Правительство Российской Федерации

Федеральное государственное автономное образовательное учреждение высшего профессионального образования

Национальный исследовательский университет Высшая школа экономики

Факультет математики

ВЫПУСКНАЯ КВАЛИФИКАЦИОННАЯ РАБОТА

На тему: Скорость сходимости в эргодической теореме для фуксовых групп

Студент группы № 4.11.1
Толмачев К. Л.

Научный руководитель
д.ф.-м.н., Буфетов А. И.

Аннотация

In this paper, we estimate the rate of convergence in the ergodic theorem for Fuchsian groups discovered by Ledrappier. Using the asymptotics of the integrals along horocycle flow in terms of finitely-additive measures on the space of rectifiable arcs obtained by Bufetov and Forni, we modify the argument of Ledrappier to obtain the rate of convergence. This rate appears to depend heavily on the spectral properties of the underlying surface. In this draft we restrict ourselves to the case of smoothened sums.

1 Introduction

In this paper, we study ergodic properties of discrete subgroups of $SL(2,\mathbb{R})$. In [1] Ledrappier formulates and proves the ergodic theorem for specific sums over the balls in discrete subgroup Γ. The main step in his proof is the translation from such sum to the integral along the horocycle on the surface $\Gamma\backslash S\mathbb{H}$. From this point one may use the well-studied ergodic theory of horocycle flows. Ledrappier, for example, applies the ergodic theorem of Dani (1982). In this paper we are concerned with the rate of convergence to the mean in the ergodic theorem, so we need results of [2]. In [2] Bufetov and Forni, in particular, discover the exact asymptotics for the ergodic integral of horocycle flows in terms of finitely-additive measures β_μ, depending on the Casimir parameter (see the next Section). We substitute their result to the argument of Ledrappier and get the error term depending on the spectral properties of Laplace operator on the surface $\Gamma\backslash S\mathbb{H}$.

2 Ergodic theorems

Let $f : \mathbb{R}^2\backslash\{0\} \to \mathbb{R}^+$ be an even function with compact support, let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ of finite covolume and without torsion. Following [1] define a norm on Γ:

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2+b^2+c^2+d^2}.$$

Let $|\cdot|$ be the standard euclidean norm on $\mathbb{R}^2\backslash\{0\}$. The following theorem is proved in [1]:

Theorem (Ledrappier)**.** *Assume that Γ-orbit of $X \in \mathbb{R}^2\backslash\{0\}$ is non-discrete. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\gamma\in\Gamma, \|\gamma\|\le T} f(\gamma X) = \frac{1}{c(\Gamma)|X|} \int \frac{f(Y)}{|Y|} dY,$$

where $c(\Gamma) = 1/4(vol(\Gamma\backslash SL(2,\mathbb{R}))$

Corollary 1 (smoothened version)**.** *Let φ be a smooth function with compact support in $[0,1]$ and unit mean with respect to Lebesgue measure, and let X be as in the above Theorem. Then*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\gamma\in\Gamma, \|\gamma\|\le T} \varphi\left(\frac{\|\gamma\|}{T}\right) f(\gamma X) = \frac{1}{c(\Gamma)|X|} \int \frac{f(Y)}{|Y|} dY$$

We want to estimate the rate of convergence in smoothened version of this ergodic theorem.

In [2] finitely-additive measures $D^{\pm}_{\mu}$ on $SL(2,\mathbb{R})$ were defined. They may be represented as $D^{\pm}_{\mu} = M^{K,\pm}_{\mu} \otimes M^{A,\pm}_{\mu} \otimes M^{N,\pm}_{\mu}$, where K, A and N are subgroups of $SL(2,\mathbb{R})$ as in standard KAN-decomposition. We are interested in measures

$$\tilde{D}^{\pm}_{\mu} = M^{K,\pm}_{\mu} \otimes M^{A,\pm}_{\mu}$$

on the space of horocycles $KA \cong \mathbb{R}^2 \backslash \{0\}$.

In [2] Bufetov and Forni also define functions $\beta^{\pm}_{\mu} : SM \times \mathbb{R} \to \mathbb{R}$, where SM is the spherization of the tangent bundle of the surface $M = \Gamma \backslash SL(2,\mathbb{R})$, and prove the following theorem:

Theorem 1 (Bufetov, Forni).

$$\left| \left(\int_0^T f \circ h^U_s(x) ds - T \int_M f(x) dx \right) - \beta_f(x,T) \right| \leq C_s \|f\|_s (1 + \log |T|),$$

where $\|\cdot\|_s$ stands for the Sobolev s-norm on the space of functions on M, f is a function on M.

$$\beta_f = \sum_{\mu \in Spec(\Box) \cap \mathbb{R}^+} D^+_{\mu}(f) \beta^+_{\mu} + D^-_{\mu}(f) \beta^-_{\mu},$$

and mean of f is understood with respect to the measure induced from Haar measure on $SL(2,\mathbb{R})$.

Corollary 2 (smoothened version).

$$\left| \left(\int_0^T \varphi(t/T) f \circ h^U_t(x) dt - T \int_M f(x) dx \right) - B^{\varphi}_f(x,T) \right| \leq C_s \|f\|_s (1 + \log |T|) T^{2/3},$$

where $B^{\varphi}_f = \sum_{\mu \in Spec(\Box) \cap \mathbb{R}^+} D^+_{\mu}(f) B^{\varphi,+}_{\mu} + D^-_{\mu}(f) B^{\varphi,-}_{\mu}$, and $B^{\varphi,\pm}_{\mu} = \int_0^T \varphi(t/T) d\beta^{\pm}_{\mu}$ – here we interpret $\beta^{\pm}_{\mu}$ as a finitely additive measure on the horocycle.

We will need the following properties of $\beta^{\pm}_{\mu}$. Define $\nu = \sqrt{1-4\mu}$.

1. Hölder property:
$$|\beta^{\pm}_{\mu}(x,T)| \leq C_{\mu} |T|^{\frac{1 \mp \Re \nu}{2}} \tag{1}$$
for $\mu \neq 1/4$,
$$|\beta^+_{\frac{1}{4}}(x,T)| \leq C|T|^{\frac{1}{2}+}, |\beta^-_{\frac{1}{4}}(x,T)| \leq C|T|^{\frac{1}{2}}.$$

2. Renormalization under geodesic flow: for $\mu \neq 1/4$
$$\beta^{\pm}_{\mu}(g_{-t}x, Te^t) = \exp(\frac{1 \mp \nu}{2} t) \beta^{\pm}_{\mu}(x,T),$$
for $\mu = 1/4$ ($\nu = 0$)
$$\begin{pmatrix} \beta^+_{1/4}(g_{-t}x, Te^t) \\ \beta^-_{1/4}(g_{-t}x, Te^t) \end{pmatrix} = \exp(\frac{t}{2}) \begin{pmatrix} 1 & -\frac{t}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^+_{1/4}(x,T) \\ \beta^-_{1/4}(x,T) \end{pmatrix}.$$

Our main result is as follows:

Theorem 2. *Put* $\gamma_0 = \sqrt{11/3} - 1$. *Let* $f : \mathbb{R}^2 \backslash 0 \to \mathbb{R}$ *be the function with compact support, let* $\mu(f)$ *be the minimal Casimir parameter from* $(0, \frac{1}{4})$ *for* $\Gamma \backslash SL(2,\mathbb{R})$. *Then for* $\gamma(\mu(f)) > \gamma_0$

$$\sum_{\gamma \in \Gamma, \|\gamma\| \leq T} \phi(\|\gamma\|/T) f(\gamma X) = \frac{T}{c(\Gamma)|X|} \int \frac{f(Y)}{|Y|} dY +$$
$$\sum_{\mu \in Spec(\Box) \cap \mathbb{R}^+, \delta \in \{+,-\}} \tilde{D}^{\delta}_{\mu}(Y) \left(f(Y) B^{\varphi,\delta}_{\mu} \left(x, \frac{T}{|X||Y|} \right) \right) + o(T^{\gamma}), \quad (2)$$

where $x \in \Gamma \backslash SL(2,\mathbb{R})$ *depends only on* X, *and for* $\gamma(\mu(f)) \leq \gamma_0$

$$\sum_{\gamma \in \Gamma, \|\gamma\| \leq T} \phi(\|\gamma\|/T) f(\gamma X) = \frac{T}{c(\Gamma)|X|} \int \frac{f(Y)}{|Y|} dY + O(T^{\gamma(\mu(f))})$$

in the above notation.

3 Proof of the main theorem

Let K be a compact in $\mathbb{R}^2 \backslash \{0\}$, f be a positive function with support in K. We will need the following data depending on an arbitrary small $r > 0$:

1. Partition $K = \sqcup_{i=1}^{N(r)} K_i^{(r)}$, $diam K_i \leq 2r, N(r) \leq C/r^2$, where dY stands for the Lebesgue measure on the plane, C is fixed.
2. Partition of unity $1 = \sum_{i=1}^{N(r)} \alpha_i$, $\alpha_i \geq 0$

$$\int_{K_i} |\alpha_i f - f| dY \leq Crr', \int_{K \backslash K_i} |\alpha_i f| dY \leq Crr', \|\alpha_i\|_1 \leq C \frac{r}{r'}, \quad (3)$$

   where $r' < r$ we are going to adjust later, and $\|\cdot\|_1$ is the Sobolev L^1-norm.

Denote $f_i = \alpha_i \tilde{f}$. Also note that $\|f_i\|_1$ has the same estimate as $|\alpha_i\|_1$ (see the Section with the proof of Ledrapie's theorem: ε there depends only on Γ, and hence the derivative of φ_ε is bounded). From [1] (see the last Section) we know that for the positive f

$$\sum_i \int_{-\frac{T-D}{|X|c(f)}}^{\frac{T-D}{|X|c(f)}} f \circ h^U_s(x) ds \leq \sum_i \sum_{\gamma \in \Gamma, \|\gamma\| \leq T} f(\gamma X) \leq \sum_i \int_{-\frac{T+D}{|X|c(f)}}^{\frac{T+D}{|X|C(f)}} f_i \circ h^U_s(x) ds,$$

where D is the constant depending solely on K, $c(f) = \inf_{Y \in \mathrm{supp}(f)} |Y|$, $C(f) = \sup_{Y \in \mathrm{supp}(f)} |Y|$, and x is the projection on M of the point on the intersection of the horocycle corresponding to X and the geodesic passing through i. Using the fact that D depends only on K we may rewrite these inequalities for not necessary positive f:

$$|\sum_i \sum_{\gamma \in \Gamma, \|\gamma\| \leq T} f(\gamma X) - \sum_i \int_{-\frac{T}{|X|c(f)}}^{\frac{T}{|X|C(f)}} f_i \circ h^U_s(x) ds| = O(\frac{1}{r^2})$$

(note that the change of limits of integration changes the summands by $O(1)$.

Using Lemma 3 from [1] (we recall the proof of this Lemma in the last Section, see Proposition 6.3) we get

$$|\sum_i \sum_{\gamma\in\Gamma, \|\gamma\|\le T} \varphi\left(\frac{\|\gamma\|}{T}\right) f_i(\gamma X) - \sum_i \int_{-\frac{T}{|X||Y_i|}}^{\frac{T}{|X||Y_i|}} \varphi\left(\frac{t}{|X||Y_i|T}\right) f_i \circ h_t^U(x)dt| = O(\frac{1}{r^2}), \tag{4}$$

where $Y_i \in \mathrm{supp}(\alpha_i)$. From the corollary of Theorem 1

$$\sum_{\gamma\in\Gamma, \|\gamma\|\le T} \varphi\left(\frac{\|\gamma\|}{T}\right) f(\gamma X) = \frac{1}{c(\Gamma)|X|}\int \frac{f(Y)}{|Y|}dY + O(rT) + \sum_i B^{\varphi}_{\tilde f}\left(x, \frac{T}{|X|c(f_i)}\right) + O\left(\frac{T^{2/3}\log T}{rr'}\right),$$

where $\tilde f$ is a lift of f on SM as in [1]. Theorem follows from the Proposition below (for appropriate r and r'):

Proposition 3.1.

$$\sum_i \tilde D^{\delta}_{\mu}(f_i) B^{\varphi,\delta}_{\mu}\left(x, \frac{T}{|X||Y|}\right) = \tilde D^{\delta}_{\mu}\left(f(Y)\tilde\beta^{\delta}_{\mu}\left(x, \frac{T}{|X||Y|}\right)\right) + c(\mu)O(T^{\gamma(\mu)}\frac{r^2}{r'}),$$

where $c(\mu)$ is such that the series $\sum_\mu c(\mu)$ is convergent and $\gamma(\mu)$ is the exponent in (1).

We will need the following

Proposition 3.2. *Let f be the positive function with compact support. Then*

$$\tilde D^{\pm}_{\mu}(f) \le C\mu^{-k}\|f\|_1, k > 1.$$

We now deduce Proposition 3.1 from Proposition 3.2. For the sake of convenience we omit indices μ, δ, φ, and denote $B(Y) = B(x, \frac{T}{|X||Y|})$. We want to estimate the Sobolev norm of the difference of left and right hand sides of Proposition 3.1. First estimate

$$\left|\sum \alpha_i f(Y)B(Y_i) - \sum \alpha_i f(Y)B(Y)\right| = \left|\sum \alpha_i f(Y)(B(Y_i) - B(Y))\right| \le C(Tr)^{\gamma(\mu)}.$$

Then compare the derivatives:

$$\left|\frac{\partial}{\partial x}\left(\sum \alpha_i f(Y)B(Y_i) - f(Y)B(Y)\right)\right| = \left|\frac{\partial}{\partial x}\left(f(Y)\left(\sum \alpha_i B(Y_i) - B(Y)\right)\right)\right| \le$$
$$\le CT^{\gamma(\mu)}r + C\left|\sum \frac{\partial \alpha_i}{\partial x}B(Y_i) - \frac{\partial B(Y)}{\partial x}\right| \le CT^{\gamma(\mu)}r +$$
$$+ C\left|\sum \frac{\partial \alpha_i}{\partial x}(B(Y_i) - B(Y))\right|. \tag{5}$$

Here we use the fact that the derivative of $B(Y)$ is bounded:

$$\left(\int_0^T \varphi(t/T)d\beta\right)' = -\left(\int_0^T \beta d\varphi(t/T)\right)' = \frac{1}{T}\left(\int_0^T \beta(t)\varphi'(\frac{t}{T})dt\right)' = o(1).$$

Now note that in $\mathrm{supp}(\alpha_i)$ $|\frac{\partial \alpha_i}{\partial x}| \leq Cr/r'$, $|B(Y_i) - B(Y)| \leq C(rT)^{\gamma(\mu)}$ and hence we get

$$\| \sum \alpha_i f B(Y_i) - \sum \alpha_i B(Y) \|_1 \leq C T^{\gamma(\mu)} \frac{r^{\gamma+1}}{r'}.$$

We have the following errors:

$$(Tr)^{\gamma}, T^{\gamma} \frac{r^{\gamma+1}}{r'}, Tr, \frac{T^{2/3}}{rr'}, \frac{1}{r^2}.$$

γ_0 is the value of γ for which maximum of these errors overpowers the main term in the asymptotics.

4 Translation to the smoothened sum

In this section we prove the estimates concerning the translation to the smoothened Birkhoff sum.

4.1 Proof of corollary 2

We want to estimate

$$|\int_0^T \phi(\frac{t}{T}) f \circ h_t^U(x) dt - T \int_M f(x) dx - \int_0^T \phi(\frac{t}{T}) d\beta_m^{\pm}|.$$

Divide the interval $[0, T]$ into intervals Δ_i of length Δ. We have

$$\begin{gathered}
\sum_i \Big(\sum_i \int_{\Delta_i} \varphi(\frac{t}{T}) f \circ h_t^U(x) dt - \delta \int_M f(x) dx \Big) = \sum_i \Big(\int_{\Delta_i} \varphi(\frac{t_i}{T}) f \circ h_t^U(x) dx - \\
- \Delta \int_M f(x) dx + O(\Delta \frac{\Delta}{T} \Big) = \sum_i \Big(\varphi(\frac{t_i}{T}) \beta(\Delta_i) + O(\Delta^2/T) + \\
+ O(\log \Delta \|f\|_1) \Big) = \sum_i \Big(\int_{\Delta_i} \varphi(\frac{t}{T}) d\beta + O(\frac{\Delta}{T} \Delta^{1+\gamma}) + \\
+ O(\frac{\Delta^2}{T}) + O(\log \Delta \|f\|_1) \Big) = \int_0^T \varphi(\frac{t}{T}) d\beta + O(\Delta^{\gamma+1}) + O(\Delta) + \frac{T}{\Delta} O(\log \Delta \|f\|_1).
\end{gathered} \tag{6}$$

As always, we omit indices of β, and γ is a Hölder exponent of β. To complete the proof of corollary 2 set $\Delta = T^{1/3}$.

Translation to the smoothened Birkhoff sum in (4) is completely analogous. Note that the $\|\gamma\|$ parameter int the sum corresponds to the parameter of integration by Proposition 6.3 (Lemma 3 in [1]).

5 Riemann-Stieltjes integral

We smoothened our ergodic sums because of the nature of distributions D_μ, which are defined on the space of smooth functions. However, one may interpret the main result in the terms of Riemann-Stieltjes integral. To do this, one should think about the sum of the form $\sum D(f_i)\beta(x, \frac{T}{|X||Y_i|})$ as of the sums approximating Riemann-Stieltjes integral $\int f dg$, f (corresponding to our β) being the Hölder function and g (dg corresponding to our D_μ) being the smooth function. We now formulate our main result in these terms:

Theorem 3. *Put $\gamma_0' = \sqrt{3}-1$. Let $f : \mathbb{R}^2 \backslash 0 \to \mathbb{R}$ be the function with compact support, let $\mu(f)$ be the minimal Casimir parameter from $(0, \frac{1}{4})$ for $\Gamma\backslash SL(2,\mathbb{R})$. Then for $\gamma(\mu(f)) > \gamma_0'$*

$$\sum_{\gamma\in\Gamma, \|\gamma\|\le T} f(\gamma X) = \frac{T}{c(\Gamma)|X|}\int \frac{f(Y)}{|Y|}dY + \sum_{\mu\in Spec(\Box)\cap\mathbb{R}^+, \delta\in\{+,-\}} \int \left(f(Y)\beta_\mu^{\varphi,\delta}\left(x, \frac{T}{|X||Y|}\right)\tilde{D}_\mu^\delta(Y)\right) + o(T^\gamma), \quad (7)$$

where $x \in \Gamma\backslash SL(2,\mathbb{R})$ depends only on X, and for $\gamma(\mu(f)) \le \gamma_0'$

$$\sum_{\gamma\in\Gamma, \|\gamma\|\le T} f(\gamma X) = \frac{T}{c(\Gamma)|X|}\int \frac{f(Y)}{|Y|}dY + O(T^{\gamma(\mu(f))}).$$

The proof follows the same lines as the smoothened version. Note that two approximating sums over the partitions of size $< r$ differ by $(rT)^\gamma r/r'$ (see [3]), and, again, we have the following errors:

$$T^\gamma \frac{r^{\gamma+1}}{r'}, Tr, \frac{1}{rr'}, \frac{1}{r^2}.$$

Again, γ_0' is the value of γ for which maximum of these errors overpowers the main term in the asymptotics.

6 Proof of the Ledrappier theorem

In this section we present the proof of Ledrappier theorem following the original one from [1]. We now recall its statement.

Let $f : \mathbb{R}^2\backslash\{0\} \to \mathbb{R}^+$ be an even function with compact support, let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ of finite covolume and without torsion.

Theorem (Ledrappier)**.** *Assume that Γ-orbit of $X \in \mathbb{R}^2\backslash\{0\}$ is non-discrete. Then*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{\gamma\in\Gamma, \|\gamma\|\le T} f(\gamma X) = \frac{1}{c(\Gamma)|X|}\int \frac{f(Y)}{|Y|}dY,$$

where $c(\Gamma) = 1/4(vol(\Gamma\backslash SL(2,\mathbb{R}))$

Without loss of generality we may assume that $-1 \in \Gamma$. Identify $\mathbb{R}^2\backslash\{0\}/\pm$ with the space of stable horocycles: point $X = (\alpha, \beta)$ corresponds to the horocycle $\Phi(X)$ on $\mathbb{H}$ given by the equation

$$(\beta x - \alpha)^2 + \beta^2 y^2 - y = 0.$$

Define $\Psi : \mathbb{R}^2\backslash\{0\}/\pm \to PSL(2,\mathbb{R})$ mapping X to the unique vector on $\Phi(X)$ lying on the geodesics passing through i. Define $s : \Gamma \times \mathbb{R}^2\backslash\{0\}/\pm \to \mathbb{R}$ given by

$$\Psi(\gamma X) = \gamma\Psi(X)h_{s(\gamma,X)}.$$

We now want to lift f to obtain the function $\tilde{f} : S\mathbb{H} \to \mathbb{R}$. Set

$$\phi_\varepsilon(t) = \max(\frac{1}{\varepsilon}(1 - \frac{|t|}{\varepsilon}), 0);$$

$$\tilde{f}(Z) = f(\Psi^{-1}(Zh_s))\phi_\varepsilon(s), Zh_s \in \Psi(\mathbb{R}^2\backslash\{0\}/\pm).$$

Proposition 6.1.

$$f(\gamma X) = \int\limits_{s(\gamma,X)-\varepsilon}^{s(\gamma,X)+\varepsilon} \tilde{f}(\gamma\Psi(X)h_s)ds = \int\limits_{-\infty}^{+\infty} \tilde{f}(\gamma\Psi(X)h_s)ds$$

for all $\gamma \in \Gamma, X \in \mathbb{R}^2\backslash\{0\}/\pm 1$.

Indeed, $f(\Psi^{-1}(Zh_s))$ is constant along the horocycle and ϕ has mean one and support in $[-\varepsilon, \varepsilon]$.

This makes the following Proposition evident:

Proposition 6.2.

$$\int \tilde{f}(Z)dZ = \int f(X)dX,$$

where dZ *stands for the Liouville measure,* dX *stands for the Lebesgue measure and integration is over whole space.*

Now set $SM = \Gamma\backslash S\mathbb{H}$, and starting from f define a function $\overline{f} : SM \to \mathbb{R}$. Let $\pi : S\mathbb{H} \to S\mathbb{M}$ be the standard projection. Define

$$\overline{f}(Z) = \sum_{Y\in\pi^{-1}(Z)} \tilde{f}(Y).$$

Note that this sum is finite because Γ is discrete.

Chose ε small enough so that for all $\gamma_1, \gamma_2 \in \Gamma$ supports of $\tilde{f} \circ \gamma_1$ and $\tilde{f} \circ \gamma_2$ do not intersect. It is possible because Γ does not contain elliptic elements. This also implies that for all $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 X \in \mathrm{supp} f$ and $\gamma_2 X \in \mathrm{supp} f$ $[s(\gamma_1, X) - \varepsilon, s(\gamma_1, X) + \varepsilon] \cap [s(\gamma_2, X) - \varepsilon, s(\gamma_2, X) + \varepsilon] = \varnothing$. Indeed, we need to show that if γX is in $\mathrm{supp} f$, then $|s(\gamma, X)| > 2\varepsilon$. But $\gamma X \in \mathrm{supp} f$ implies that $X \in \mathrm{supp} f \circ \gamma$. Now note that intersection of the horocycle corresponding to X with $\mathrm{supp}\tilde{f} \circ \gamma$ is $\{\Psi(X)h_s, s \in [-\varepsilon, \varepsilon]\}$ and our claim follows from the fact that $\tilde{f} \circ \gamma$ have disjoint support.

Write

$$\sum_{\|\gamma\|\le T,\gamma\in\Gamma} f(\gamma X) = \sum_{\|\gamma\|\le T,\gamma\in\Gamma} \int\limits_{s(\gamma,X)-\varepsilon}^{s(\gamma,X)+\varepsilon} \tilde{f}(\gamma\Psi(X)h_s)ds =$$

$$= \sum_{\|\gamma\|\le T,\gamma\in\Gamma} \int\limits_{s(\gamma,X)-\varepsilon}^{s(\gamma,X)+\varepsilon} \overline{f}(\pi(\Psi(X))h_s)ds. \quad (8)$$

We will need

Proposition 6.3.

$$|X|^2|\gamma X|^2 s^2(\gamma, X) = \|\gamma\|^2 - \frac{|X|^2}{|\gamma X|^2} - \frac{|\gamma X|^2}{|X|^2}.$$

This implies

Corollary 3. *There is a constant $K(X, suppf)$ such that for $\gamma X \in suppf$ we have*

$$|\|\gamma\|^2 - |X|^2|\gamma X|^2 s^2(\gamma, X)| \le K(X, suppf).$$

We prove this proposition with (almost) direct computation.

$$s\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (1,0)^t\right).$$

This gives us two equations: on the imaginary and real parts,

$$\gamma h_s g_t i = i.$$

This may be understood as follows: we identified $(1,0)^t$ with a horizontal horocycle passing through i and a vertical vector Z_0 at i is identified with the unit in $PSL(2,\mathbb{R})$, so that $\Psi((0,1)^t) = Z_0$. We shift it by γ (recall that Γ acts on $PSL(2,\mathbb{R})$ with left shifts). We want to find s such that geodesics, passing through $\gamma Z_0 h_s$ passes throgh i, and that's expressed in the equation.

Expanding, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} i = \begin{pmatrix} a & as+c \\ c & cs+d \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} i = \begin{pmatrix} e^t a & e^{-t}(as+c) \\ e^t c & e^{-t}(cs+d) \end{pmatrix} i =$$
$$= \frac{e^t ai + e^{-t}(as+c)}{e^t ci + e^{-t}(cs+d)} = i; e^t a = e^{-t}(cs+d), e^{-t}(as+c) = -e^t c. \quad (9)$$

Solving these equations we get

$$s = -\frac{ab+cd}{a^2+c^2}.$$

Now substitute s to the statement of the Proposition and obtain the desired formula.

Note that for every $X \in \mathbb{R}^2 \backslash \{0\}$ there exist $k \in K$, $a \in A$, such that $X = ka(1,0)^t$. It is easy to see that for all $\gamma_1, \gamma_2 \in \Gamma, s(\gamma_1\gamma_2, X) = s(\gamma_2, X) + s(\gamma_1, \gamma_2 X)$. Nota also that if $k \in K$, then $s(k, X) = 0, |kX| = |X|$ (point i is fixed). So $s(\gamma k, X) = s(\gamma, kX)$. Moreover, $\|\gamma k\| = \|k\gamma\| = \|\gamma\|$ (k preserves the norm of row vector while acting from the right) which implies that if statement of the Proposition holds for γ then it holds for γk. Hence it is enough to prove the Proposition for $X = (e^\lambda, 0)^t = g_\lambda(1,0)^t$. But

$$s(\gamma, g_\lambda(1,0)^t) = s(\gamma g_\lambda, (1,0)^t) - s(g_\lambda, (1,0)^t) = s(\gamma g_\lambda, (1,0)^t) = e^{2\lambda} s(\gamma, (1,0)^t)$$

.

Proposition implies the following estimates for $\gamma \in \Gamma$ such that $\gamma X \in \text{supp} f$.

$$\exists D > 0, |s(\gamma, X)| \le \frac{T-D}{|X||\gamma X|} \Rightarrow \|\gamma\| \le T,$$

$$\|\gamma\| \le T \Rightarrow |s(\gamma, X)| \le \frac{T+D}{|X||\gamma X|}.$$

We use the fact that D depends only on support of f and that for all functions with support within the support of f we may chose the same D. Set $k(f) = \inf_{Y \in \mathrm{supp} f} |Y|, K(f) = \sup_{Y \in \mathrm{supp} f} |Y|$. Applying the above inequalities we get

$$2 \int\limits_{-\frac{T-D}{|X|K(f)}}^{\frac{T-D}{|X|K(f)}} \overline{f} \circ h_s(x) ds \le \sum_{\|\gamma\| \le T, \gamma \in \Gamma} f(\gamma X) \le 2 \int\limits_{-\frac{T+D}{|X|k(f)}}^{\frac{T+D}{|X|k(f)}} \overline{f} \circ h_s(x) ds.$$

Multiplier 2 pops up because $\gamma, -\gamma \in \Gamma$ act on $\mathbb{H}$ in the same way. Applying the ergodic theorem for horocycle flow we get

$$4\frac{1}{|X|} \int \frac{\overline{f}(Z)}{K(f)} dZ \le \lim_{T \to \infty} \frac{1}{T} \sum_{\|\gamma\| \le T, \gamma \in \Gamma} f(\gamma X) \le 4\frac{1}{|X|} \int \frac{\overline{f}(Z)}{k(f)} dZ.$$

Partition f into the sum of f_i with small supports. We obtain

$$4\frac{1}{|X|} \sum_i \int \frac{\overline{f_i}(Z)}{K(f_i)} dZ \le \lim_{T \to \infty} \frac{1}{T} \sum_{\|\gamma\| \le T, \gamma \in \Gamma} f_i(\gamma X) \le 4\frac{1}{|X|} \sum_i \frac{1}{|X|} \int \frac{\overline{f_i}(Z)}{k(f_i)} dZ.$$

Desired sum is bounded by riemannian sums for the function $\dfrac{f(Z)}{|Z|}$. Passing to the limit and noticing that

$$\int \overline{f}(Z) dZ = \frac{1}{c(\Gamma)} \int f(X) dX.$$

we get the statement of the Ledrappier theorem.

Список литературы

[1] F. Ledrappier, Distribution des orbites des réseaux sur le plan réel, C. R. Acad. Sci 329 (1999), 61–64.

[2] A. Bufetov, G. Forni, Limit theorems for horocycle flows, arxiv.org/pdf/1104.4502.

[3] L. C. Young, An Inequality of the Hölder Type, Connected with Stieltjes Integration.