

Attribute Exploration of Properties of Functions on Ordered Sets

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Abstract. An approach for studying relations between properties of functions on ordered sets is proposed. The approach is based on Attribute Exploration. 16 properties of functions are considered, among them monotonicity, idempotency, path independence, exchange properties, convexity, etc. Example functions are partially automatically generated on the powersets of sets with 2, 3 and 4 elements. The protocol of Attribute Exploration, which is run on examples of functions as objects and 16 function properties as attributes, is considered. The current Duquenne-Guigues implication base is presented. The list of proved implications is presented and discussed.

Key words: Properties of Functions; Attribute Exploration; Implication Base

1 Introduction

Many authors have investigated interrelations between properties of functions. For example, in [AM81], [AA95] this issue was studied within choice theory, in [Bir67], [GW96] the authors considered properties of functions in the lattice theory and formal concept analysis, in [Kos99], [MR00] problems about relations between properties arise when considering connection between choice and lattice theories. In this work we propose a clear and easy way for generating statements about properties of functions. For this purpose we use the method called "Attribute exploration" described in [GW96]; an application of this method to boolean algebras and lattices can be found in works [KPR06] and [Dau00].

The result of applying the method is a set of rules which should further be proved or rejected theoretically. If all the rules are either proved or rejected, we will have the complete set of all possible implications between sets of properties under investigation. For making easier the proofs or rejections of implications from the Duquenne-Guigues base, we compute minimal generators of their premises.

The paper is organized as follows. In section 2 we recall some important properties of functions, definitions of implication, their bases, introduce Attribute Exploration. A modification of implication bases is proposed in 2.1. In subsection 2.2 we present and discuss the main results. The list of proved implications is given in subsection 2.3.

2 Main Definitions

Let S be a finite set. A *function* f on 2^S is a map from 2^S to 2^S ($f : 2^S \rightarrow 2^S$), where 2^S is the set of all subsets of S (so-called powerset of S).

Denote by fg the composition of two functions f and g on set 2^S , $fg(A) = f(g(A))$. If $f = g$, then instead of fg , one usually writes f^2 .

Definition 1 ([AA95], [AM81], [Sen71], [Kos99], [Mou85], [Plo73]). *Let f be a function on 2^S ($f : 2^S \rightarrow 2^S$). Consider the following properties for arbitrary $A, B, C \subseteq S$, $x, y \in S$, $x \neq y$,*

<i>Acr.</i>	<i>Name</i>	<i>Definition</i>
<i>ID</i>	<i>Idempotency</i>	$f^2(A) = f(A)$
<i>INT</i>	<i>Intensity</i>	$f(A) \subseteq A$
<i>EXT</i>	<i>Extensivity</i>	$A \subseteq f(A)$
<i>M</i>	<i>Monotonicity</i>	$A \subseteq B \Rightarrow f(A) \subseteq f(B)$
<i>AT</i>	<i>Antitonicity</i>	$A \subseteq B \Rightarrow f(B) \subseteq f(A)$
<i>O</i>	<i>Outcast property</i>	$f(B) \subseteq A \subseteq B \Rightarrow f(B) = f(A)$
<i>H</i>	<i>Heritage property</i>	$A \subseteq B \Rightarrow f(B) \cap A \subseteq f(A)$
<i>C</i>	<i>Concordance</i>	$f(A) \cap f(B) \subseteq f(A \cup B)$
<i>CS</i>	<i>Constancy</i>	$A \subseteq B \Rightarrow \begin{cases} f(B) = \emptyset \Rightarrow f(A) = \emptyset, \\ f(B) \cap A \neq \emptyset \Rightarrow f(A) = f(B) \cap A \end{cases}$
<i>IT</i>	<i>Intransigence</i>	$A \subseteq B \Rightarrow f(A) = f(B) \cap A$
<i>EX</i>	<i>Exchange property</i>	$x, y \notin f(A), y \in f(A \cup \{x\}) \Rightarrow x \in f(A \cup \{y\})$
<i>AEX</i>	<i>Anti-exchange property</i>	$x, y \notin f(A), y \in f(A \cup \{x\}) \Rightarrow x \notin f(A \cup \{y\})$
<i>IAEX</i>	<i>Inverse anti-exchange property</i>	$x, y \in f(A), y \notin f(A - x) \Rightarrow x \in f(A - y)$
<i>CV</i>	<i>Convexity</i>	$[f(A) = f(B) \text{ and } A \subseteq C \subseteq B] \Rightarrow f(A) = f(C)$
<i>PI</i>	<i>Path independency</i>	$f(A \cup B) = f(f(A) \cup f(B))$
<i>CG</i>	<i>Congruency</i>	$f(A) = f(B) \Rightarrow f(A \cup C) = f(B \cup C)$

In the work [Plo73] property PI was defined as: $f(A \cup B) = f(f(A) \cup B)$, but in this work we follow [MR00].

Remark 1. Authors do not claim this list of properties is complete. One can easily find properties not included in this list. Authors have chosen the properties that they find most useful and studied by many authors.

Let G, M be sets. Let $I \subseteq G \times M$ - be a binary relation between G and M . Triple $\mathbf{K} := (G, M, I)$ is called a (*formal*) *context*. Set G is called a set of *objects*, set M - a set of *attributes*.

Consider mappings $\varphi : 2^G \rightarrow 2^M$ and $\psi : 2^M \rightarrow 2^G$: $\varphi(A) := \{m \in M \mid gIm \text{ for all } g \in A\}$, $\psi(B) := \{g \in G \mid gIm \text{ for all } m \in B\}$. Any $A_1, A_2 \subseteq G$, $B_1, B_2 \subseteq M$ satisfy:

1. $A_1 \subseteq A_2 \Rightarrow \varphi(A_2) \subseteq \varphi(A_1)$

2. $B_1 \subseteq B_2 \Rightarrow \psi(B_2) \subseteq \psi(B_1)$
3. $A_1 \subseteq \psi\varphi(A_1)$ and $B_1 \subseteq \varphi\psi(B_1)$

Mappings φ and ψ define *Galois connection* between $(2^G, \subseteq)$ and $(2^M, \subseteq)$. Usually, instead of φ and ψ a single notation $(\cdot)'$ is used.

(Formal) *concept* is a pair (A, B) : $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. Concepts are partially ordered by relation $(A_1, B_1) \geq (A_2, B_2) \iff A_1 \supseteq A_2$ ($B_2 \supseteq B_1$).

Implication $A \rightarrow B$, where $A, B \subseteq M$ takes place if $A' \subseteq B'$, i.e. every object, that has all the attributes from A , also has all the attributes from B . Implications satisfy *Armstrong rules*:

$$\frac{}{X \rightarrow X} \quad , \quad \frac{X \rightarrow Y}{X \cup Z \rightarrow Y} \quad , \quad \frac{X \rightarrow Y, Y \cup Z \rightarrow W}{X \cup Z \rightarrow W}$$

Implication base is a set of implications, from which using Armstrong rules any correct implication for a given context can be deduced and none of the subsets of this set of implications is an implication base.

A minimal in the size implication base was defined in [GD86] and is known as *Duquenne-Guigues implication base*. In paper [Gan84] the premises of implications from the minimal base were characterized in terms of pseudo-intents. A subset of attributes $P \subseteq M$ is called *pseudo-intent*, if $P \neq P''$ and for every such pseudo-intent Q such that $Q \subset P$, one has $Q'' \subset P$. Duquenne-Guigues implication base looks then as follows: $\{P \rightarrow (P'' \setminus P) \mid P - \text{pseudo-intent}\}$.

2.1 Modification of Bases

Proposition 1. *Let \mathfrak{G} be a Duquenne-Guigues implication base for a context \mathbf{K} that contains an implication $H \rightarrow H'' \setminus H$. If $X \subseteq H$ and $X'' = H''$ then an implications set \mathfrak{G}_0 , constructed from \mathfrak{G} by substitution implication $X \rightarrow H'' \setminus H$ for implication $H \rightarrow H'' \setminus H$, is a minimal implications base for the context \mathbf{K} .*

Proof. 1. **Implication $X \rightarrow H''$ is correct in context \mathbf{K} .** Indeed, $X' = X''' = H'''$.

2. **\mathfrak{G}_0 is an implications base.** Indeed, as $X \subseteq H$ then (using second Armstrong rule $\frac{X \rightarrow Y}{X \cup Z \rightarrow Y}$) from $X \rightarrow H''$ can be deduced $H \rightarrow H''$ and, consequently, from \mathfrak{G}_0 can be deduced any implication, that can be deduced from \mathfrak{G} . Moreover, as shown above implication $X \rightarrow H''$ is correct in the context \mathbf{K} , so there doesn't appear any implication that can be deduced from \mathfrak{G}_0 but cannot be deduced from \mathfrak{G} .
3. **Implication base \mathfrak{G}_0 is minimal in the number of implications.** Indeed, the size of \mathfrak{G}_0 is equal to the size of the Duquenne-Guigues base \mathfrak{G} which is minimal in the number of implications.

As proved in proposition 1 the implication base can be changed, namely the premises of the implications can be reduced. It is worth saying that conditions $X \subseteq H$ and $X'' = H''$ may be satisfied by different subsets of the set of attributes

M . Hence, there may be a certain level of freedom in choosing the exact form of an implication base with reduced premises.

If an implication with reduced premise is proved, then, obviously, a proof is correct for an implication with a bigger premise as well. If a counter-example has been found, then it changes both bases; indeed, any implications base defines the whole set of correct implications, so the changes in the set of correct implications result in changes in the implication base.

Hence, one may change the form of the base depending on personal needs. Studying relations between function properties, it is of course easier to prove implications with reduced premises and to find counter-examples for implications with reduced premises.

The pseudocode of the programme we used for finding minimal premises looks as follows (we already had Duquenne-Guigues base computed):

consider implication $a \rightarrow b$,

1. $i := 1, x := \emptyset$
2. **while** $\neg(x \rightarrow a)$ **do**
3. $x := next(x, i)$
4. **return** x

$$next(x, i) = \begin{cases} y, & \text{if } y \text{ is lexicographically next element of } 2^a \text{ of size } i; \\ next(x, i + 1), & \text{if } y \text{ is not defined.} \end{cases}$$

2.2 Implication Bases

In this work we study Duquenne-Guigues implication bases for contexts, where objects are example functions, attributes are properties of functions (definition 1) and a binary relation indicates whether a function has or does not have a property. Example functions were generated on sets $M_2 = 2^{\{0,1\}}$, $M_3 = 2^{\{0,1,2\}}$ and $M_4 = 2^{\{0,1,2,3\}}$, where 0, 1, 2, 3 are incomparable elements.

Functions $f : M_i \rightarrow M_i$ can be represented as M_i -tuples of outputs, if we make an agreement about the input tuple. As sets M_i can be lexicographically ordered, M_i -tuples can be also lexicographically ordered. For each set M_3 and M_4 , due to resource constraints we have taken first 100000 lexicographically ordered M_i -tuples and, first ordered the input tuple lexicographically and obtained first 100000 functions, then reversed the input tuple and obtained another 100000 functions. On set M_2 all the functions were generated. Functions generated on M_4 did not add new implications to the implication base generated for M_2 and M_3 .

The context of functions on M_2 after row reducing is shown in Table 2. The context of functions on both sets M_2 and M_3 after row reduction is shown in Table 1. The functions in the tables are numbered, the properties are written on the top of every column (for definitions see definition 1). Row reduction was performed for both contexts.

Implications that we obtain are rules of the form "if a function has a set of properties A , then it also has a set of properties B ", where $A, B \subseteq M$, $A \cap B = \emptyset$.

Though the Duquenne-Guigues base is minimal in the number of implications, it is not guaranteed that the premises of implications are minimal. However, as shown in proposition 1, the premises of implications in the base may be made smaller. Below the bases with reduced premises for contexts in Table 1 and Table 2 are presented.

The Duquene-Guigues base with reduced premises for context of functions on sets M_2 and M_3 (Table 1):

1. $H \rightarrow IAEX$
2. $CG \rightarrow CV$
3. $M \rightarrow C, CV$
4. $ID, EX, AEX \rightarrow IAEX$
5. $AT \rightarrow H, EX, IAEX, AEX, CV$
6. $INT \rightarrow AEX$
7. $INT, EX \rightarrow H, ID, IAEX$
8. $H, INT \rightarrow EX, ID$
9. $AT, C \rightarrow CG$
10. $, EX, AEX \rightarrow C$
11. $H, ID, , AEX \rightarrow C$
12. $INT, \rightarrow ID, CV$
13. $, CV, EX \rightarrow C$
14. $, AEX, CG \rightarrow IAEX, C$
15. $CS \rightarrow H, IAEX, AEX, C,$
16. $PI \rightarrow ID, IAEX, CG, C, CV,$
17. $EXT \rightarrow H, IAEX,$
18. $IT \rightarrow H, EX, CS, ID, IAEX, PI, CG, M, AEX, C, INT, CV,$
19. $ID, CV \rightarrow$
20. $INT, CG \rightarrow H, EX, CS, ID, IAEX, PI, C,$
21. $M, EX \rightarrow$
22. $H, M \rightarrow$
23. $EXT, CS \rightarrow EX, ID$
24. $H, ID, CV \rightarrow C$
25. $INT, CV, EX \rightarrow CS, PI, CG$
26. $ID, CG \rightarrow IAEX, PI, C$
27. $M, AEX, EX \rightarrow IAEX$
28. $ID, M, EX \rightarrow IAEX$
29. $PI, EX \rightarrow H$
30. $M, EXT \rightarrow CG$
31. $PI, EXT \rightarrow M$
32. $M, CS \rightarrow ID$
33. $H, M, EX \rightarrow CG$
34. $M, AT \rightarrow ID, PI$
35. $AT, EXT \rightarrow ID, PI, M$
36. $PI, AT \rightarrow M$
37. $INT, M, PI \rightarrow IT$
38. $M, EXT, CS \rightarrow IT, INT$

The Duquene-Guigues base with reduced premises for context of functions on set M_2 (Table 2):

1. $H \rightarrow \text{IAEX}$
2. $\text{CG} \rightarrow \text{CV}$
3. $M \rightarrow \text{C}, \text{CV}$
4. $\text{AT} \rightarrow \text{H}, \text{EX}, \text{IAEX}, \text{AEX}, \text{CV}$
5. $\text{INT} \rightarrow \text{AEX}, \text{C}(+)$
6. $\text{INT}, \text{EX} \rightarrow \text{H}, \text{ID}, \text{IAEX}$
7. $\text{H}, \text{INT} \rightarrow \text{EX}, \text{ID}$
8. $\text{AT}, \text{C} \rightarrow \text{CG}$
9. $(-\text{EX}, \text{AEX}) (-\text{EX}, \text{CV}) \rightarrow \text{C}$
10. $\text{INT}, \rightarrow \text{ID}, \text{CV}$
11. $, \text{AEX}, \text{CG} \rightarrow \text{IAEX}, (-\text{C})$
12. $\text{CS} \rightarrow \text{H}, \text{IAEX}, \text{AEX}, \text{C},$
13. $\text{PI} \rightarrow \text{ID}, \text{IAEX}, \text{CG}, \text{C}, \text{CV},$
14. $\text{EXT} \rightarrow \text{H}, \text{IAEX}, \text{C}(+),$
15. $\text{IT} \rightarrow \text{H}, \text{EX}, \text{CS}, \text{ID}, \text{IAEX}, \text{PI}, \text{CG}, \text{M}, \text{AEX}, \text{C}, \text{INT}, \text{CV},$
16. $\text{ID}, \text{CV} \rightarrow , \text{C}(+)$
17. $\text{INT}, \text{CG} \rightarrow \text{H}, \text{EX}, \text{CS}, \text{ID}, \text{IAEX}, \text{PI}, (-\text{C})$
18. $\text{M}, \text{EX} \rightarrow$
19. $\text{H}, \text{M} \rightarrow , \text{CG}(+)$
20. $\text{EXT}, \text{CS} \rightarrow \text{EX}, \text{ID}$
21. $\text{INT}, \text{H}, \rightarrow \text{CS}, \text{PI}, \text{CG}$
22. $\text{ID}, \text{CG} \rightarrow \text{IAEX}, \text{PI} (-\text{C})$
23. $\text{M}, \text{AEX}, \text{EX} \rightarrow \text{IAEX}$
24. $\text{PI}, \text{EX} \rightarrow \text{H}$
25. $\text{PI}, \text{EXT} \rightarrow \text{M}$
26. $\text{M}, \text{CS} \rightarrow \text{ID}, \text{PI}(+)$
27. $\text{M}, \text{AT} \rightarrow \text{ID}, \text{PI}$
28. $\text{AT}, \text{EXT} \rightarrow \text{ID}, \text{PI}, \text{M}$
29. $\text{PI}, \text{AT} \rightarrow \text{M}$
30. $\text{INT}, \text{M}, \text{PI} \rightarrow \text{IT}$
31. $\text{M}, \text{EXT}, \text{CS} \rightarrow \text{IT}, \text{INT}$
32. $\text{ID}, \text{EX} (-\text{AEX}) \rightarrow \text{IAEX}, \text{C}(+)$
33. $\text{AT}, \text{CG} \rightarrow \text{C}(*)$
34. $\text{M}, \text{INT}, \text{CG} \rightarrow \text{IAEX}(*)$
35. $\text{AEX}, \text{EX}, \text{CG} \rightarrow \text{IAEX}(*)$

The implications that are not in the base for context of functions on M_2 and M_3 (first list in this subsection) are marked with "(*)". The properties that are present in some implication in the base of functions on M_2 , but are absent in corresponding implication in the base for context of functions on M_2 and M_3 are marked with "(+)", finally, "(-)" marks the properties that are absent in some implication in the base of functions on M_2 , but are present in corresponding implication in the base for context of functions on M_2 and M_3 .

Implication 9 has two corresponding implications in the base of functions on M_2 and M_3 (first list in this subsection): implication 10 and implication 13.

The list of implications that are present in the base for the context of functions on M_2 and M_3 (Table 1), but are absent in the base for the context of functions on M_2 (Table 2):

1. H, ID, , AEX \rightarrow C
2. H, ID, CV \rightarrow C
3. ID, M, EX \rightarrow IAEX
4. M, EXT \rightarrow CG
5. H, M, EX \rightarrow CG

As the context of functions on M_2 and M_3 (Table 1) is an extension of context of functions on M_2 (Table 2) in some sense, then the above implications list should be possible to derive from the base on M_2 . Indeed, the implication "H, ID, , AEX \rightarrow C" can be derived from the implication " \rightarrow C" (implication 9). Implication "H, ID, CV \rightarrow C" can be derived from implication "ID, CV \rightarrow , C" (implication 16). Implication "ID, M, EX \rightarrow IAEX" can be derived from implication "ID, EX \rightarrow IAEX, C" (implication 32). Implication "M, EXT \rightarrow CG" can be derived from implication "EXT \rightarrow H, IAEX, C, " (implication 14) and implication "H, M \rightarrow , CG" (implication 19). Finally, implication "H, M, EX \rightarrow CG" can be derived from implication "H, M \rightarrow , CG" (implication 19).

Now we prove a proposition that helps to explain the difference between bases.

Proposition 2. *If function f on M_2 has a property it also has a property C.*

Proof. First we recall the properties:

Outcast property (O): $f(B) \subseteq A \subseteq B \Rightarrow f(B) = f(A)$

Concordance (C): $f(A) \cap f(B) \subseteq f(A \cup B)$

Let us consider all possible cases:

1. $A \subseteq B$: then $f(A \cup B) = f(B)$, property C is satisfied.
2. A is incomparable with B : then let $A = \{0\}$ and $B = \{1\}$. $f(A \cup B) = f(\{0, 1\})$, $A, B \subset \{0, 1\}$,
 - (a) $f(\{0, 1\}) = \{0, 1\}$, property C is satisfied.
 - (b) $f(\{0, 1\}) \subset \{0, 1\} \Rightarrow f(\{0, 1\}) \subseteq A \subseteq \{0, 1\}$ (A is taken for certainty, but if the condition is satisfied for B , then in every expression below we should substitute B for A), considering property O: $f(A) = f(\{0, 1\}) \Rightarrow f(A) \cap f(B) \subseteq f(\{0, 1\}) = f(A \cup B)$, which was to be proved. \square

The proved proposition accounts for the changes in implications 9, 5, 11, 14, 16, 19, 22, 32 in the base for the context of functions on M_2 . The proved proposition is only correct for the functions on M_2 , there are counter-examples for functions on M_3 (see Table 1 26th function).

Function	H	EX	CS	ID	IAEX	PI	CG	EXT	M	IT	AEX	C	AT	INT	CV	O
1				✓	✓				✓		✓	✓		✓	✓	✓
2	✓	✓	✓	✓	✓	✓	✓				✓	✓		✓	✓	✓
3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓
4				✓	✓	✓	✓		✓		✓	✓			✓	✓
5	✓			✓	✓	✓	✓	✓	✓		✓	✓			✓	✓
6	✓	✓	✓	✓	✓			✓			✓	✓			✓	✓
7					✓				✓		✓	✓		✓	✓	
8				✓					✓		✓	✓		✓	✓	✓
9	✓	✓		✓	✓						✓	✓		✓		
10	✓		✓	✓	✓	✓	✓		✓		✓	✓			✓	✓
11	✓	✓		✓	✓	✓	✓	✓	✓			✓			✓	✓
12	✓	✓			✓		✓	✓			✓	✓			✓	✓
13		✓		✓	✓				✓		✓	✓			✓	✓
14	✓	✓			✓		✓	✓	✓		✓	✓			✓	✓
15	✓	✓	✓	✓	✓			✓			✓	✓				✓
16	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓
17					✓		✓		✓		✓	✓			✓	
18		✓					✓		✓			✓			✓	✓
19	✓	✓			✓		✓				✓	✓	✓		✓	
20	✓	✓	✓	✓	✓	✓	✓		✓		✓	✓	✓		✓	✓
21		✓			✓		✓		✓		✓	✓			✓	✓
22		✓									✓	✓			✓	✓
23	✓	✓			✓						✓	✓	✓		✓	
24	✓	✓	✓		✓		✓				✓	✓	✓		✓	✓
25	✓	✓		✓	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓
26				✓	✓						✓			✓	✓	✓
27	✓	✓		✓	✓						✓			✓		
28	✓		✓	✓	✓				✓		✓	✓			✓	✓
29		✓		✓								✓			✓	✓
30							✓		✓		✓	✓			✓	
31	✓	✓			✓		✓				✓		✓		✓	
32		✓					✓				✓	✓			✓	
33	✓	✓		✓	✓			✓								✓
34	✓				✓			✓			✓				✓	✓
35	✓				✓		✓	✓							✓	✓

Table 1. Row reduced context of functions on M_2 and M_3

Function	H	EX	CS	ID	IAEX	PI	CG	EXT	M	IT	AEX	C	AT	INT	CV	O
1				✓	✓				✓		✓	✓		✓	✓	✓
2	✓	✓	✓	✓	✓	✓	✓				✓	✓		✓	✓	✓
3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓
4				✓	✓	✓	✓		✓		✓	✓			✓	✓
5	✓			✓	✓	✓	✓	✓	✓		✓	✓			✓	✓
6	✓	✓	✓	✓	✓			✓			✓	✓			✓	✓
7					✓				✓		✓	✓		✓	✓	
8				✓					✓		✓	✓		✓	✓	✓
9	✓	✓		✓	✓						✓	✓		✓		
10	✓		✓	✓	✓	✓	✓		✓		✓	✓			✓	✓
11	✓	✓		✓	✓	✓	✓	✓	✓			✓			✓	✓
12	✓	✓			✓		✓	✓			✓	✓			✓	✓
13		✓		✓	✓				✓		✓	✓			✓	✓
14	✓	✓			✓		✓	✓	✓		✓	✓			✓	✓
15	✓	✓	✓	✓	✓			✓			✓	✓				✓
16	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓
17					✓		✓		✓		✓	✓			✓	
50	✓			✓	✓						✓					
18		✓					✓		✓			✓			✓	✓
19	✓	✓			✓		✓				✓	✓	✓		✓	
20	✓	✓	✓	✓	✓	✓	✓		✓		✓	✓	✓		✓	✓
51							✓				✓	✓			✓	
21		✓			✓		✓		✓		✓	✓			✓	✓
52	✓	✓			✓		✓				✓				✓	
22		✓									✓	✓			✓	✓
23	✓	✓			✓						✓		✓		✓	
24	✓	✓	✓		✓		✓				✓	✓	✓		✓	✓
25	✓	✓		✓	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓

Table 2. Row reduced context of functions on M_2

2.3 List of Proved Implications

Currently we know that the following implications hold:

- $\text{EXT} \rightarrow \text{O}, \text{H}, \text{IAEX}$
- $\text{INT} \rightarrow \text{AEX}$
- $\text{M} \rightarrow \text{CV}, \text{C}$
- $\text{IT} \rightarrow \text{INT}, \text{CS}, \text{ID}, \text{PI}, \text{O}, \text{C}, \text{H}, \text{M}$
- $\text{PI} \rightarrow \text{ID}, \text{CG}$ [MR00], [DK06]
- $\text{H} \rightarrow \text{IAEX}$
- $\text{ID}, \text{CV} \rightarrow \text{O}$
- $\text{ID}, \text{M} \rightarrow \text{O}$ [MR00]
- $\text{ID}, \text{CG} \rightarrow \text{PI}$
- $\text{CS} \rightarrow \text{O}, \text{H}, \text{C}$ [AA95]
- $\text{INT}, \text{PI} \leftrightarrow \text{INT}, \text{CG}$ [DK06]
- $\text{INT}, \text{ID}, \text{CV} \leftrightarrow \text{INT}, \text{O}$ [MR00]
- $\text{INT}, \text{PI} \leftrightarrow \text{INT}, \text{H}, \text{O}$ [AM81]
- $\text{EXT}, \text{PI} \leftrightarrow \text{EXT}, \text{ID}, \text{M}$ [MR00]
- $\text{INT}, \text{H}, \text{M} \leftrightarrow \text{INT}, \text{IT}$

Let us prove the implications that are not proved in the literature.

As can be seen from the definitions of properties, property IT is stronger than property CS, so property IT also implies properties H, C, O ([AA95]).

Choice functions are called functions that have property INT.

Theorem 1. *Let f be a choice function on 2^S . Then the following conditions are equivalent:*

- f has properties H and M;
- f has property IT.

Proof. $\text{INT}, \text{H}, \text{M} \rightarrow \text{INT}, \text{IT}$. Having M and INT we obtain $A \subseteq B \Rightarrow f(A) \subseteq f(B) \Rightarrow f(A) \cap A \subseteq f(B) \cap A \Rightarrow f(A) \subseteq f(B) \cap A$. Having H we obtain $f(B) \cap A \subseteq f(A)$. Hence, we have $f(B) \cap A \subseteq f(A) \subseteq f(B) \cap A$, then $f(A) = f(B) \cap A$, which is property IT.

$\text{INT}, \text{IT} \rightarrow \text{INT}, \text{H}, \text{M}$. Conversely, IT implies $A \subseteq B \Rightarrow f(A) = f(B) \cap A \Rightarrow f(A) \subseteq f(B)$, which is property M. Moreover, IT implies $f(A) = f(B) \cap A \Rightarrow f(B) \cap A \subseteq f(A)$, which is property H. \square

Remark 2. Note that implication $\text{IT} \rightarrow \text{H}, \text{M}$ holds for every function.

Theorem 2. *Let f be a function on 2^S , then the following statements hold:*

1. *If f has property EXT, then f has properties O, H and IAEX;*
2. *If f has property INT, then f has properties AEX;*
3. *If f has property H, then f has property IAEX;*
4. *If f has properties ID and CV, then f has property O;*
5. *If f has properties ID and CG, then f has property PI;*
6. *If f has property M, then f has properties CV and C;*

7. If f has property IT, then f has properties M, INT, PI, ID.

Proof. 1. Having EXT $A \subseteq f(A)$ we obtain $f(B) \cap A \subseteq f(A)$, which is property H.

H implies IAEX, so EXT implies IAEX (see 3 in this theorem).

If $f(B) \subseteq A \subseteq B$, then, having EXT $B \subseteq f(B)$ we obtain $f(B) = A = B$, therefore $f(A) = f(B)$, which is property O.

2. If $x, y \notin f(A)$ and f has property INT, then $f(A) \subseteq A \Rightarrow x, y \notin A$, so $x \notin A \cup \{y\}$, $f(A \cup \{y\}) \in A \cup \{y\} \Rightarrow x \notin f(A \cup \{y\})$.

3. Having H we obtain $(A - x) \subseteq A \Rightarrow f(A) \cap (A - x) \subseteq f(A - x)$, and, if $x, y \in f(A)$, $y \notin f(A - x)$, then $y \notin A - x$, and, as $x \neq y$, $y \notin A$, then $f(A - y) = f(A)$, so $x \in f(A - y) = f(A)$, q.e.d.

4. If $f(B) \subseteq A \subseteq B$, having ID $f(f(B)) = f(B)$ and CV we obtain $f(f(B)) = f(A) = f(B)$, which is property O.

5. Having ID and CG we obtain $f(f(A)) = f(A) \Rightarrow f(f(A) \cup f(B)) = f(A \cup f(B))$, applying the same conversion once more we obtain $f(f(B)) = f(B) \Rightarrow f(A \cup f(B)) = f(A \cup B)$, finally we get $f(f(A) \cup f(B)) = f(A \cup f(B)) = f(A \cup B)$, q.e.d.

6. Having M we obtain $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(A \cup B) \forall A, B \subseteq S$, then $f(A) \cap f(B) \subseteq f(A \cup B)$, which is property C.

If $A \subseteq X \subseteq B$, then, having M we obtain $f(A) \subseteq f(X) \subseteq f(B)$, but $f(A) = f(B)$, so $f(X) = f(A) = f(B)$.

7. Having IT we obtain $A \subseteq B \Rightarrow f(A) = f(B) \cap A \Rightarrow f(A) \subseteq f(B)$, which is property M.

Having IT we obtain $A \subseteq B \Rightarrow f(A) = f(B) \cap A \Rightarrow f(A) \subseteq A$, which is property INT.

Property IT implies properties H, O and INT, and these three properties imply property PI ([AM81]). \square

3 Conclusions

Implication bases for context of functions on the powerset of two-element set and for context of functions on powerset of two- and three-element sets were obtained. The difference in bases is presented and partly explained. Some implications in these bases were proved in the literature, for some of implications we gave the proofs. The remaining implications are to be either proved or rejected. We computed minimal generators of the premises closures as alternative to pseudo-intents. The use of minimal generators instead of pseudo-intents may make proofs and counterexamples of implications easier. We presented the complete list of proved implications and the contexts used in our study, so one can test the results and continue proving or refuting the implications between function properties.

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