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THE SHARE OF MANIPULABLE OUTCOMES IN SOCIAL CHOICE PROBLEM

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We introduce and study two specific types of manipulation in social choice problem. These are the standard manipulation with the restriction that the coalitions can be formed only by the candidates with the same first alternative in their preferences. The second type also demands that after the manipulation the top alternative in the preferences of the coalitions' participants must win the election. The probabilities that such manipulation will occur in a 3-candidate election of Borda type are computed. An algorithm for producing necessary and sufficient conditions for a profile to be manipulable under weighted scoring voting rules is presented.

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В работе определяются два специальных типа манипулирования в теории коллективного выбора. Вычислены доли профилей, допускающих манипулирование первого или второго типа, при подсчете голосов по правилу Борда и условии, что число избирателей стремится к бесконечности. Описан алгоритм получения систем и совокупностей линейных уравнений и неравенств, позволяющих определить манипулируемость профиля для позиционных правил с весами (число избирателей стремится к бесконечности).

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1 Introduction

The problem of manipulation in social choice theory has attracted attention during last decades. A number of studies has been done on the evaluation of the degree of manipulability of social rules [1,3,9-11] and a geometric approach to the analysis of manipulability of voting rules was suggested (the description can be found in [2]).

In [9] the share of manipulable outcomes (the result is called *manipulable* if there exists a subset of the set of voters such that the preferences of all the voters outside the subset remain the same, while the preferences of voters within the subset can be altered in such a way that the winner changes, and each of the voters from the subset is 'happy about the change') for plurality rule, anti-plurality rule, plurality with runoff, anti-plurality with runoff was computed, and more recently in [7] and [10] the corresponding result for Borda rule was obtained[1].

Below we introduce and study two specific types of manipulation, the second of which includes not only the fact of manipulability but the manipulability which leads to the winning of the desired alternative as well[2].

To be precise, we call the result *significantly* manipulable if there exists a group of voters (with the same prior candidate in their preferences) whose candidate did not emerge as the winner, nevertheless, they could have chosen different preferences such that if the other voters' preferences remained the same, the candidate would win. In case (under the same assumption on coalitions) the weaker property that the new winner is more preferable by all the members of the coalition holds, we call the profile *manipulable with respect to restricted coalitions.* It is shown that approximately 30.6% of results in the elections with three alternatives and Borda rule turn out to be significantly manipulable and 38% - manipulable with respect to restricted coalitions.

In Section 2 we recall the geometric representation of voting outcomes and the technique for counting probabilities of manipulable results. In Section 3 this technique is used for evaluating the probabilities of a profile to be manipulable with respect to restricted coalitions or significantly manipulable. An algorithm for obtaining the conditions for a profile to be manipulable is presented in Section 4. The method can be used for any weighted scoring voting rule.

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[1]We performed an independent calculation (described in Section 4) and came up with the same result of approximately 50.25 %.

[2]This idea was suggested by F.Aleskerov.

2 Geometry of Voting

The voters' preferences are assumed to be linear orders. The number of voters is denoted by n and the number of alternatives - by m. A *profile* p is an $m!$-tuple of non-negative integer numbers $(n_1, n_2, \ldots, n_{m!})$ (each n_i is equal to the number of voters with preferences of type i) such that $\sum n_i = n$. Another natural assumption throughout the paper is

Impartial Anonymous Culture (IAC): each possible preference profile is equally likely.

Next we assume that in all elections there are three alternatives (candidates) and hence $6 = 3!$ possible preferences. The outcome of an election is said to be *significantly manipulable* if there exists a candidate i such that all members of the electorate for whom she is the best alternative can change their preferences in such a way that this candidate wins the election (the preferences of the rest of the electorate remain the same).

The voting procedure used hereafter is the Borda count.

Example 1. Let the candidates be A,B and C. The number of people with preference (A, B, C) is n_1, (A, C, B) - n_2, (B, A, C) - n_3, (B, C, A) - n_4, (C, A, B) - n_5, (C, B, A) - n_6. According to this procedure a candidate ranked i gets $(3 - i)$ points (if the preference is (A, B, C), A is awarded 2 points, B - 1 point and C does not receive any points at all). To show a manipulable outcome, we set $n_1 = 5, n_3 = 4, n_i = 0, i \neq 1, 3$. By the definition of Borda count candidate A gets 14, B - 13 and C - 0 points. But if three voters chose (B, C, A) instead of (B, A, C), and the others kept their preferences unchanged, then the final result would be A - 11, B - 13, C - 3 in favor of candidate B.

As the following example shows, the condition of being manipulable with respect to restricted coalitions does not necessarily imply significant manipulability.

Example 2. Set $n_1 = n_2 = n_4 = 8, n_6 = 5, n_3 = n_5 = 0$. Then the final ranking is A - 32, B - 29, C - 26. If the group of 8 people with preferences (B, C, A) formed a coalition and changed to (C, B, A), C would be two points ahead of A and thirteen points ahead of B, making the result manipulable

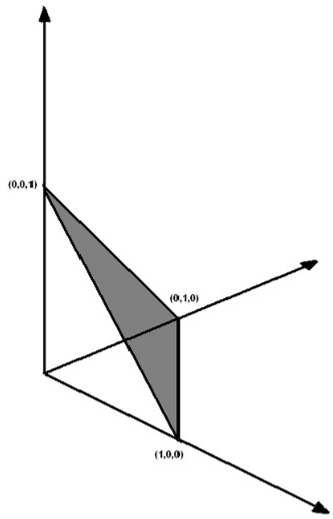

Figure 1: The region of all possible profiles

with respect to restricted coalitions (all the members of the coalition prefer C to A). Nevertheless, our definition, obviously, does not recognize it as significantly manipulable.

Now the natural question that arises is what is the probability that the result of an election will be significantly manipulable. The approach described in [2] tells us an effective way to tackle this problem for the three candidates case. When we have more than three alternatives, however, this algorithm becomes too complicated in practice.

Let us have a look at the geometry behind the elections. Consider a 6-dimensional vector space with the basis enumerated by all possible preferences. Then as all the profiles are linear combinations of preferences with nonnegative integer coefficients that sum up to n, they lie within the simplex $\sum n_i = n,\ n_i \geq 0$. The picture is similar to figure 1 (the only difference is dimension).

But we are interested in certain types of situations, in fact, in those points in the simplex, which correspond to manipulable profiles. How can these points be separated from the others? Every condition on the profile ('A gets more points than B', 'B gets more points than C', etc.) can be written as a linear inequality. Thus, manipulable profiles correspond to the points in the simplex, which are common solutions of certain linear inequalities, in other words, those profiles above or below (according to the sign of the inequality) the hyperplane determined by it (for each of the inequalities). Thus, the simplex is cut into two (convex) polytopes, when the first inequality is taken into consideration. The next inequality determines another hyperplane which slices the polytope obtained on the first step into two, etc. For example, when candidate A has a score greater or equal than candidate B, the inequality responsible for that is

$$2(n_1 + n_2) + n_3 + n_5 \geq 2(n_3 + n_4) + n_1 + n_6.$$

Before presenting the system of all inequalities that the result satisfies if (and only if) it is significantly manipulable, we make the following observation.

Observation. The group of voters, having the last candidate prior in their preferences can not significantly manipulate an election whichever the number of candidates is. Otherwise, as after such a manipulation the last candidate would have the same number of points (his number of points is not influenced by the manipulation) all the other candidates would have to undergo a simultaneous loss of points. That is clearly impossible, because the number of rearranged points equals the number of points initially arranged between the alternatives.

Therefore, in the case of 3 candidates only the group of the second candidate's supporters has to be considered as potential manipulators (according to our definition). We should also distinguish between the case of a finite number n of voters and 'the limit case' $n \to \infty$. The principal difference is that in a finite type election the number of manipulable results is found by counting lattice points inside the polytopes. As for the limit case - normalize the n_i by setting $p_i = n_i/n$ (this obviously does not affect the final ranks), and as n is growing, more points fall inside the polytope (the lattice is somehow 'diminished'), until finally (in the limit) every point of the polytope becomes a result of some voting (with infinite number of voters). Then the answer is given in terms of volumes of certain polyhedra.

Counting the number of lattice points inside a polytope $N(P)$ is much more complicated than evaluating its volume. If all the vertices of the polytope are points of the lattice (such a polytope is called *integral*), the number $N(P)$ can be expressed via a polynomial $E(t, P)$, where (t, P) corresponds to the polytope formed by multiplying each vertex coordinates by t. This was proved by Ehrhart ($E(t, P)$ is called Ehrhart's polynomial). If the polytope is not integral, $E(t, P)$ becomes a quasi-polynomial (the coefficients are periodic functions with integral period)[5,6].

3 Main Results and Computation

Now we will compute the share of significantly manipulable profiles in three-candidate elections with Borda count. Without loss of generality, assume that candidate A won the election, B was the second and C - the third. The result will be multiplied by 6 afterwards, because in each of the 6 regions (corresponding to all possible final arrangements of candidates) the procedure below is carried out independently and defines polytopes of equal volume, to achieve the final answer. We have the following system of inequalities:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 & \\ p_1 + 2p_2 - p_3 - 2p_4 + p_5 - p_6 > 0, & \text{A beats B} \\ p_1 - p_2 + 2p_3 + p_4 - 2p_5 - p_6 > 0, & \text{B beats C} \\ -p_1 - 2p_2 + 2p_3 + 2p_4 - p_5 + p_6 > 0, & \\ -p_2 + p_3 + p_4 - p_5 > 0. & \end{cases}$$

This system determines a polytope and we need to compute its volume.

The linear equation (the first line of the system) represents the fact that the sum of the normalized preferences is equal to 1. The first and second inequalities in the system stand for the ranking of the candidates. The result of Borda count is obviously transitive meaning that if A has a higher rank than B, and B has a higher rank than C then A already beats C. Thus the additional inequality which guarantees that A wins over C is automatically obeyed. The next step is to understand when the supporters of candidate B can manipulate. Actually, all the freedom they possess is to rearrange $p_3 + p_4$ points between A and C. The last two inequalities correspond to the situation when $p_3 + p_4$ points have yet to be arranged between candidates B and C. The third inequality checks whether candidate B has more points than candidate A. The last inequality holds if (and only if) the sum of the differences in points between B and A, and B and C is greater than $p_3 + p_4$.

Thus the task is reduced to finding the 5-dimensional polytope and computing its volume. This can be done as follows. A convex polytope is uniquely determined by the set of its vertices (as a convex hull of them). To evaluate the volume, however, all of the faces of the polytope must be described. This faces are again convex polytopes of lower dimensions and thus are convex hulls of certain subsets of the set of the vertices of the polytope.

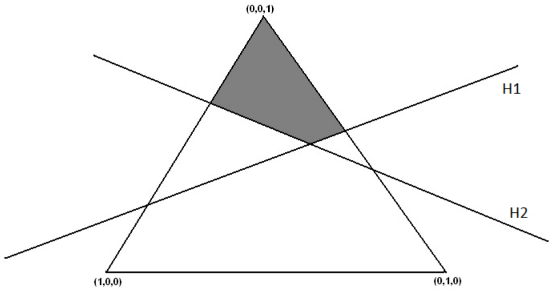

Figure 2: The region of profiles, satisfying certain conditions

Therefore the procedure of obtaining the required polytope is recursive. On each step the polytope obtained on the previous step is cut with the next hyperplane and its faces of all dimensions are determined (the picture below provides a low-dimension illustration). The new vertices are all of the old ones that are above or on the hyperplane (without loss of generality assume that the inequality is of the type 'something is greater than zero') plus the vertices that appear when an edge which is formed by one vertex above and one below the hyperplane is sliced by it[3]. The faces of other dimensions (vertices are faces of dimension 0) can be obtained similarly (either they are above the hyperplane, cut by the hyperplane themselves or a face one dimension greater is cut by the hyperplane). We complete our description of this procedure by giving a useful remark. More details can be found in [2].

Remark 1. One criteria that the number of faces of each dimension was counted correctly is $\sum(-1)^i f_i = 2$, where f_i denotes the number of faces of dimension[4] i.

Having these polytopes, the next step is to calculate their volumes. The basic idea is to cut the polytope into pyramids and evaluate their volumes using the fact that $V = \frac{1}{n} h V_{base}$, where V is the volume of an n - dimensional pyramid and V_{base} is the volume of its base. After that we need to find the

[3]To understand whether a vertex is above, below or on the hyperplane, use the scalar product of the radius-vector formed by the coordinates of the vertex with the normal vector to the hyperplane.

[4]This is due to the fact that a convex polytope is homeomorphic to a sphere of dimension 1 less than the dimension of the polytope and therefore has the same Euler-Poincare characteristics (χ) as this sphere. In our case the sphere has dimension 4 and $\chi(S^4) = 2$ (see [4]).

volume of the base (which is of dimension $n-1$). It is computed exactly the same way. The process terminates when the base becomes a polygon on the plane and its 2 - dimensional volume (area) is evaluated. Having found the areas of all of the required polygons, compute the volumes of 3-dimensional pyramids, etc. The volume of the polytope on each step is the sum of the volumes of the pyramids it is cut into. To make the procedure more effective, take on each step the vertex that is adjacent to the maximal number of facets (faces of the greatest dimension). This procedure is also outlined in [2].

Now we are ready to state our first result. Let V_1 denote the volume [5] of the polytope obtained by cutting the standard simplex in $\mathbb{R}^6$ with the hyperplanes from the system of inequalities and V_s - the volume of the simplex. Then $V_1 = \frac{11}{155520}$. The portion of significantly manipulable profiles is

$$M = \frac{6V_1}{V_s} = 6 \times 720 \times \frac{11}{155520} \approx 0.3056.$$

It is interesting to compare the value of M with the share of results, manipulable with respect to restricted coalitions $M^{'}$. To compute $M^{'}$ we need to compute the volumes of the polytopes[6] V_2 and V_3, consisting of the profiles manipulable by the coalitions of voters with preferences (C, B, A) and (B, C, A), respectively[7]. These polytopes are defined by the hyperplanes in the systems below

$$\begin{cases} p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 & \\ p_1 + 2p_2 - p_3 - 2p_4 + p_5 - p_6 > 0, & \text{A beats B} \\ p_1 - p_2 + 2p_3 + p_4 - 2p_5 - p_6 > 0, & \text{B beats C} \\ -p_1 - 2p_2 + p_3 + 2p_4 - p_5 + 2p_6 > 0 & \\ -p_2 + p_3 + p_4 - p_5 + p_6 > 0, & \text{B wins after the manipulation} \end{cases}$$

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[5] Instead of the volumes of polytopes mentioned we compute the volumes of pyramids with the apex located at $(0, 0, 0, 0, 0, 0)$ and these polytopes as bases. However, it does not affect any of the results since all the polytopes lie in the same hyperplane, therefore all the heights are equal, hence do not change the ratio of volumes.

[6] The volumes V_1, V_2 and V_3 were found with the use of [8].

[7] As the polytope V_1 contains the results manipulable by the group of voters with favorite alternative B, it remains to find the profiles, manipulable by the '(B, C, A) - coalition' (by making C the winner), if the restriction on coalitions is that B is the top candidate in all preferences inside them. Due to the observation on page 6, only the '(C, B, A) - coalition' can possibly manipulate a profile (by making B above A and C), when the coalitions are restricted to the preferences with candidate C as the first choice.

$$\begin{cases} p_1+p_2+p_3+p_4+p_5+p_6=1 \\ p_1+2p_2-p_3-2p_4+p_5-p_6>0, & \text{A beats B} \\ p_1-p_2+2p_3+p_4-2p_5-p_6>0, & \text{B beats C} \\ -p_1+p_2-2p_3+2p_4+2p_5+p_6>0 \\ -2p_1-p_2-p_3+2p_4+p_5+2p_6>0 \\ -p_1-p_3+p_4+p_5+p_6>0, & \text{C wins after the manipulation.} \end{cases}$$

The last two inequalities of the first system correspond to the situation when the voters with preferences (C, B, A) have changed their top candidate to B, but are undecided about the second and third places (p_4 points have to be arranged between alternatives A and C). The inequalities verify that B has more points than A, and p_4 points can be distributed between A and C, not violating that B has more points than each of them, respectively. The last three inequalities in the second system stand for analogous reasons. Thus all the manipulable outcomes with respect to restricted coalitions are seen to be inside the union of polytopes of the total volume:

$$V = V_1 + V_2 + V_3 - V_1 \bigcap V_2 - V_1 \bigcap V_3 - V_2 \bigcap V_3 + V_1 \bigcap V_2 \bigcap V_3.$$

$$6\frac{V}{V_s} = 6 \times 720 \times (\frac{11}{155520} + \frac{17}{373248} + \frac{29}{2612736} - \frac{29}{4572288} - \frac{23}{4898880} - \\ - \frac{83}{2592000} + \frac{1553}{431101440}) \approx 37.98\%$$

The points inside each of the intersections of the polytopes correspond to the profiles, which are not only manipulable with respect to restricted coalitions, but there exist manipulations which result in two or more rankings of candidates (different from the initial one). For example, any profile inside $V_2 \bigcap V_3$ can be manipulated by the '(C, B, A)- or (B, C, A)-coalition', so that the final ranking changes either to the one with C or B on the first place.

The result may seem totally unexpected: more than a third (38 per cent) of the outcomes are manipulable with respect to restricted coalitions and a little less than a third (30.6 per cent) of them can be manipulated in such a way that the most desirable candidate for the coalition wins!

4 General Method

This section describes an algorithm on how to write systematically the conditions a profile must satisfy if and only if it is manipulable. Throughout the section all voting procedures are weighted scoring rules (with nonnegative integer weights)[8], the number of alternatives is m, the number of agents is n and $n \to \infty$.

First, let us note that exactly $m-1$ different coalitions can be formed (for each candidate, who has not won the election, there is a group of voters, who rank him higher than the winner). As any coalition is determined by the candidate it would prefer to have as the winner of the election, the candidates' names will be used for the notations of the coalitions hereafter. Such a candidate will be called *unifying* for the coalition. We claim that for every coalition there exists a system of inequalities, which determines all the profiles (the polytope, inside which they lie), manipulable by it. Therefore, using the inclusion-exclusion principle, we can find the share of manipulable outcomes by computing the volumes of $2^m - 1$ polytopes. The idea is to look at the situation when the members of the coalition are undecided about all the places in their preferences accept the first one (which is given to the unifying alternative). Obviously, if we count the number of points for each alternative prior to the coalition participants' arrangement of all the places except the first, the unifying candidate must have the greatest number of points. Thus, there are $m-1$ inequalities $d(C, A_i) > 0, A_i \neq C$ (where $d(C, A_i)$ denotes the difference in points between the unifying candidate C and candidate A_i prior to the coalition participants' arrangement of all the places except the first). Another $m-1$ inequalities are responsible for the initial results of the election (the first candidate has more points than the second, the second - more points than the third, etc.). Only one inequality must be added to complete our system[9]. But this inequality is the most difficult to construct. We give a description of how it can be obtained and illustrate it via two examples below.

Without loss of generality, suppose that the coalition was unified by candidate C. As shown above, all the differences $d(C, A_i)$ must be positive. We arrange them in a non-increasing order. After that the question of whether it is possible for the coalition to choose the remaining part of their preferences (places from the second to the last) in such a way that C has still more points than any other alternative is equivalent to the following combinatorial problem.

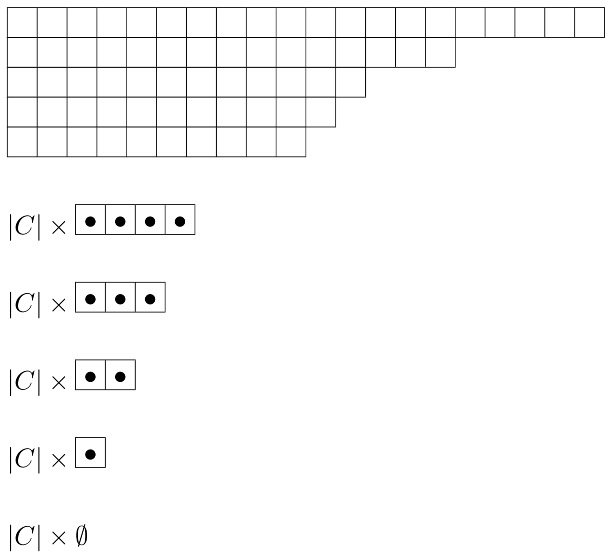
Problem. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ represent the differences in points between candidate C and other alternatives in a non-increasing order. Under

[8]This condition is not very restrictive, since if the weights are rational or negative, we can multiply them by the common denominator or add the number equal to the least of them, accordingly.

[9]This inequality includes maximum and minimum, each of which is a system of two inequalities itself.

what conditions on $(\alpha_1, \alpha_2, \dots, \alpha_{m-1})$ is it possible to put $|C|$ sets of $m-2$ black balls, $|C|$ sets of $m-3$ black balls, ..., $|C|$ sets of 1 black ball and $|C|$ empty sets on $m-1$ shelves with $(\alpha_1, \alpha_2, \dots, \alpha_{m-1})$ boxes[10] so, that each box admits only one ball, the sets of balls are unbreakable (if a set is chosen all the balls from this set must occupy subsequent boxes on the same shelf), no more than $|C|$ sets can be put on the same shelf?

One such problem for $m = 6$ candidates, the Borda rule and a coalition of $|C|$ people is illustrated on the picture below:



Now the optimal arrangement of sets of balls is presented. We put[11]

$$k_1 := min([\frac{\text{number of boxes on the shelf}}{m-2}], |C|)$$

sets of $m-2$ black balls on the top shelf. This is optimal, because if the minimum above is equal to $|C|$ than the greatest possible total number of

[10] The numbers of balls in the sets are defined by the weights of the voting rule (there are $|C|$ sets of w_i balls for all weights w_i, $i \neq 1$). In the text we work with the example of the Borda count (the weights are $(m-1, m-2, \dots, 1, 0)$).

[11] [a] denotes the greatest integer less than or equal to a. When we normalize the n_i's (dividing by n and substituting with p_i's) the number of sets we put on the top shelf becomes $min(\frac{\text{length of the shelf}}{m-2}, |C|)$.

balls will be put on the shelf, otherwise no more than $m-3$ boxes will remain empty, but this is a constant number, which is neglected (becomes 0), when $n \to \infty$. The boxes on the second shelf are filled with

$$k_2 := min([\frac{\text{number of boxes on the shelf}}{m-2}], |C| - k_1)$$

sets of $m-2$ black balls and

$$k_3 := min([\frac{\text{number of boxes on the shelf} - (m-2)k_2}{m-3}], |C| - k_2)$$

sets of $m-3$ black balls, which is justified by an argument, analogous to the one above, etc. Finally, if

$$\sum k_i = (m-2+m-3+\ldots+1+0)|C| = \frac{m(m-1)}{2}|C|$$

($\frac{m(m-1)}{2}$ is the total number of black balls for the voters from the coalition to be arranged between the candidates), the coalition will be able to manipulate the result and otherwise it will not.

Example 3. Consider the election with three alternatives A, B, C and the Borda rule. In this example we show the conditions (both necessary and sufficient) for an outcome to be manipulable. The number of voters is denoted by n, while each n_i corresponds to the number of voters with coinciding preferences of type i (as in the examples from the beginning of the second section). Again, without loss of generality, assume that candidate A won the election, B was the second and C - the third. The result will be multiplied by 6 afterwards, because in each of the 6 regions (corresponding to all possible final arrangements of candidates) the procedure below is carried out independently, to achieve the final answer. As discussed earlier (the second passage of section 3), the corresponding inequalities are

$$\begin{cases} n_1 + 2n_2 - n_3 - 2n_4 + n_5 - n_6 > 0, & \text{A beats B} \\ n_1 - n_2 + 2n_3 + n_4 - 2n_5 - n_6 > 0, & \text{B beats C.} \end{cases}$$

Obviously, only two coalitions can be formed: by those members of the electorate with either B or C higher than A in their preferences (they will be better of, if the winner changes). The first group consists of $n_3 + n_4 + n_6$ members, and the second - of $n_4 + n_5 + n_6$ members. The best strategy for all the participants of the first coalition is to put B on the top place in their preferences. According to the algorithm above, we need to assure that the difference in points between candidates B and A is positive, when the $n_3+n_4+n_6$ voters are undecided on which alternative is their second choice, and which is the third[12]. The corresponding inequality is

$$d(B,A) = 2(n_3+n_4+n_6) + n_1 - 2(n_1+n_2) - n_5 > 0.$$

Next, the remaining $n_3+n_4+n_6$ points have to be arranged between candidates A and C in such a way, that B has still more points than both of them. This is possible, whenever $d(B,A) + d(B,C) - (n_3+n_4+n_6) = 4(n_3+n_4+n_6) + 2n_1 - 2(n_1+n_2) - n_5 - 2n_5 - n_2 - (n_3+n_4+n_6) = 3(n_3+n_4+n_6-n_2-n_5) > 0 \Leftrightarrow n_3+n_4+n_6-n_2-n_5 > 0$. Therefore, the profiles, manipulable by the first coalition (with unifying candidate B) are exactly the solutions of the following system of inequalities

$$\begin{cases} n_1+n_2+n_3+n_4+n_5+n_6 = n \\ n_1+2n_2-n_3-2n_4+n_5-n_6 > 0 \\ n_1-n_2+2n_3+n_4-2n_5-n_6 > 0 \\ 2(n_3+n_4+n_6)+n_1-2(n_1+n_2)-n_5 > 0 \\ n_3+n_4+n_6-n_2-n_5 > 0. \end{cases}$$

The results, manipulable by the second coalition (with unifying candidate C) are determined by the system below, which is obtained similarly to the previous one (the fourth and fifth lines of the system are responsible for $d(C,A) > 0$ and $d(C,B) > 0$, the last inequality guaranties that $d(C,A) + d(C,B) > n_4+n_5+n_6$):

$$\begin{cases} n_1+n_2+n_3+n_4+n_5+n_6 = n \\ n_1+2n_2-n_3-2n_4+n_5-n_6 > 0 \\ n_1-n_2+2n_3+n_4-2n_5-n_6 > 0 \\ 2(n_4+n_5+n_6)+n_2-2(n_1+n_2)-n_3 > 0 \\ 2(n_4+n_5+n_6)+n_2-2n_3-n_1 > 0 \\ n_4+n_5+n_6-n_1-n_3 > 0. \end{cases}$$

When the number of voters $n \to \infty$, each of the two systems determines a polytope (as earlier, we substitute each n_i with $p_i = \frac{n_i}{n}$, the first lines in both systems transform into $\sum p_i = 1$). The sum of the volumes of these polytopes minus the volume of their intersection gives the volume of the

[12]The difference in points between candidates B and C ($d(B,C)$) is greater than 0, as B was above C after the election.

polytope, formed by manipulable profiles. When the result is divided by the volume of the simplex (all possible profiles) and multiplied by 6 (all possible final arrangements of alternatives), we arrive with the same value of approximately $0,5025$ as in [7,10].

Example 4. We illustrate the algorithm via one more example, namely, the case of four alternatives, with the Borda rule as before. There are 24 preferences, n_i (p_i) is the number (normalized) of voters with the i's preference in the alphabetical order on preferences according to alternatives' names (n_1 people have voted (A, B, C, D), n_2 - (A, B, D, C), $\dots$, n_{24} - (D, C, B, A)). Again, without loss of generality, the candidates' initial arrangement is A, B, C, D. This time three coalitions can be formed: unifying for alternative B - voters with 12 types of preferences (in which B is above A); unifying for alternative C - voters with 12 types of preferences (in which C is above A) and unifying for alternative D - voters with 12 types of preferences (in which D is above A). The inequalities responsible for the initial arrangement are

$$\begin{cases} \sum\limits_{i=1}^{24} p_i = 1 \\ 3\sum\limits_{i=1}^{6} p_i + 2(p_7 + p_8 + p_{13} + p_{14} + p_{19} + p_{20}) - 3\sum\limits_{i=7}^{12} p_i + \\ +(p_9 + p_{11} + p_{15} + p_{17} + p_{21} + p_{23}) - 2(p_1 + p_2 + p_{15} + p_{16} + p_{21} + p_{22}) - \\ -(p_3 + p_5 + p_{13} + p_{18} + p_{19} + p_{24}) > 0 \\ 3\sum\limits_{i=7}^{12} p_i + 2(p_1 + p_2 + p_{15} + p_{16} + p_{21} + p_{22}) - 3\sum\limits_{i=13}^{18} p_i + \\ +(p_3 + p_5 + p_{13} + p_{18} + p_{19} + p_{24}) - 2(p_3 + p_4 + p_9 + p_{10} + p_{23} + p_{24}) - \\ -(p_1 + p_6 + p_7 + p_{12} + p_{20} + p_{22}) > 0 \\ 3\sum\limits_{i=13}^{18} p_i + 2(p_3 + p_4 + p_9 + p_{10} + p_{23} + p_{24}) - 3\sum\limits_{i=19}^{24} p_i + \\ +(p_1 + p_6 + p_7 + p_{12} + p_{20} + p_{22}) - 2(p_5 + p_6 + p_{11} + p_{12} + p_{17} + p_{18}) - \\ -(p_2 + p_4 + p_8 + p_{10} + p_{14} + p_{16}) > 0. \end{cases}$$

The systems of inequalities, responsible for the positivity of differences in points between the unifying candidate and all the other alternatives (prior to the unifying coalition participants' arrangement of all the places except the first) for the coalitions B, C and D, respectively, are

$$\begin{cases} 3(\sum_{i=7}^{12} p_i + p_{15} + p_{16} + p_{18} + p_{21} + p_{22} + p_{24}) + 2(p_1 + p_2) + \\ +(p_3 + p_5 + p_{13} + p_{19}) - 3\sum_{i=1}^{6} p_i - 2(p_{13} + p_{14} + p_{19} + p_{20}) - \\ -(p_{17} + p_{23}) > 0, d(B, A) > 0. \end{cases}$$

$$\begin{cases} 3(\sum_{i=13}^{18} p_i + p_9 + p_{10} + p_{12} + p_{22} + p_{23} + p_{24}) + 2(p_3 + p_4) + \\ +(p_1 + p_6 + p_7 + p_{20}) - 3\sum_{i=1}^{6} p_i - 2(p_7 + p_8 + p_{19} + p_{20}) - \\ -(p_{11} + p_{21}) > 0, d(C, A) > 0 \\ 3(\sum_{i=13}^{18} p_i + p_9 + p_{10} + p_{12} + p_{22} + p_{23} + p_{24}) + 2(p_3 + p_4) + \\ +(p_1 + p_6 + p_7 + p_{20}) - 3\sum_{i=7}^{12} p_i - 2(p_1 + p_2 + p_{21}) - \\ -(p_3 + p_5 + p_{19}) > 0, d(C, B) > 0 \end{cases}$$

$$\begin{cases} 3(\sum_{i=19}^{24} p_i + p_{10} + p_{11} + p_{12} + p_{16} + p_{17} + p_{18}) + 2(p_5 + p_6) + \\ +(p_2 + p_4 + p_8 + p_{14}) - 3\sum_{i=1}^{6} p_i - 2(p_7 + p_8 + p_{13} + p_{14}) - \\ -(p_9 + p_{15}) > 0, d(D, A) > 0 \\ 3(\sum_{i=19}^{24} p_i + p_{10} + p_{11} + p_{12} + p_{16} + p_{17} + p_{18}) + 2(p_5 + p_6) + \\ +(p_2 + p_4 + p_8 + p_{14}) - 3\sum_{i=7}^{12} p_i - 2(p_1 + p_2 + p_{15}) - \\ -(p_3 + p_5 + p_{13}) > 0, d(D, B) > 0 \\ 3(\sum_{i=19}^{24} p_i + p_{10} + p_{11} + p_{12} + p_{16} + p_{17} + p_{18}) + 2(p_5 + p_6) + \\ +(p_2 + p_4 + p_8 + p_{14}) - 3\sum_{i=13}^{18} p_i - 2(p_3 + p_4 + p_9) - \\ -(p_1 + p_6 + p_7), d(D, C) > 0 \end{cases}$$

Now the algorithm described above is used to come up with the last inequality which is satisfied whenever the unifying coalition for C can manipulate.

The corresponding inequalities for coalitions for B and D are the same with α, β and γ equal to the 'first, second and third maximum' among the differences in points (and also number of boxes on the first, second and third shelves) between this candidates and the others. Indices 1 and 2 (to the right of Greek letters) below are responsible for the number of sets of one black ball and two black balls on the shelves, respectively

$$\alpha := max_1(d(C,A), d(C,B), d(C,D)),$$
$$\beta := max_2(d(C,A), d(C,B), d(C,D)),$$
$$\gamma := max_3(d(C,A), d(C,B), d(C,D))$$
$$\alpha_2 := min(\frac{\alpha}{2}, |C|)$$
$$\beta_2 := min(|C| - \alpha_2, \frac{\beta}{2})$$
$$\beta_1 := min(|C| - \beta_2, \beta - 2\beta_2)$$
$$\gamma_2 := min(|C| - \alpha_2 - \beta_2, \frac{\gamma}{2})$$
$$\gamma_1 := min(\gamma - 2\gamma_2, |C| - \beta_1, |C| - \gamma_2)$$
$$\beta_1 + \gamma_1 + 2\beta_2 + 2\gamma_2 + 2\alpha_2 = 3|C|.$$

5 Conclusion

In the paper we find the percentage of manipulable outcomes in three-alternative elections with Borda count. This is done for two different definitions of manipulability: significant and with respect to restricted coalitions. The computed values show that both manipulations are likely to occur. An algorithm for testing the manipulability of results of elections under weighted scoring rules is developed and demonstrated on the examples of elections of Borda type for three and four alternatives.

References

[1] Aleskerov, F., Kurbanov, E. Degree of manipulability of social choice procedures. Current Trends in Economics. Berlin: Springer, 1999

[2] Cervone, D., Gehrlein W. and Zwicker, W. What Scoring Rule Maximizes Condorcet Efficiency Under IAC, Theory and Decision, Volume 58, 2005, 145-185.

[3] Chamberlin, J. Investigation into the relative manipulability of four voting systems. Behaviour Science, 1985, 30. 195-203.

[4] Early, E. On the Euler Characteristic. MIT Undergraduate Journal of Mathematics. 39-40.

[5] Ehrhart, E. Sur un probl‘eme de gŕeomŕetrie diophantienne linŕeaire. I. Poly‘edres et rŕeseaux, J. Reine Angew. Math., 1967, 226. 1-29.

[6] Ehrhart, E. Sur un probl‘eme de gŕeomŕetrie diophantienne linŕeaire. II. Syst‘emes diophantiens linŕeaires, J. Reine Angew. Math., 1967, 227. 25-49.

[7] Favardin, P., Lepelley, D., Serais, J. Borda rule, Copeland method and strategic manipulation. Review of Economic Design, 2002, 7(2). 213-228.

[8] Franz, M. Convex - a Maple package for convex geometry, version 1.1, 2009, available at http://www-fourier.ujf-grenoble.fr/ franz/convex

[9] Lepelley, D., Mbih, B. The vulnerability of four social choice functions to coalitional manipulation of preferences. Social Choice and Welfare 11, 1994. 253-265.

[10] Lepelley, D., Louichi, A., Smaoui, H. On Ehrhart polynomials and probability calculations in voting theory. Social Choice and Welfare, 2008, 30(3). 363-383.

[11] Slinko, A. How Large a Coalitional Should Be to Manipulate an Election? Mathematical Social Sciences, 2004, 47(3). 289-293.

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