Aggregating and Updating Information☆

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Abstract

We study information aggregation problems where to a set of measures a single measure of the same dimension is associated. The collection of measures could represent the beliefs of agents about the state of the world, and the aggregate would then represent the beliefs of the population. Individual measures could also represent the connectedness of agents in a social network, and the aggregate would reflect the importance of each individual. We characterize the aggregation rule that resembles the Nash welfare function. In the special case of probability aggregation problems, this rule is the only one that satisfies Bayesian updating and some well-known axioms discussed in the literature.

Keywords: belief aggregation, belief updating, Nash welfare function
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1. Introduction

We study information aggregation problems where to a set of measures a single measure of the same dimension is associated. The collection of measures could represent the beliefs of agents about the state of the world, and the aggregate would then represent the beliefs of the population. Individual measures could also represent the connectedness of agents in a social

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network, and the aggregate would reflect the importance of each individual. We characterize the aggregation rule (the *Nash rule*) that resembles the Nash welfare function Kaneko (1979).

This is done both in the case the measures are probability distributions and in the case of non-normalized measures. In the special case of probability aggregation problems, the Nash rule rule is the only one that satisfies Bayesian updating on top of some standard axioms. While probability measures seem natural in belief aggregation problems, non-normalized measures could be better suited in some network applications, for example.

Crès *et.al* (2011) and Gilboa *et.al* (2004) are recent papers where belief aggregation or belief *and* preference aggregation problems are studied. In Crès *et.al* (2011) there is decision maker and a number of experts who all have the same utility function but different set of prior beliefs (probability measures) over the states. The problem is how to determine the beliefs for the decision maker in a reasonable way. Gilboa *et.al* (2004) study utilitarian aggregation of preferences and beliefs in the social choice context: when are society's welfare function and beliefs representable as weighted averages of those of individual agents.

The same machinery that for decades has been used to analyze social choice problems can be applied to all kinds of belief or opinion aggregation problems. Recently these tools have been applied to the analysis and construction of citation indices and internet search engines (see Palacios-Huerta and Volij 2004; Slutski and Volij 2006). For recent papers dealing with judgement aggregation from the logical point of view, see List and Polak (2010) or Nehring and Puppe (2010).

The paper is organized in the following way. In Section 2 the notation and aggregation rules are introduced. The axioms are introduced in Section 3. The main results are given in Section 4.

2. Preliminaries

A *measure* μ on S satisfies (i) $\mu(E) \geq 0$, for each *event* $E \subset S$; (ii) $\mu(\emptyset) = 0$; and (iii) $\mu(E \cup E') = \mu(E) + \mu(E')$ for all disjoint events $E, E' \subset S$.

We may denote the measure of singletons $\{s\}$ by $\mu(s)$ instead of $\mu(\{s\})$. Inequality $m(E) < m(E')$ means $m_i(E) < m_i(E')$ for all $i \in N$; inequality $m_i < m'_i$ means $m_i(E) < m'_i(E)$ for all nonempty $E \subset S$; inequality $m < m'$ means $m_i < m'_i$ for all $i \in N$. Given a measure μ on S and $E \subset S$, the *restriction* of μ to E is a measure $\mu_{|E}$ on S defined by $\mu_{|E}(A) = \mu(A \cap E)$ for every $A \subset S$.

Let $supp(\mu) = \{s \in S \,|\, \mu(S) > 0\}$ be the *support* of a measure μ on S. A measure with an empty support is called a *null measure* and we denote it by μ^0. Given a profile $m = (m_i)_{i \in N}$ of measures on S, let $supp(m) = \{s \in S \,|\, m_i(s) > 0,\ \text{for all } i \in N\}$ be the intersection of the supports of the measures m_i. If there is any risk of confusion we will state explicitly whether $supp(m)$ means the support of a single measure or a profile of measures.

An *aggregation problem* is a triple $P = (N, S, m)$, where N is a nonempty finite subset of natural numbers $\mathbb{N} = \{0, 1, \ldots\}$, S is a nonempty finite set, and $m = (m_i)_{i \in N}$ is a *profile* of measures m_i on S. We denote the set of all aggregation problems (or simply problems) by $\mathcal{P}$. We may study the subclass of problems with a common support ($supp(m_i) = supp(m_j)$, for all $i, j \in N$) denoted by $\mathcal{P}^{cs}$, and a special case of this, the problems with full support ($supp(m_i) = S$, for all $i \in N$) denoted by $\mathcal{P}^{+}$. If we want to study subclasses of problems with a given set of agents N or states S, we may denote these classes by $\mathcal{P}^{N}$, $\mathcal{P}^{N,S}$, $\mathcal{P}^{+,N}$ *e.t.c.*

An interpretation of the model is that N is the set of agents, S is the set of states of the nature, and m_i is the measure for agent i representing his beliefs about what is the true state s. Another possible interpretation is that S is the set of alternatives from which the society must choose one element, and the measure m_i represents the preferences of agent i. A third interpretation is that N is the set of authors, S is the set of articles in academic journals, and $m_i(s)$ denotes the number of times author i has cited article s. More generally, since tastes and beliefs are opinions and citations reflect opinions as well, we may say that the measures m_i represent the opinions of the agents.

An *aggregation rule* is a function f sucht that $f(P)$ is a measure on

S for each aggregation problem $P = (N, S, m) \in \mathcal{P}$. Depending on the interpretation of the aggregation problem P, $f(P)$ may be interpreted as an aggregate belief of the society, or as a social preference, or as a "general opinion".

We say that a problem $P = (N, S, m)$ is a *probability aggregation problem*, if each m_i is a probability measure and $f(P)$ should also be a probability measure. Note that this subclass of problems is different than the ones defined above, since the definition also restricts the class of feasible rules.

2.1. Some well-known aggregation rules

The *Average rule* f^A is the best known rule. It is defined by $f^A(P)(s) = \frac{1}{n}\sum_{i\in N} m_i(s)$ for every $s \in S$, for each problem $P = (N, S, m)$.

The Median rule f^M is defined as follows for every problem $P = (N, S, m)$ (see *e.g.* Balinski and Laraki 2007; Barthelemy and Monjardet 1981. Given $s \in S$, let $f^M(P)(s)$ be the median of the components of the vector $m(s)$. In case where the successive elimination of greatest and least values of the coordinates of the vector $m(s)$ leaves us with two components $m_i(s)$ and $m_j(s)$, we define the median to be the average of these values. For example, if $m(s) = (1, 1, 3)$, then the median is 1, but if $m(s) = (1, 1, 3, 3)$, then the median is 2.

The *Borda rule* f^B is also quite well-known (see *e.g.* Nurmi and Salonen 2008; Saari 2006; Young 1974. Let $b_i(P)(s) = |\{s' \in S \,|\, m_i(s') \leq m_i(s)\}|$ for all $s \in S$, and let $f^B(P)(s) = \frac{1}{n}\sum_{i\in N} b_i(P)(s)$, for all problems $P = (N, S, m)$. Note that if for each i the measures $m_i(s)$ are different for different states s, we get the standard form of the Borda rule. The Borda rule is often defined as the sum $\sum_{i\in N} b_i(P)(s)$. For all practical purposes the two versions are the same.

The *Nash rule* f^G is based on the Nash welfare function Kaneko (1979), and the idea can be applied in the present context as well. It is defined by $f^G(P)(s) = \sqrt[n]{\prod_{i\in N} m_i(s)}$ for each $s \in S$, for each problem $P = (N, S, m)$. The superscript G refers to the fact that $f^G(P)(s)$ is the geometric average of the individual $m_i(s)$ -values.

The *Norm rules* f^{EN}, f^{SN} are based on the Euclidean norm and *sup* -norm, respectively. The rule f^{EN} is defined by $f^{EN}(P)(s) = n^{-1/2}\sqrt{m_1(s)^2 + \cdots + m_n(s)^2}$ for each $s \in S$, for each problem $P = (N, S, m)$. Define f^{SN} by $f^{SN}(P)(s) = \sup\{|m_1(s)|, \ldots, |m_n(s)|\}$ for each $s \in S$, for each problem $P = (N, S, m)$. Note that the norm rule corresponding to the city block norm $|m_1(s)| + \cdots + |m_n(s)|$ is the Average rule f^A.

The rules defined above can be defined in such a way that they are applicable in probability aggregation problems as well. There are many ways to do it. Suppose the subclass of problems is such that $f(P)(S) > 0$, and each m_i is a probability measure, for every problem $P = (N, S, m)$ in this subclass. Then a probability aggregation rule $f^\times$ can be defined by $f^\times(P)(s) = f(P)(s)/f(P)(S)$ for every $s \in S$. We call $f^\times$ the *multiplicative normalization* of f.

3. Properties of aggregation rules

Now we present some properties or axioms that aggregation rules could satisfy. For a more comprehensive treatment of different aggregation procedures and their properties, see *e.g.* Nurmi (2002).

We don't specify in each case the subclass of problems where the axiom in question should be applicable. Instead, we specify in our theorems the subclass where the rules are defined, and axioms are then restricted to this subclass as well. This way we may use the axioms in a more flexible manner. For example, if we analyze the class of problems with full support, then *Regularity* (defined below) has no bite. Notable exception to this practice is the axiom *Bayesian updating* that is designed specifically for probability aggregation problems.

Given agent sets N and M with equally many members, let $\pi : N \longrightarrow M$ be any bijection, and given an n -tuple m of measures, let πm be an n -tuple of profiles such that $\pi m_{\pi(i)} = m_i$. In other words, the agent $\pi(i)$ has the same measure in profile πm as person i has in profile m. Given an aggregation problem $P = (N, S, m)$ and a bijection $\pi : N \longrightarrow M$, define another aggregation problem $Q = (M, S, \pi m)$, which is otherwise the same as P except that agent $\pi(i) \in M$ has been given the measure m_i of agent $i \in N$.

Axiom 1 (*Anonymity*, **AN**)**.** For every bijection $\pi : N \longrightarrow M$ and aggregation problems $P = (N, S, m)$ and $Q = (M, S, \pi m)$, it holds that $f(Q) = f(P)$.

Let S and T be two finite sets with the same number of elements. Given an aggregation problem $P = (N, S, m)$ and a bijection $\pi : S \longrightarrow T$, define another aggregation problem $Q = (N, T, m^\pi)$, which is otherwise the same as P except that elements $s \in S$ are replaced by elements $\pi(s) \in T$, and $m^\pi(\pi(s)) = m(s)$, for all $s \in S$.

Axiom 2 (*Neutrality*, **NE**)**.** For every bijection $\pi : S \longrightarrow T$ and aggregation problems $P = (N, S, m)$ and $Q = (N, T, m^\pi)$, it holds that $f(P)(s) = f(Q)(\pi(s))$ for every $s \in S$.

Anonymity says that the labels of the agents do not matter, while *Neutrality* says that labels of the states do not matter. All the rules defined in Section 2.1 satisfy *Neutrality* and *Anonymity*. These rules satisfy also the following axiom called *Unanimity*.

Axiom 3 (*Unanimity*, **UN**)**.** If $m_1 = \cdots = m_n = \mu$ in an aggregation problem $P = (N, S, m)$, then $f(P) = \mu$.

These three axioms are standard in the literature.

Axiom 4 (*Common Scale Covariance*, **CSC**)**.** If $P = (N, S, p)$ and $Q = (N, S, q)$ are two problems such that $q = ap$ for some $a > 0$, then $f(Q) = af(P)$.

Common scale covariance says that if we multiply the opinions of all agents by the same constant then the aggregate opinion will be multiplied by the same constant. All rules in Section 2.1 except the Borda rule satisfy this axiom.

Axiom 5 (*Individual Scale Covariance*, **ISC**)**.** If problems $P = (N, S, p)$ and $Q = (N, S, q)$ are such that for some $i \in N$, $q_i = ap_i$ for some $a > 0$, and $q_j = p_j$ for all $j \neq i, j \in N$, then $f(Q) = \alpha_i^P(a)f(P)$. The function $\alpha_i^P : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is strictly increasing and continuous.

Individual scale covariance says that if we multiply agent i's opinions by a positive constant, then the aggregated beliefs are also multiplied by some positive constant. This axiom is needed in applications where only the ratios $m_i(s)/m_i(s')$ and $f(P)(s)/f(P)(s')$ of individual and aggregate opinions matter. Because if **ISC** is satisfied, scaling the measure m_i up or down has no effect on these ratios. All rules in Section 2.1 except the Borda rule satisfy **CSC** but only the Nash rule f^G satisfies both **ISC** and **SC**.

The next one is a version of the well-known axiom the appears already in Arrow's seminal work Arrow (1963).

Axiom 6 (*Independence of Irrelevant Alternatives*, **IIA**)**.** Let $P = (N, S, p)$ and $Q = (N, S, q)$ be two aggregation problems such that $p(s) = q(s)$ and $p(s') = q(s')$ for some $s, s' \in S$. Then $f(P)(s) < f(P)(s')$ if and only if $f(Q)(s) < f(Q)(s')$.

All the rules defined in Section 2.1 satisfy this axiom except the Borda rule. The following axiom is closely related to **IIA**.

Axiom 7 (*Updating*, **UP**)**.** If $P = (N, S, m)$ and $Q = (N, E, m_{|E})$ are such that $E \subset S$ and $E \neq \emptyset$, then $f(Q) = f(P)_{|E}$.

We will show in Section 4 that every rule that satisfies **UP** satisfies also **IIA**. The only rule in Section 2.1 that does not satisfy **UP** is the Borda rule. The following axiom is the well-known Bayesian updating property. It's domain is the class of probability aggregation problems.

Axiom 8 (*Bayesian Updating*, **BUP**)**.** Let $P = (N, S, p)$ and $Q = (N, E, q)$ be two probability aggregation problems such that $E \subset S$, $p_i(E) > 0$ and q_i is derived from p_i by the Bayes rule for all $i \in N$. Then the probability measure $f(Q)$ is derived from the probability measure $f(P)$ by the Bayes rule.

Bayesian updating seems so natural that one may wonder whether it has any bite at all. However, even the best-known aggregation rule, the *Average rule*, fails to satisfy this axiom. The following axiom makes sense in all kinds of aggregation problems.

Axiom 9 (*Expert Proofness*, **EP**)**.** Let $P = (N, S, p)$ and $Q = (N \setminus \{i\}, S, q)$ be two aggregation problems such that $i \in N$, $p_j = q_j$, for all $j \in N \setminus \{i\}$, and $p_i = f(Q)$. Then $f(P) = f(Q)$.

If agent i adopts the *aggregated opinions* of the other agents $j \in N \setminus \{i\}$, then the aggregated opinions of the enlarged population N are the same as the aggregated opinions of $N \setminus \{i\}$. One interpretation is that the public already has a quite good idea of what the opinions in the society are, and they may have adjusted a little bit their own views as a response. Then if an expert comes and makes the societys opinions common knowledge, the public has no reason to adjust their opinions any more. We will show in Section 4 that all the rules defined in Section 2.1 satisfy **EP**.

Axiom 10 (*Regularity*, **REG**)**.** Given $P = (N, S, m)$, it holds that $f(P)(s) = 0$ for a given $s \in S$, if $m_i(s) = 0$ for all $i \in N$.

Regularity is satisfied in the class $\mathcal{P}$ by all the other rules defined in Section 2.1.

4. Results

All the rules defined in Section 2.1 are *Expert proof.*

Lemma 1. *The Average rule, the Borda rule, the Median rule, the Nash rule, and the Norm rules f^{EN} and f^{SN} satisfy **EP**.*

Proof. Let $P = (N, S, p)$ and $Q = (N \setminus \{i\}, S, q)$ be two problems as in the axiom **EP**.

It is straightforward to verify that the Average rule satisfies **EP**.

Take the Borda rule f^B and consider the problems $P' = (N, S, b(P))$ and $Q' = (N, S, b(Q))$, where $b_i(P)$ is the measure derived from agent i's

Borda scores $b_i(P)(s)$ in the problem $P = (N, S, p)$, and $b_j(Q)$ is derived from agent j's Borda scores $b_j(Q)(s)$ in the problem $Q = (N \setminus \{i\}, S, q)$.

Since $p_i(s') < p_i(s)$ iff $b_i(P)(s') < b_i(P)(s)$, the problem P' is obtained from P by applying strictly increasing transformations to the measures p_i. Similarly, Q' is obtained from Q by applying strictly increasing transformations to the measures q_j. Since the Borda scores b_i depend only on the ordinal ranking of states s, the individual Borda scores satisfy $b_i(P')(s) = b_i(P)(s)$ and $b_j(Q')(s) = b_j(Q)(s)$.

Since $p_i(s) = f^B(Q)(s) = \frac{1}{n-1}\sum_{j\in M} b_j(Q)(s)$ and $b_j(P) = b_j(Q)$ for every $j \in M$, we have

$$\frac{1}{n}\sum_{j\in N} b_j(P)(s) = \frac{1}{n}\Big[\sum_{j\in M} b_j(Q)(s) + \frac{1}{n-1}\sum_{j\in M} b_j(Q)(s)\Big],$$

which implies $f^B(P')(s) = f^B(Q')(s)$. Since $b_i(P')(s) = b_i(P)(s)$ and $b_j(Q')(s) = b_j(Q)(s)$, we have that $f^B(P)(s) = f^B(Q)(s)$, so the Borda rule satisfies **EP**.

The Median rule has the property that for all $s \in S$, $p_i(s) = f^M(Q)(s)$ and this is the median of the vector $q(s)$. But then $p_i(s)$ is the median of the coordinates of the vector $p(s)$ as well, and so f^M satisfies **EP**.

Let $P = (N, S, p)$ and $Q = (N \setminus \{i\}, S, q)$ be as stated in **EP**, and define $M = N \setminus \{i\}$. Then for each $s \in S$,

$$f^G(P)(s) = \left[\prod_{j\in M} p_j(s)\left(\prod_{j\in M} p_j(s)\right)^{1/(n-1)}\right]^{1/n} = f^G(Q)(s),$$

and therefore the Nash rule satisfies **EP**.

If we use the Norm rule f^{EN}, we have

$$f^{EN}(Q)(s) = (n-1)^{-1/2}\sqrt{\sum_{j\neq i} m_j(s)^2}\,,$$

which implies

$$f^{EN}(P)(s) = n^{-1/2}\sqrt{\sum_{j\neq i} m_j(s)^2 + (n-1)^{-1}\sum_{j\neq i} m_j(s)^2}\,,$$

but then $f^{EN}(P)(s) = f^{EN}(Q)(s)$ as desired.

The proof for *sup* -norm rule f^{SN} is easy and omitted. $\square$

We show next that the axiom **UP** implies **IIA**.

Lemma 2. *If a rule f satisfies **UP**, then it satisfies **IIA**.*

Proof. Suppose f satisfies **UP**. Let $P = (N, S, p)$ and $Q = (N, S, q)$ be two aggregation problems as in the statement of **IIA**: $p(s) = q(s)$ and $p(s') = q(s')$ for two members $s, s' \in S$. Let $E = \{s, s'\}$, and $P' = (N, E, p_{|E})$, and $Q' = (N, E, q_{|E})$. Then by **UP**, $f(P') = f(P)_{|E}$ and $f(Q') = f(Q)_{|E}$. But $P' = Q'$ because $p_{|E} = q_{|E}$, and therefore f satisfies **IIA**. $\square$

The next lemma is needed in the proofs of the main results.

Lemma 3. *Suppose f satisfies **NE**, **ISC**, and **UP**. Then the function α_i in the axiom **ISC** does not depend on P.*

Proof. Suppose problems $P = (N, S, p)$ and $Q = (N, S, q)$ are such that for some $i \in N$, $q_i = ap_i$ for some $a > 0$, and $q_j = p_j$ for all $j \neq i, j \in N$. Given $s \in S$, define $P^s = (N, \{s\}, p_{|\{s\}})$ and $Q^s = (N, \{s\}, q_{|\{s\}})$. By **UP**, $f(P)(s) = f(P^s)(s)$ and $f(Q)(s) = f(Q^s)(s) = \alpha_i^P f(P^s)(s)$.

Let $P' = (N, X, p')$ and $Q' = (N, X, q')$ be such that a) for some $x \in X$, $p'(x) = p(s)$, b) $q'_i = ap'_i$, and $q_j = p_j$ for all $j \neq i, j \in N$. Define $P'^x = (N, \{x\}, p'_{|\{x\}})$ and $Q'^s = (N, \{x\}, q'_{|\{x\}})$. Then by **NE**, $f(P'^x) = f(P^s)$ and $f(Q'^x) = f(Q^s) = \alpha_i^P f(P^s)$. By **UP**, $f(P')(x) = f(P'^x)(x)$, and $f(Q')(x) = f(Q'^x)(x) = \alpha_i^P f(P')(x)$. By **ISC**, $f(Q') = \alpha_i^{P'} f(P')$. But then $\alpha_i^P = \alpha_i^{P'}$, and we are done. $\square$

We give next an axiomatic characterization of the Nash rule on the class of full support aggregation problems $\mathcal{P}^+$. First we characterize a one-parameter family of rules.

Theorem 1. *Let f be a rule satisfying **AN**, **ISC**, **CSC**, **NE**, and **UP** on the class of full support problems $\mathcal{P}^{+,N}$ with a given set N of agents. Then for some $a > 0$, $f = af^G$, or $f(P)$ is the null measure μ^0 for all $P \in \mathcal{P}^{+,N}$.*

Proof. Clearly the rule that assigns the null measure to every problems satisfies these axioms. So suppose f is another rule satisfying the axioms **AN**, **ISC**, **CSC**, **NE** and **UP**.

Let $P = (N, S, m) \in \mathcal{P}^{+,N}$ be any problem and take any $s \in S$. By **UP**, $f(Q) = f(P)_{|\{s\}}$ where $Q = (N, \{s\}, m_{|\{s\}})$. By the full support assumption, $m_i(s) > 0$ for every $i \in N$.

Let q^i be the vector such that $q^i_i = 1$ and $q^i_j = m_j(s)$ for all $j \neq i$, and let $Q^i = (N, \{s\}, q^i)$. Note that $m_i(s) = m_i(s)q^i_i$. Then by **ISC**, $f(Q) = \alpha_i(m_i(s))f(Q^i)$, where α_i is the continuous strictly increasing function in the axiom **ISC**. By Lemma 3, α_i does not depend on P.

Let q^{ij} be the vector such that $q^{ij}_i = q^{ij}_j = 1$ and $q^{ij}_k = m_k(s)$ for all $k \neq i, j$, and let $Q^{ij} = (N, \{s\}, q^{ij})$. Then $f(Q) = \alpha_j(m_j(s))\alpha_i(m_i(s))f(Q^{ij})$ by **ISC**.

Let P' be a problem that is otherwise like P except in problem P' player i has a measure $m'_i = m_j$ and player j has the measure $m'_j = m_i$. Derive Q' from P' in the same ways as Q was derived from P above, and construct q'^{ij} in the same fashion as q^{ij}.

By **AN**, $f(P) = f(P')$ and $f(Q) = f(Q')$, and therefore we must have $\alpha_j(m_j(s))\alpha_i(m_i(s)) = \alpha_j(m_i(s))\alpha_i(m_j(s))$. Since $m_i(s)$ and $m_j(s)$ are arbitrary positive numbers, and functions α_i and α_j are strictly increasing, we must have that $\alpha_i = \alpha_j$. Since players i and j were arbitrarily chosen, $\alpha_1 = \cdots = \alpha_n \equiv \alpha$.

Applying **ISC** recursively, we get that $f(Q) = \prod_i \alpha(m_i(s))f(Q^N)$, where $Q^N = (N, \{s\}, \mathbf{1})$ and $\mathbf{1} = (1, \ldots, 1)$. In the special case $m_1(s) = \cdots = m_n(s) = a$, we get that $f(Q) = \alpha(a)^n f(Q^N)$. But by **CSC**, we must have $\alpha(a)^n = a$, or equivalently $\alpha(a) = \sqrt[n]{a}$.

It follows that

$$f(P)(s) = \left[\sqrt[n]{\prod_{i=1}^{n} m_i(s)} \right] f(Q^N). \tag{1}$$

Now the value $f(Q^N)$ must be the same for all $s \in S$ by **NE**, so $f(Q^N)(s) = a$, for some $a > 0$, for all $s \in S$. But the constant a must be the same for all problems $P' = (N, S', m')$.

To see this, note that in the axiom **ISC** the functions α_i of agents $i \in N$ were defined to be the same for all problems. In particular, α_i did not depend on the profile of measures m or the state space S. If we have some other problem $P' = (N, S', m')$, then by **UP** and **NE**, we get again that equation (1) holds, when m_i is replaced by m'_i and $Q^N = (N, \{s\}, \mathbf{1})$ is replaced by $Q'^N = (N, \{s'\}, \mathbf{1})$. But **NE** implies that these can be viewed as the same problem and hence they must have the same solution, so $f(Q^N) = f(Q'^N)$. So the values $f(P)$ and $f(P')$ are different only if m and m' are different □

Remark 1. If $f(P) = \mu^0$ for all P, then $f = 0 \cdot f^G$, so the theorem gives a characterization of a one-parameter family $\mathcal{F} = \{af^G \mid a \geq 0\}$ of rules.

Remark 2. Theorem 1 does not say whether or not the parameter a of the family $\mathcal{F}$ depends on N.

If we add *Unanimity* to the list of axioms of Theorem 1, the only possible solution is the Nash rule f^G and the agent set N need not be the same in every problem. Moreover, we can replace the full support assumption by the common support assumption.

Lemma 4. *If a rule f satisfies **UP** and **UN**, then f satisfies **REG**.*

Proof. Let $P = (N, S, m)$ be such that $m_i(s) = 0$ for all $i \in N$, for some $s \in S$. Then by **UP**, $f(P')(s) = f(P)(s)$, where $P' = (N; \{s\}, m_{|\{s\}})$. By **UN**, $f(P')(s) = 0$. □

Lemma 4 implies that in common support problems $P = (N, S, m)$, $f(P)(s) = 0$ for all $s \notin supp(m)$.

Theorem 2. *Let f be a rule satisfying **AN**, **ISC**, **CSC**, **NE**, **UN** and **UP** on the class of common support problems $\mathcal{P}^{cs}$, then $f = f^G$.*

Proof. If $m_1 = \cdots = m_n = \mu$ in equation (1), then by **UN** we get that $f(Q^N) = 1$. Since this holds independently of N, we are done. □

Remark 3. The Nash rule f^G satisfies all the axioms mentioned in Theorem 2 in the class $\mathcal{P}$ of all problems. At the moment I don't know if there are other rules satisfying these axioms as well.

Here is our main result concerning probability aggregation problems. Let $f^{G\times}$ be the multiplicative normalization of the Nash rule f^G.

Theorem 3. *Suppose f is a rule that satisfies **AN**, **ISC**, **CSC**, **NE**, **UN** and **UP** on the class of common support problems $\mathcal{P}^{cs}$, and that its multiplicative normalization $f^\times$ satisfies **BUP** on the class of probability aggregation problems in $\mathcal{P}^{cs}$. Then $f = f^G$ and $f^\times = f^{G\times}$.*

Proof. It follows from Theorem 2 that $f = f^G$, so we just have to show that its multiplicative normalization $f^{G\times}$ satisfies **BUP**.

Take any probability aggregation problem $P = (N, S, p)$ with a common support. By Lemma 4 and **UP**, we may assume $S = supp(p)$. For any $s \in S$ we have $p_i(s) > 0$, and so $f^G(P)(s) = \left[\prod_i p_i(s)\right]^{1/n} > 0$ and $f^G(P)(A) = \sum_{s\in A} \left[\prod_i p_i(s)\right]^{1/n} > 0$ for every $A \subset S$. By the definition of the multiplicative normalization we have for any $s \in S$

$$f^{G\times}(P)(s) = \frac{f^G(P)(s)}{f^G(P)(S)}.$$

Now update $f^{G\times}(P)$ on the nonempty event $E \subset S$ by using the Bayes rule:

$$f^{G\times}(P)(s \mid E) = \frac{f^G(P)(s)/f^G(P)(S)}{f^G(P)(E)/f^G(P)(S)}) = \frac{f^G(P)(s)}{f^G(P)(E)}. \tag{2}$$

Let $Q = (N, E, q)$ be related to P as in the axiom **BUP**. So q is derived from p by applying the Bayes rule: $q_i(s) = p_i(s)/p_i(E)$, for all $i \in N$, for all $s \in E$. Therefore $f^G(P)$ is computed by

$$f^G(Q)(s) = \frac{\left[\prod_i p_i(s)\right]^{1/n}}{\left[\prod_i p_i(E)\right]^{1/n}},\ \forall s \in E.$$

The corresponding multiplicative normalization is computed by

$$f^{G\times}(Q)(s) = \frac{\left[\prod_i p_i(s)\right]^{1/n} \Big/ \left[\prod_i p_i(E)\right]^{1/n}}{\sum_{s\in E}\left[\prod_i p_i(s)\right]^{1/n} \Big/ \left[\prod_i p_i(E)\right]^{1/n}},\ \forall s \in E. \tag{3}$$

But the right hand sides of equations 2 and 3 are the same. Therefore $f^{G\times}$ satisfies **BUP**. □

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