x}_2^\alpha} - \tilde{x}_2. \tag{29}$$

The Theorem can be then reformulated in the following form:

Corollary. *Consider rent-seeking unfair contest of two players with payoff functions (29).*
There is an EinSS (which coincides with Nash equilibrium):

$$\tilde{x}_1/R_1 = \tilde{x}_2/R_2 = \frac{\alpha R_1^\alpha R_2^\alpha}{(R_1^\alpha + R_2^\alpha)^2}, \quad \alpha \leqslant \frac{R_1^\alpha + R_2^\alpha}{\max\{R_1^\alpha, R_2^\alpha\}} \tag{30}$$

There are two monopolistic EinSS:

$$\begin{gathered} \left\{ \left(0,\ \frac{R_1}{\alpha R_2}\,(\alpha-1)^{\frac{\alpha-1}{\alpha}}\right) \right\},\ \alpha > 1, \frac{(\alpha-1)^{(\alpha-1)/\alpha}}{\alpha} < \frac{R_2}{R_1} \\ \left\{ \left(\frac{R_2}{\alpha R_1}\,(\alpha-1)^{\frac{\alpha-1}{\alpha}}\ ,\ 0\right) \right\},\ \alpha > 1, \frac{(\alpha-1)^{(\alpha-1)/\alpha}}{\alpha} < \frac{R_1}{R_2} \end{gathered} \tag{31}$$

Any other EinSS x^* must satisfy the following conditions:

$$\forall i \in \{1,2\}: \quad \tilde{x}_i^*/R_i \geqslant \xi^{-1}(\tilde{x}_{-i}^*/R_{-i}) > 0 \tag{32}$$

and at least for one player this inequality is binding.
If $\alpha > \frac{R_1^\alpha + R_2^\alpha}{\max\{R_1^\alpha, R_2^\alpha\}}$ there are no EinSS other than monopolistic ones (31).

Conclusion

The concept of equilibrium in secure strategies allows to discover a different type of equilibria in the rent-seeking game, those for which one player exerts a high level of effort to keep the resource, while the other player has zero level of effort. In these equilibria the first player prefer to fix his or her secure monopolistic position and not reduce the effort because such a move would be insecure and be subjected to exploitation by the other player. Thus the first player imposes an entry barrier. When power parameter $\alpha > 2$ and there is no Nash equilibrium

the monopolistic situation is the only stable position in the game in terms of secure strategies. Moreover it provides more efficient solution than the mixed-strategy Nash equilibrium in terms of the rent dissipation. The logic of the best responses can not reveal the possibility of such kind of "monopolistic" equilibria since it does not take into account the security considerations and assumes the players would choose the most profitable but insecure and possibly eventually not-profitable for them strategies.

The total rent dissipation in the found EinSS depends on the quality of contest. If the contest is highly sensitive to player effort (which corresponds to high α in the model), the one-player equilibrium is more efficient, and rent is not fully dissipated, while the symmetric Nash equilibrium is either less efficient, or, for $\alpha > 2$, does not exist (for the latter case, rent is fully dissipated in a mixed-strategy equilibrium). If α is small, then the one-player equilibrium is less efficient. For almost all values of α, the payoff of the monopolist in the one-player equilibrium is higher than in the Nash equilibrium (where both players exert effort).

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APPENDIX A

Verification of multiple EinSS in Theorem 2

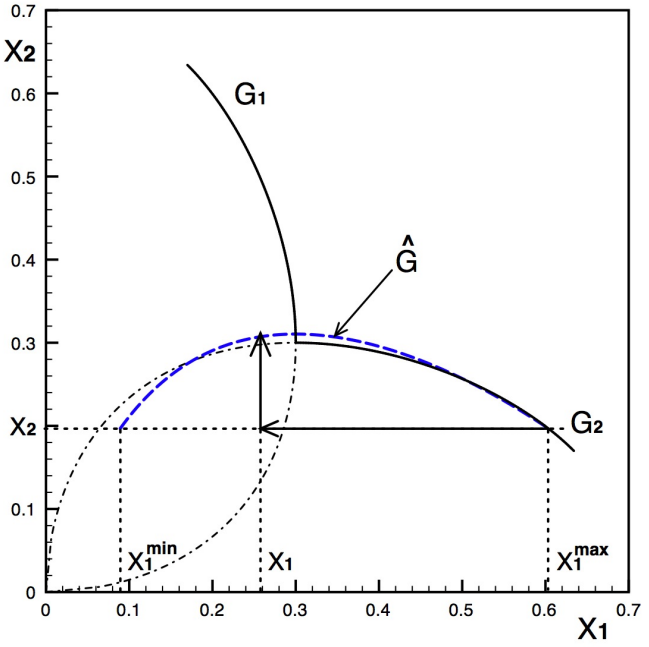

Fig. 7. Verification of multiple EinSS. Any deviation $x_1^{max} \to x_1$ of player 1 can be effectively "punished" by the response deviation $x_2 \to \hat{x}_2(x_1) + \epsilon$ of player 2 just slightly above the indifference curve $\hat{G}$ of player 1.

According to Theorem 2 when $1 \leqslant \alpha \leqslant 2$ all EinSS other than $(\alpha/4, \alpha/4)$, $(0, \bar{x})$ and $(\bar{x}, 0)$ lie on the curves $G_1 = \left\{(x_1,\ \xi^+(x_1)) : \frac{\alpha-1}{\alpha} \leqslant x_1 < \frac{\alpha}{4}\right\}$ and $G_2 = \left\{(\xi^+(x_2),\ x_2) : \frac{\alpha-1}{\alpha} \leqslant x_2 < \frac{\alpha}{4}\right\}$. Consider for example profiles on the curve G_2. These profiles are best responses

for player 2 who can make no profitable deviation. Therefore the definition of EinSS is satisfied for player 2. Consider player 1. At each point of G_2 draw the indifference curve for player 1 (see Fig.7: G_1 and G_2 are plotted by solid lines and indifference curve $\hat{G}$ by dashed line): $\hat{G}(x_2) = \left\{(x, \hat{x}_2(x)) : F_1(x, \hat{x}_2(x)) = \frac{(\xi^+(x_2))^\alpha}{(\xi^+(x_2))^\alpha + x_2^\alpha} - \xi^+(x_2) \equiv F_0(x_2)\right\}$, $\frac{\alpha-1}{\alpha} \leqslant x_2 \leqslant \frac{\alpha}{4}$. $=> \frac{x^\alpha}{x^\alpha + \hat{x}_2^\alpha(x)} - x = F_0$. From here we obtain an explicit equation of the indifference curve: $\hat{x}_2(x_1) = x_1 \left(\frac{1}{F_0 + x_1} - 1\right)^{1/\alpha}$. It is a strictly concave one-picked function with its maximum on the curve G_1. Let us prove that at any (profitable) deviation x_1 of player 1 player 2 can make (profitable) deviation into the vicinity of $\hat{G}$ and effectively "punish" player 1 (so that in the new position $F_1 < F_0$). Formally we shall prove that $F_2(x_1, \hat{x}_2(x_1)) > F_2(x_1, x_2)$ for all x_1 such that $F_1(x_1, x_2) > F_0$. Then the definition of EinSS will be satisfied for player 1 either. For the function $\Phi(x_1) \equiv F_2(x_1, \hat{x}_2(x_1)) - F_2(x_1, x_2)$ we have to check the following condition:

$$\Phi(x_1) > 0 \text{ at } x_1^{min} < x_1 < \xi^+(x_2),\ F_1(x_1^{min}, x_2) = F_0,$$
$$\frac{\alpha - 1}{\alpha} \leqslant x_2 < \frac{\alpha}{4},\ 1 \leqslant \alpha < 2$$

This condition has been verified numerically for all values of parameters (x_1, x_2, α) on the mesh of $500 \times 500 \times 500$ points. For computing very small values of Φ we have used Taylor expansion. The profiles of $\Phi(x_1)$ for different x_2 at $\alpha = 1.0001, 1.01, 1.1, 1.2, 1.4, 1.9$ are plotted in Fig.8. The function $\Phi(x_1)$ is rather small but has no singularities in the plain (x_1, x_2). It decreases sharply to zero when $\alpha \to 1$. Therefore arbitrary profile on G_2 is an EinSS. Symmetrically all profiles on G_1 are EinSS.

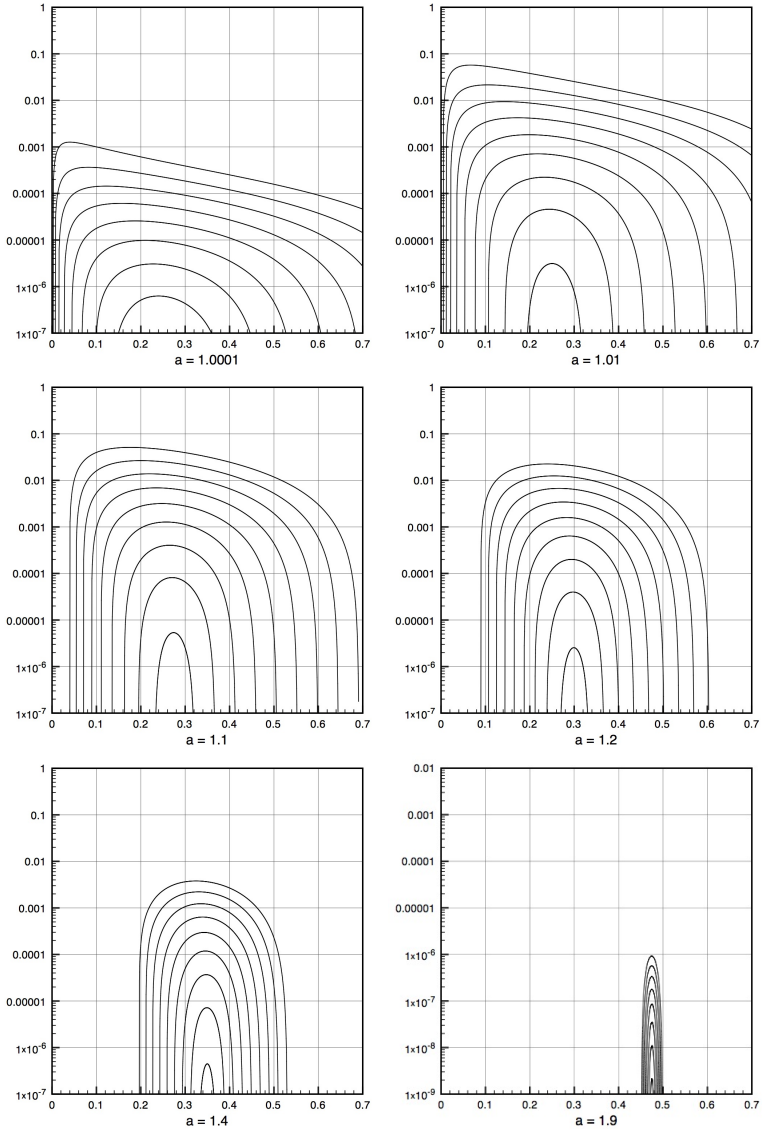

Fig. 8. The profiles of $\Phi(x_1)$ for different x_2 at $\alpha = 1.0001, 1.01, 1.1, 1.2, 1.4, 1.9$

APPENDIX B

Proof of the estimations in Theorem 2

Let us prove the following estimations for the payoff functions U_i ($i \in \{1,2\}$) given by (3) and functions $\xi^{\pm}$ given by (4):

$$\text{when } \ 0<\alpha<1, \quad 0<x_i<\alpha/4:$$
$$U_{-i}(x_i,\xi^-(x_i)) > U_{-i}(x_i,\xi^+(x_i)), \quad i\in\{1,2\}$$
$$\text{when } \ 1<\alpha<2, \quad \frac{\alpha-1}{\alpha}<x_i<\alpha/4:$$
$$U_{-i}(x_i,\xi^-(x_i)) < U_{-i}(x_i,\xi^+(x_i)), \quad i\in\{1,2\}$$

Proof. Choose for example $i=1$.
Then $\xi^{\pm}(x_1) \equiv \left(\frac{x_1^{\alpha-1}}{2}\left(\alpha-2x_1 \pm \sqrt{\alpha^2-4\alpha x_1}\right)\right)^{1/\alpha}$. Notice that $x_2^+x_2^- = \xi^+(x_1)\xi^-(x_1) = x_1^2$. Therefore, for convenience we introduce the variable $y \equiv \frac{1}{2x_1}\left(\alpha-2x_1+\sqrt{\alpha^2-4\alpha x_1}\right)$. If $0<x_1<\alpha/4$ we get $1<y<+\infty$. Then $x_2^+ = x_1y^{1/\alpha}$, $x_2^- = x_1y^{-1/\alpha}$. Notice also that $y+y^{-1} = \frac{\alpha}{x_1}-2$ or $(1+y)(1+y^{-1}) = \frac{\alpha}{x_1}$. Then:

$$U_2^+ \equiv \frac{(x_2^+)^\alpha}{x_1^\alpha+(x_2^+)^\alpha} - x_2^+ = \frac{y}{1+y} - x_1y^{1/\alpha};$$
$$U_2^- \equiv \frac{(x_2^-)^\alpha}{x_1^\alpha+(x_2^-)^\alpha} - x_2^- = \frac{y^{-1}}{1+y^{-1}} - x_1y^{-1/\alpha}$$

$$\Rightarrow U_2^+ - U_2^- = -(x_2^+-x_2^-) + \frac{y-y^{-1}}{(1+y)(1+y^{-1})} =$$
$$= x_1\left(\frac{1}{\alpha}(y-y^{-1}) - (y^{1/\alpha}-y^{-1/\alpha})\right).$$

If $\alpha = 1$ the function $\Phi \equiv (U_2^+ - U_2^-)/x_1 \equiv 0$.

$$\frac{d\Phi}{dy} = \frac{1}{\alpha y}\Big((y + y^{-1}) - (y^{1/\alpha} + y^{-1/\alpha})\Big).$$

If $0 < \alpha < 1$ $(1/\alpha > 1)$: $y^{1/\alpha} > y > 1 \ \Rightarrow \ y^{1/\alpha} + y^{-1/\alpha} > y + y^{-1}$ and $\frac{d\Phi}{dy} < 0 \ \forall y > 1$.
If $\alpha > 1$ $(0 < 1/\alpha < 1)$: $1 < y^{1/\alpha} < y \ \Rightarrow \ y^{1/\alpha} + y^{-1/\alpha} < y + y^{-1}$ and $\frac{d\Phi}{dy} > 0 \ \forall y > 1$.
Since $\Phi(y = 1) = 0 \ \ \forall \alpha > 0$, then if $0 < \alpha < 1 : \ U_2^+ < U_2^-$, and if $\alpha > 1 : \ U_2^+ > U_2^-$. The estimations are proven. □

APPENDIX C

Proof of Theorem 4

Proof. The proof is similar to the proof of Theorem 2. Let us prove that monopolistic profiles (27) are EinSS. Consider for example profile $(0, \ \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}})$. Player 1 can not increase his payoff in it by whatever deviation. Therefore the profile $(0, \ \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}})$ satisfies the definition of EinSS for player 1. Consider player 2. Any deviation into $x_2 > \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$ is not profitable for him. Deviation into $x_2 = 0$ is not secure for player 2 since player 1 can in response decrease his payoff to zero by arbitrarily small deviation. Let us prove that deviation of player 2 into $0 < x_2 < \bar{x}_2 \equiv \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} < 1$ is not a secure deviation either. Indeed, player 1 in response can deviate into the position arbitrarily close to $x_1 > \frac{\alpha-1}{\alpha} : \ U_1(x_1, x_2) = 0$. Expressing x_2 through x_1 one gets $x_2 = \gamma_1^{1/\alpha} x_1 \big(\frac{1-x_1}{x_1}\big)^{1/\alpha}, \ x_1 > \frac{\alpha-1}{\alpha}$. Let us prove that in this case $U_2(x_1, x_2) - U_2(0, \bar{x}_2) =$

$\bar{x}_2 - x_1 - x_2 < 0$ for all $x_2 \in (0, \bar{x}_2)$, or

$$\gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} < x_1 \left(1 + \gamma_1^{1/\alpha} \left(\frac{1 - x_1}{x_1} \right)^{1/\alpha} \right) \equiv f(x_1)$$

$$\text{for all} \quad \frac{\alpha - 1}{\alpha} < x_1 < 1 \qquad (*)$$

One can easily check that $f''(x_1) = \frac{\gamma_1^{1/\alpha}(1-\alpha)}{\alpha^2 x_1 (1-x_1)^2} \left(\frac{1-x_1}{x_1} \right)^{1/\alpha} < 0$ at $\alpha > 1$. Therefore $\min\limits_{\frac{\alpha-1}{\alpha} < x_1 < 1} f(x_1) = \min\left\{ f\left(\frac{\alpha-1}{\alpha}\right), f(1) \right\} = \min\left\{ \frac{\alpha-1}{\alpha} + \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, \ 1 \right\}$ and estimation $(*)$ is true. The deviation of player 2 into $0 < x_2 < \bar{x}_2$ is not a secure deviation. Thus no player can make secure deviation in the profile $(0, \ \gamma_1^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}})$ if $\alpha > 1$ and it is an EinSS by definition. By symmetry the profile $(\gamma_2^{1/\alpha} \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, \ 0)$ is either an EinSS. Other conditions can be verified like in the proof of Theorem 2. $\square$

REFERENCES

1. Alexeev M., Leitzel J. (1996). Rent shrinking. *Southern Economic Journal* 62: 620–626.

2. Baik K., Lee S. (1997). Collective Rent Seeking with Endogenous Group Sizes. *European Journal of Political Economy* 13:121–130.

3. Baye M.R., Kovenock D. and de Vries C.G. (1996). The all-pay auction with complete information. *Economic Theory* 8: 362–380.

4. Baye M.R., Kovenock D. and de Vries C.G. (2005). Comparative analysis of litigation systems: An auction-theoretic approach. *Economic Journal* 115: 583–601.

5. Baye M.R., Hoppe H.C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior* 44: 217–226.

6. Burkov V.N., Kondratyev V.V. (1981). *Control mechanisms in organizational systems,* Nauka, Moscow, 319–331 (in Russian).

7. Clark D.J., Riis C. (1996). Contest success functions: an extension. *Economic Theory* 11: 201–204.

8. Che Y.-K., I. Gale (1997). Rent dissipation when rent seekers are budget constrained. *Public Choice* 92: 109–126.

9. Cowling K., Mueller D.C. (1978). The Social Costs of Monopoly Power. *The Economic Journal* 88: 727–748.

10. Epstein G., Nitzan S. (2002). Asymmetry and cjrrective public policy in contests. *Public Choice* 113: 231–240.

11. Garfinkel M. R., Skaperdas S. (2006). Economics of Conflict: An Overview, in T. Sandler and K. Hartley (eds.) *Handbook of Defense Economics Vol. 2.* North-Holland.

12. Hillman A., and Riley J.G. (1989). Politically contestable rents and transfers. *Economics and Politics* 1: 17–40.

13. Iskakov, M.B. (2005). Equilibrium in Safe Strategies. *Automation and Remote Control* 66(3): 465–478.

14. Iskakov M., Iskakov A. (2012). Solution of the Hotelling's game in secure strategies. *Economics Letters* 117: 115–118.

15. Iskakov M., Iskakov A. (2012). Equilibrium in secure strategies. *CORE Discussion Paper* 2012/61.

16. Konrad K. (2007). *Strategy in contests: An introduction* WZB Berlin.

17. Krueger A. (1974). The political economy of the rent-seeking society. *American Economic Review* 64: 291–303.

18. Laband D.N., Sophocleus J.P. (1998). The social cost of rent-seeking: First estimates. *Public Choice* 58(3): 269–275.

19. Lim W. and Matros A. (2009). Contests with a stochastic number of players. *Games and Economic Behavior* 67(2): 584–597.

20. Linster B.G. (1993). Stackelberg rent-seeking. *Public Choice* 77(2): 307–321.

21. Nitzan S. (1994). Modelling rent-seeking contests. *European Journal of Political Economy* 10(1): 41–60.

22. Polterovich V. (2001). Rent seeking, taxpolicy, and economic growth. *NES working paper* 2001/25.

23. Posner R. A. (1975). The social costs of monopoly and regulation. *Journal of Political Economy* 83: 807–827.

24. Skaperdas S. (1994). Contest success functions. *Economic Theory* 7: 283–290.

25. Schmalensee R. (1972). *The Economics of Advertising.* North-Holland, Amsterdam-London.

26. Shoonbeek L., Kooperman P. (1997). Tullok's rent-seeking contest with minimum expenditure requirement. *Public Choice* 93: 477–486.

27. Szymanski S. (2003). The economic design of sporting contests. *Journal of Economic Literature* 41: 1137–1187.

28. Tullock G. (1967). The welfare cost of tari§s, monopolies, and theft. *Western Economic Journal* 5: 224–232.

29. Tullock G. (1980). Efficient rent seeking. In: J. Buchanan, R. Tollison and G. Tullock, (eds.) *Towards a Theory of the Rent-Seeking Society*. College Station, Texas A&M University Press, 97–112.

30. Warneryd K. (2000). In defense of lawyers: moral hazard as an aid to cooperation. *Games and Economic Behavior* 33 (1): 145–158.

Искаков, М., Искаков, А., Захаров, А. Состязание за ренту Таллока и его решение в безопасных стратегиях : препринт WP7/2013/01 [Текст] / М. Искаков, А. Искаков, А. Захаров ; Нац. исслед. ун-т «Высшая школа экономики». – М. : Изд. дом Высшей школы экономики, 2013. – 45 с. (in English).

Хорошо известно, что в состязании Таллока за ренту двух игроков не существует равновесия Нэша в чистых стратегиях, если показатель функции успеха α превышает двойку. В работе представлен анализ состязания на основе концепции Равновесий в безопасных стратегиях (РБС), которая является обобщением равновесия Нэша. РБС определяется двумя условиями: (1) никто не может улучшить свое положение односторонним отклонением, ухудшив при этом положение другого игрока, и (2) никто не может улучшить свое положение односторонним отклонением, не создавая при этом угрозы потерять больше, чем он выигрывает. Показано, что РБС в состязании Таллока всегда существует. Более того, при $\alpha > 2$ оно единственно с точностью до перестановки игроков и имеет более низкую диссипацию ренты, чем соответствующее равновесие Нэша в смешанных стратегиях.

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