The Majority Decision Function on Median Semilattices

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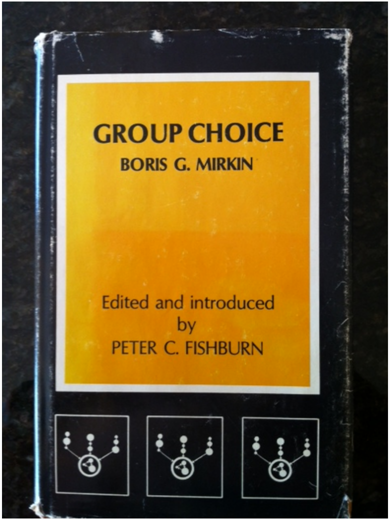
Inspiration from the early 1980's



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For this presentation, I will focus on motivation using the majority decision function. All sets are finite.

The Majority Decision Function of K. May, 1952

Assume $S = \{x, y\}$ alternatives and $K = \{1, \dots, k\}$ voters. Each voter $i \in K$ is required to reveal a preference weak order on S,

$$D_i = \begin{cases} 1 & \text{if } i \text{ prefers } x \text{ to } y, \\ 0 & \text{if } i \text{ is indifferent to } x \text{ and } y, \\ -1 & \text{if } i \text{ prefers } y \text{ to } x. \end{cases}$$

The Majority Decision Function of K. May

The *Simple Majority Decision Function* is defined as follows:

$M : \{-1, 0, 1\}^k \to \{-1, 0, 1\}$, such that $M(D_1, \ldots, D_k) = D$, where

$$D = \begin{cases} 1 & \text{if } \sum_{i=1}^{k} D_i > 0, \\ 0 & \text{if } \sum_{i=1}^{k} D_i = 0, \\ -1 & \text{if } \sum_{i=1}^{k} D_i < 0. \end{cases}$$

Axioms

Let $f : \{-1, 0, 1\}^k \to \{-1, 0, 1\}$ be a "group decision function". Then reasonable properties that f may or may not satisfy are the following.

(A) For any k-tuple $P = (D_1, \ldots, D_k)$ and for any permutation α of K,

$$f(D_{\alpha(1)}, \ldots, D_{\alpha(k)}) = f(D_1, \ldots, D_k).$$

(N) For any k-tuple $P = (D_1, \ldots, D_k)$,

$$f(-D_1, \ldots, -D_k) = -f(D_1, \ldots, D_k).$$

(PR) For any k-tuples $P = (D_1, \ldots, D_k)$ and $P' = (D'_1, \ldots, D'_k)$,

$$\text{if } f(D_1, \ldots, D_k) \in \{0, 1\}, D'_i = D_i \text{ for all } i \neq i_0, \text{ and } D'_{i_0} > D_{i_0},$$

then

$$f(D'_1, \ldots, D'_k) = 1.$$

May's Theorem

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This result has stimulated research into various extensions for more than 50 years. Just pick up recent copies of *Mathematical Social Sciences*, *Social Choice and Welfare*, *Economic Letters*, etc.

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Our goal is to extend May's theorem to an order-theoretic case, with no restrictions (other than being finite) on the number of alternatives or voters.

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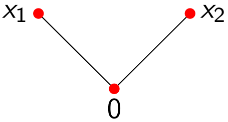
Consider the following ordered set:



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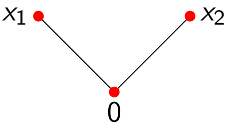

Think $x_1 = 1, x_2 = -1$.
Let $P = (z_1, \ldots, z_k)$ where $z_i \in \{0, x_1, x_2\}$ and set

$$K_{x_i}(P) = \{i : x_i = z_i\}.$$

Another view

Then the majority decision function on two alternatives is given by:

$$M(P) = \begin{cases} x_1 & \text{if } |K_{x_1}(P)| > |K_{x_2}(P)| \\ x_2 & \text{if } |K_{x_2}(P)| > |K_{x_1}(P)| \\ 0 & \text{if } |K_{x_1}(P)| = |K_{x_2}(P)| \end{cases}$$

Meet Semilattice

A **meet semilattice** is a partially ordered set $(X, \leq)$ in which any two elements $u, v \in X$ have a *meet* (greatest lower bound) denoted by $u \wedge v$.

If u and v have a *join* (least upper bound), then it is denoted by $u \vee v$.

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In general for any subset A of X, the meet of A is denoted by $\bigwedge A$ and the join (if it exists) is denoted by $\bigvee A$. If the join of A does not exist, then we write $\bigvee A$ *dne*.

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Moreover, $\bigwedge X = \bigvee \emptyset$ is the least element of X and is denoted by 0. Thus $0 \leq x$ for all $x \in X$.

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A meet semilattice X satisfies the **join-Helly property** if, for all $x, y, z \in X$, whenever $x \vee y$, $x \vee z$, and $y \vee z$ exist, then $x \vee y \vee z$ exists. In this case, by an induction argument, if $x \vee y$ exists for all $x, y \in A$, then $\bigvee A$ exists, for any subset A of X.

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A distributive lattice is a simple example of a median semilattice. Our interest, however, will be in median semilattices that are not lattices.

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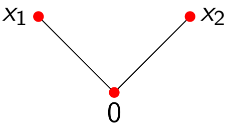
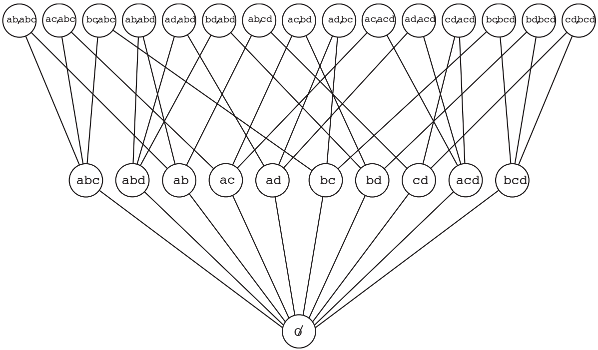
Let J be the set of all join irreducible elements of X. Notice that $0 \notin J$ and $x = \bigvee\{s \in J | s \leq x\}$ for all $x \in X$.

Example 1

A median semilattice X with x_1 and x_2 as join irreducibles.



Example 2: Median semilattice of hierarchies on {a,b,c,d}



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- The set of weak orders on a set. ($W \leq W'$ if every class of W is the union of classes of W'. Join-irreducibles are the two-class weak orders. Max elements are the linear orders.)

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- Rooted trees.

See the many papers of Barthélemy, Leclerc, Monjardet ...

Terminology and Notation

Let $X^* = \bigcup_{k>0} X^k$. So $P \in X^*$ if there exists a positive integer k such that $P \in X^k$. The vector $P \in X^k$ is called a **profile** and $\ell(P) = k$ is the **profile length**.

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For any profile $P = (x_1, \ldots, x_k) \in X^*$ and for any join irreducible $s \in J$, set

$$K_s(P) = \{i | s \leq x_i\} \text{ and } \overline{K}_s(P) = \{i | x_i \vee s \text{ dne}\}$$

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Thus $K_s(P) \cap \overline{K}_s(P) = \emptyset$ and $K_s(P) \cup \overline{K}_s(P) \subseteq \{1, \ldots, \ell(P)\}$.

Majority Decision

The **majority decision function** is the function $M : X^* \to X$ defined by

$$M(P) = \bigvee\{s \in J : |K_s(P)| > |\overline{K}_s(P)|\}$$

for all $P \in X^*$.

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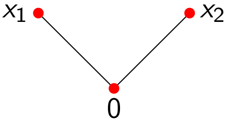
M is well-defined, in the sense that this join does in fact exist.

The function M in action

Suppose X is the median semilattice shown below:

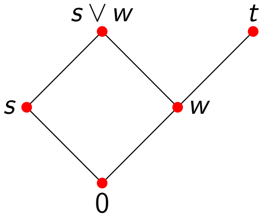


Then

$$M(P) = \begin{cases} x_1 & \text{if } |K_{x_1}(P)| > |K_{x_2}(P)| \\ x_2 & \text{if } |K_{x_2}(P)| > |K_{x_1}(P)| \\ 0 & \text{if } |K_{x_1}(P)| = |K_{x_2}(P)| \end{cases}$$

The function M in action.

Let X be the median semilattice



The set of join-irreducibles is $J = \{s, w, t\}$.Consider the simple profiles $P = (s, t)$ and $Q = (s, s, t)$. Since the join irreducible w is join compatible with s and is less than t, it follows from the definition of M that $M(P) = w$ and $M(Q) = s \vee w$.

Our Problem

Find properties that distinguishes the simple majority decision function M from any other function $F : X^* \to X$.

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Find properties that distinguishes the simple majority decision function M from any other function $F : X^* \longrightarrow X$.

i.e., characterize M using axioms that have some intuitive appeal in decision making.

Strong Pareto

The function $F : X^* \to X$ satisfies the **strong Pareto** axiom (SP) if, for any $s \in J$ and for any $P \in X^*$, then

$$(K_s(P) \neq \emptyset \text{ and } \overline{K}_s(P) = \emptyset) \Rightarrow s \leq F(P).$$

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The axiom (SP) says that if a join irreducible is under at least one element in the profile and is join compatible with every element in the profile, then this join irreducible should be under (i.e., "in") the output.

Weak Decisive Neutrality

A function $F : X^* \to X$ satisfies **weak decisive neutrality** (WDN) if, for all $s, s' \in J$ and for all $P, P' \in X^*$ with $\ell(P) = \ell(P')$;

$$K_s(P) = K_{s'}(P') \text{ and } \overline{K}_s(P) = \overline{K}_{s'}(P') \Rightarrow [s \leq F(P) \Leftrightarrow s' \leq F(P')]$$

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Informally, the axiom (WDN) states that if two profiles have the same length and they "agree" on a pair of join irreducibles, then the outputs should agree on this pair.

If the condition $\overline{K}_s(P) = \overline{K}_{s'}(P'$ is dropped, then (WDN) is the classic *decisive neutrality*. See, for example, B. Monjardet, Math. Social Sciences, 20(1990).

Notation and Simple Profiles

For any $k \geq 2$ and for any profile $P = (x_1, x_2, \ldots, x_k)$ let $P^{-1} = (x_2, \ldots, x_k)$, $P^{-2} = (x_1, x_3, \ldots, x_k)$, $\ldots$, $P^{-k} = (x_1, x_2, \ldots, x_{k-1})$.
In other words, P^{-i} is the profile belonging to X^{k-1} obtained by deleting the i^{th} component from P.

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In other words, P^{-i} is the profile belonging to X^{k-1} obtained by deleting the i^{th} component from P.

A profile P is **simple** if there exist $s, t \in J$ such that $s \vee t$ dne and $P \in \{0, s, t\}^k$ for some positive integer k.

Simple Recursion

A function $F : X^* \to X$ satisfies **simple recursion** (SR) if for any $k \geq 2$ and for any simple profile $P \in X^k$,
$F(P) = F(F(P^{-1}), F(P^{-2}), \ldots, F(P^{-k}))$.

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Axiom (SR) is analogous to the "reducibility to subsocieties" axiom introduced by G. Woeginger, Economics Letters (2003) and we think of it as an iterated stability condition.

Our Result

R.C. Powers and I proved

Theorem

Let X be a median semilattice that is not a lattice and $F : X^* \to X$. Then F satisfies (WDN), (SP), and (SR) if and only if F is the majority decision function M.

This will appear in *Mathematical Social Sciences* (2013).

Next step

Investigate other variants of simple majority decision on median semilattices. For example, define $F : X^* \to X$ by

$$F(P) = \bigvee \{s \in J : |K_s(P)| > |K_t(P)| \ \forall \ t \in J \text{ such that } s \vee t \ dne\}.$$

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