1. Organizational and methodological issues

- *The aim of the course.* The course "Complex Analysis" is aimed at mastering basic concepts and tools of modern one-dimensional Complex Analysis as well as basic principles of one-dimensional complex geometry, understanding of the role these concepts play in Mathematics and Science.
- *Objectives of the course.* Upon successful completion of the course, the students will be able to understand mathematical results (talks, papers, etc.), in which complex analytic tools are applied to problems of geometry, dynamical systems and mathematical physics. Student will be able to use the language of complex analysis as it applies in many branches of Science.

  A student will *know* the following concepts: Möbius transformations and their properties, power series expansions of basic elementary functions, 1-forms on the plane and their integrals over oriented curves, 1-chains, properties of closed 1-forms, complex differentiability, Cauchy–Riemann relations, the Cauchy integral formula, general properties of holomorphic functions (the maximum principle, derivatives estimates, power series expansions, the Removable Singularity Theorem, the Liouville theorem), harmonic functions and the Poisson formula, calculus of residues, equicontinuity, normal families of holomorphic functions, the Riemann mapping theorem, multivalued analytic functions, the monodromy groups, Riemann surfaces and the uniformization theorem.

  A student will be *able to*: perform computations with elementary functions of complex variables, use complexification to solve problems stated in real terms, analyze the dynamical behavior of fractional linear transformations, work with power series and Laurent series expansions, use these expansions to compute integrals, solve differential equations, etc., compute residues of meromorphic 1-forms, investigate and classify branch points of multivalued analytic functions, perform computations with local branches of multivalued functions and the monodromy group, find explicit expressions for the Riemann mappings of some simple shapes.
- *Original methodological approaches used in the course.* Complex analysis is a universal language of Science. It appears, through the use of special functions, in many applied areas as well as in most branches of modern Mathematics. In addition to being mathematically elegant, complex analysis provides a powerful tool for solving problems that are either very difficult or virtually impossible to solve in any other way.

  Our exposition of complex analysis has a distinct geometric flavor. We use visualization tools (including computer graphics) systematically. A general teaching philosophy, according to which this course has been developed, is the following: the exposition should be arranged as the way from particular examples to general theorems. We consistently consider particular examples as being more important than general results.

  Every technique we use is first explained in the simplest partial case, which makes the presentation clearer at the cost of being less "efficient". We rely on the principle that it is better to know different proofs of the same theorem rather than the same proof of different theorems. Sometimes the students are requested to fill in details of proofs.

- *The place of the course in the system of innovative qualifications that are formed in the course of study.* The course is offered to the first year Master of Science students in Mathematics. This is an international M.Sc. programme conducted by the department of Mathematics in English. The course is joint with the Math in Moscow programme, a student internship programme of the NRU HSE and the Independent University of Moscow, which attracts mostly North-American students (whose tuition is often covered by NSF grants and NCERC grants).

  We expect to have students with very different background in Mathematics. The topic of the course shows the unity of Mathematics. The main qualification that the students are suppose to receive is the ability to find connections between different mathematical subjects using the universal idea of complexification (whose practical meaning is usually a simplification).

2. The content of the course

What makes this course unique (description of scientific and methodological features, comparison with similar courses offered by the NRU HSE and other universities in Russia and worldwide). The course "Complex Analysis" is among the traditional courses offered by the Math in Moscow programme (a student internship programme of the Independent University of Moscow and the NRU HSE). In 2012, our department converted the Master of Science programme in Mathematics into an international programme, conducted in English, and this course is offered as a course of students' choice in the cycle "Analysis", which is one of the three major cycles (along with "Algebra" and "Topology"). On the one hand, the organization of this course follows the main highlights developed by the Math in Moscow programme. On the other hand, the scientific content and the method of presentation correspond to the author's teaching philosophy and are different from those employed by different instructors of this course in recent years. The author has created an independent, original syllabus, and will be implementing it for the first time in Fall 2013.

Introductory one-dimensional Complex Analysis is being taught as a standard undergraduate university course in most departments of Mathematics, Physics, and Engineering. More advanced graduate-level courses are much less common in Russian universities. On the other hand, most mathematical departments in the USA offer graduate-level courses of one-dimensional complex analysis on a regular basis. The content of these courses is often a part of the Ph.D. qualifying exams. Complex analysis courses are often bundled with real analysis courses, which defines some specific approach to their syllabi.

The curriculum of the Master of Science programme in Mathematics and of the Math in Moscow programme shows the Complex Analysis course as one of the basic Master's-level courses. Therefore, some minimal requirements to the content have been developed. They take into account that the students attending this course may have different backgrounds.

The course is made as self-contained as possible to compensate for the different backgrounds of students. All necessary notions will be rigorously defined. However, to make it possible to cover a more advanced material going beyond the undergraduate curriculum, it is necessary to assume a very active participation on the part of the audience. We assume that students will recover the proofs of some standard facts using undergraduate textbooks, with consultation with the instructor.

A more specific feature of this course programme is that it is based on an "interdisciplinary" approach. Complex Analysis is viewed not only as a self-contained section of analysis but also and more importantly as the language that is useful in the study of many other mathematical subjects. We often start with applications, motivate a certain concept of a tool, then introduce it in the simplest possible set-up, after which we either proceed with a general discussion or just hint at how a general theory may develop. A traditional for this course methodological trick is the following: many particular examples are worked out; sometimes the same theorem is proved in different ways; sometimes a particular case is proved before the general statement, etc.

Plan of the course. Allocation of classroom hours among the topics and types of assignments.

No	Topic	weeks	Total hours	Lectures	Recitation sessions
1	Möbius transformations	2	20	4	4
2	Elementary functions	2	20	4	4
3	1-forms and integration.	1	10	2	2
4	The Cauchy integral	2	30	4	4
5	Properties of holomorphic functions.	3	30	6	6
6	Normal families.	2	30	4	4
7	Multivalued analytic functions.	2	20	4	4
8	Riemann surfaces and uniformization.	2	20	4	4
Total		16	180	32	32

The structure of the syllabus.

Section 1: Möbius transformations. In this section, we define Möbius transformations as transformations of the sphere that map circles to circles. A theorem of Möbius states that, using a complex coordinate on the sphere (given by a stereographic projection onto the plane of complex numbers), any Möbius transformation can be written either as a fractional linear map, or as a map, whose complex conjugate is fractional linear. We discuss a proof of this classical theorem of Möbius. We consider the classification of fractional linear transformations according to their dynamical behavior (loxodromic, hyperbolic, elliptic and parabolic transformations).

References: [1, 3, 9]

Section 2: Elementary functions of a complex variable. We discuss elementary functions of a complex variable, including polynomials, the exponential, the logarithm, trigonometric functions. Power series expansions play the major role in this discussion.

References: [1, 2, 3, 4, 9]

Section 3: Integration of 1-forms. Complex analysis uses integration of 1-forms over curves in a very essential way. We discuss the necessary background To make the treatment as short and as down-to-earth as possible, we restrict our attention to 1-forms on the plane. On the other hand, we will not pretend (as most complex analysis courses do) that we deal with something specifically complex. There is nothing specific to complex analysis here, and all properties of 1-forms we need are also true for real forms.

References: [1, 2, 3, 4, 8]

Section 4: The Cauchy integral.

The Cauchy integral is the most important technical tool of complex analysis. We derive the Cauchy integral formula and draw standard consequences of it, including the upper bounds for the derivatives, the power series expansion, etc. An important technique stemming out of the Cauchy integral formula is the the calculus of residues. It helps e.g. to perform explicit computations of definite integrals and study solutions of analytic differential equations.

References: [1, 2, 3, 4, 8]

Section 5: Properties of holomorphic functions. The properties we will discuss are: the maximum modulus principle, Taylor series and Laurent series expansions, the open domain theorem, the argument principle, etc. We will also discuss the Schwartz lemma, the contraction of the Poincaré metric under holomorphic univalent maps, convergence of sequences of holomorphic functions, the Removable Singularity theorem, the Liouville theorem, etc.

References: [1, 2, 4, 7, 9]

Section 6: Normal families. Compactness considerations are important for proving existence results. Normal families of holomorphic functions were introduced by Montel exactly as a tool for compactness arguments. We will review this theory of Montel, including his fundamental criterion of normality, and give applications such as the Koebe distortion theorem and, most importantly, the Riemann mapping theorem.

References: [1, 4, 7]

Section 7: Multivalued analytic functions. The notion of a multivalued analytic function is due to Weierstrass. Although this elegant theory was created more than a century ago, it sounds very modern, and is taught nowadays in an almost unchanged form. We will discuss particular examples of multivalued analytic functions: most importantly, algebraic functions and inverse functions to certain automorphic functions.

References: [1, 5, 12]

Section 8: Riemann surfaces and uniformization The central result of this section is the uniformization theorem, one of the most important milestones of XIXth century complex analysis. The uniformization theorem provides a complete description of all simply connected Riemann surfaces. It also provides a description of any Riemann surface as a quotient of the sphere, the plane, or the disk by a discrete group of holomorphic automorphisms. If time permits, we may discuss applications of the uniformization theorem to complex dynamical systems.

References:

[4, 12]

References

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Sample topics of course projects.

Topic 1: Dynamics of Schottki groups.
Topic 2: The Mandelbrot set.
Topic 3: Lattés examples.
Topic 4: Elliptic curves: analysis and arithmetic.
Topic 5: The Bieberbach conjecture.
Topic 6: The Carathéodori theory.

3. Grading policy

We will use the following means of evaluation: homework assignments, quizzes, the written midterm test, the written final exam.

Quiz: 1–2 questions for 5 minutes,
Homework: 3–6 problems for one week,
Midterm and Final: 8 problems for 3 hours.

The intermediate grade is computed at the end of the first module by the formula "0.5 times the average homework grade in the first module plus 0.5 times the midterm grade". The final grade is computed at the end of the term by the formula "0.3 times the cumulated grade plus 0.3 times the intermediate grade plus 0.4 times the final exam grade", where the cumulated grade is the average homework grade in the second module. Quiz results and classroom participation are taken into account as bonus points when computing the cumulated grade.

4. Problems for quizzes, tests, homework assignments

The following is a pull of problems, from which homework assignments and questions for quizzes may be taken. This is just a sample. It can also be used by students as a list of practice problems while preparing for a midterm or the final exam.

4.1. Questions for quizzes.

- Let a, b, c and d be real numbers such that $c^2 + d^2 \neq 0$. Find the real part of the complex number
$$\frac{a+bi}{c+di}.$$
- Find $|(1+i)^6|$.
- True or false: $\Re(zw) = \Re(z)\Re(w)$ for all z, $w \in \mathbb{C}$?
- True or false: the function
$$f(x,y) = \frac{y}{x^2+y^2} + \frac{ix}{x^2+y^2}$$
is differentiable for all $z = x + iy \neq 0$?

- Find $\exp(1 + \pi i/3)$.
- True or false: the product of two harmonic functions is always harmonic?
- True or false: the product of two holomorphic functions is always holomorphic?
- Find the residue of $\frac{dz}{z(z-1)}$ at $z = 0$.
- Is the function $\sqrt{z^4}$ a multivalued analytic function?

Sample homework problems

4.2. **Complex numbers.**

Problem 1. Represent the product

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

as a sum of squares $z_1^2 + z_2^2$, where z_1 and z_2 are polynomials in x_1, x_2, y_1, y_2.

Problem 2. If we consider two-dimensional vectors x and y as complex numbers, then their dot product can be written as $x \cdot y = \Re(x\overline{y})$.

For three different points z_1, z_2, z_3 in $\mathbb{C}$, define their *simple ratio* as

$$(z_1, z_2, z_3) = \frac{z_1 - z_3}{z_2 - z_3}.$$

Problem 3. Three points z_1, z_2, z_3 are collinear if and only if

$$(z_1, z_2, z_3) \in \mathbb{R}.$$

Problem 4. Find all bilinear maps $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with the property

$$|B(x, y)| = |x| \cdot |y|,$$

where x, y are arbitrary vectors, and $|x|$ is the Euclidean length of the vector x.

Answer: $B(x, y) = x(Ay)$, where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal map, and the multiplication is in the sense of complex numbers.

Problem 5. Consider a $\mathbb{R}$-linear map $A : \mathbb{C} \to \mathbb{C}$ that preserves angles and preserves the orientation. Prove that A is the multiplication by some complex number.

Problem 6. Consider a $\mathbb{R}$-linear map $A : \mathbb{C} \to \mathbb{C}$ that multiplies all distances by the same factor and preserves the orientation. Prove that A is the multiplication by some complex number.

Problem 7. Prove that the area of the parallelogram spanned by $a \in \mathbb{C}$ and $b \in \mathbb{C}$ is equal to

$$\frac{a\bar{b} - b\overline{a}}{2i}.$$

Problem 8. Let a and b be points of the unit circle $|z| = 1$, and c be the intersection point of the tangent lines of the unit circle at the points a b. Prove that c is the harmonic mean of a and b, i.e.,

$$c^{-1} = \frac{a^{-1} + b^{-1}}{2}.$$

Problem 9. Let a, b, c, d be four different points on the unit circle. Prove that the intersection point of the lines ab and cd is given by the formula

$$\frac{(\overline{a}+\overline{b})-(\overline{c}+\overline{d})}{\overline{ab}-\overline{cd}}.$$

Problem 10. **Newton's theorem.** Consider a quadrilateral circumscribed about the circle $|z|=1$ (i.e., the edges of the quadrilateral are tangent to this circle). Prove that the line connecting the midpoints of the diagonals of this quadrilateral contains the origin.

Recall that the *cross-ratio* of four points z_1, z_2, z_3, $z_4 \in \mathbb{C}$ is defined by the formula

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}.$$

Problem 11. Four points lie on the same circle (or on the same line) if and only if their cross-ratio is real.

Problem 12. Find the vertices of the regular pentagon inscribed into the unit circle $\{|z| = 1\}$ such that -1 is one of the vertices. Find the lengths of all diagonals of this pentagon.

Problem 13. For three given complex numbers z_1, z_2, z_3, find a complex number z_4 such that z_1, z_2, z_3 and z_4 are vertices of a parallelogram that appear in this circular order.

Problem 14. Find a rational function R such that $\tan(n\alpha) = R(\tan(\alpha))$ for every $\alpha \in \mathbb{R}$.

Answer:

$$R(t) = \frac{\sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} t^{2k+1}}{\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} t^{2k}}$$

Problem 15. Find $\cos\frac{\pi}{16}$.

Answer:

$$\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

Problem 16. Find the real and the imaginary parts of $\sqrt{a+bi}$.

Problem 17. Suppose that complex numbers $z_1, \ldots, z_k$ lie in one open half-plane bounded by a line passing through 0. Prove that complex numbers

$$\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_k}$$

have the same property.

Problem 18. Find the image of the upper half-plane $\Im(z) > 0$ under the map

$$f(z) = z + \frac{1}{z}.$$

Problem 19. Find the images of circles under the map

$$f(z) = z + \frac{1}{z}.$$

Problem 20. Does the series
$$\sum_{n=1}^{\infty} \frac{n \sin(ni)}{3^n}$$
converge?

Problem 21. Find all complex numbers z such that $e^z = i$.

Problem 22. Find all complex numbers z such that $\sin(z) = 3$.

4.3. **Möbius transformations.**

Problem 23. Prove that a fractional linear transformation, i.e., a transformation of the form
$$z \mapsto \frac{az+b}{cz+d}$$
is a composition of the maps $z \mapsto \frac{\lambda}{z}$, an parallel translations $z \mapsto z+\lambda$. Deduce that every fractional linear transformation maps lines or circles to lines or circles.

Problem 24. Consider the fractional linear transformation
$$f(z) = \frac{i-z}{i+z}.$$
Find the images under f of the following sets: $\Re(z) \geq 0$, $\Im(z) \geq 0$, $|z| < 1$.

Problem 25. Prove that fractional linear transformations preserve cross-ratios.

Problem 26. Find all fractional linear transformations with fixed points 1 and -1.

Problem 27. * Find all pairs of commuting fractional linear transformations.

Problem 28. * Find the general form of a fractional linear transformation corresponding to a rotation of a sphere under the stereographic projection onto $\overline{\mathbb{C}}$.

Problem 29. Determine the type (elliptic, parabolic, loxodromic, hyperbolic) of the following fractional linear transformations:
$$\frac{z}{2z-1}, \quad \frac{2z}{3z-1}, \quad \frac{3z-4}{z-1}, \quad \frac{z}{2-z}.$$

Problem 30. Suppose that a fractional linear transformation f satisfies the identity $f(f(z)) = z$. Prove that f is elliptic.

Problem 31. **The Apollonius circle theorem.** Let two different points a, $b \in \mathbb{C}$ and a positive real number $r > 0$ be given. Prove that the locus of points $z \in \mathbb{C}$ such that
$$\frac{|z-a|}{|z-b|} = r$$
is a circle.

Problem 32. **The Steiner net.** Find two families of circles that are invariant under any fractional linear transformation with fixed points ± 1 (in other words, every fractional linear transformation f such that $f(\pm 1) = \pm 1$ takes every circle of one of the two families to a circle of the same family).

4.4. **Taylor and Laurent series.**

Problem 33. Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^a} \quad (a \in \mathbb{R})$$

Problem 34. Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{z^n}{\log n^2}, \quad \sum_{n=1}^{\infty} (3+(-1)^n) z^n, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$$

Problem 35. Let the radius of convergence of a series $\sum_{n=1}^{\infty} u_n z^n$ be R. Find the radii of convergence of the following series:

$$\sum_{n=1}^{\infty} (2^n - 1) u_n z^n, \quad \sum_{n=1}^{\infty} u_n \frac{z^n}{n!}, \quad \sum_{n=1}^{\infty} n^n u_n z^n.$$

Problem 36. Find a nonzero power series $y(x)$ that solves the following differential equation:

$$x^2 y'' + x y' + (x^2 - 1) y = 0.$$

What is its radius of convergence?

Problem 37. Find power series expansions of the following functions:

$$\int_0^z e^{x^2} dx, \quad \int_0^z \frac{\sin x}{x} dx.$$

Find the radii of convergence of these power series.

Problem 38. Suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in the disk $|z| < 1 + \varepsilon$, where $\varepsilon > 0$. Find the area of the image of the unit disk $|z| < 1$ under f in terms of the coefficients a_n.

Problem 39. Suppose that the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ converges in the annulus $r - \varepsilon < |z| < R + \varepsilon$, where $\varepsilon > 0$. Let A denote the annulus $r < |z| < R$. Find the area of $f(A)$ in terms of the coefficients a_n.

Problem 40. Find the Laurent series for the function $\sin(z) \sin \frac{1}{z}$ in a suitable annulus around 0.

4.5. **Properties of holomorphic functions.**

Problem 41. Which of the following functions are real parts of holomorphic functions in the unit disk $\{|x+iy| < 1\}$:

(1) $u(x,y) = x^2 - axy + y^2$,
(2) $u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$,
(3) $u(x,y) = x^3$,
(4) $u(x,y) = \log((x-2)^2 + y^2)$?

Rigorously justify your answer.

Problem 42. For every linear polynomial $u(x,y) = ax + by + c$, find a holomorphic function $f(z)$ such that the real part of $f(x+iy)$ coincides with u.

Problem 43. Write the Cauchy–Riemann equations in polar coordinates.

Answer:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \phi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \phi}.$$

Problem 44. Prove that

$$\sum_{n=1}^{\infty} \frac{u^{2n}}{(n!)^2} = \frac{1}{2\pi}\int_0^{2\pi} e^{2u\cos\theta} d\theta.$$

Problem 45. * **The Hadamard Three Circle Theorem.** Let f be a holomorphic function on some annulus around the origin. Set $M(r) = \log \sup_{|z|=e^r} |f(z)|$. Prove that the function M is convex.

Hint: apply the Maximum Modulus Principle to functions of the form $z^a f(z)$.

Problem 46. Find the singularities in $\overline{\mathbb{C}}$ of the following functions:

$$ze^z, \quad \frac{1}{e^z-1} - \frac{1}{z}, \quad \frac{1-e^z}{1+e^z}, \quad e^{-1/z^2}, \quad \frac{e^{\frac{1}{z-1}}}{e^z-1}.$$

Problem 47. Find the order of the zero at $z = 0$ of the function $f(z) = 6\sin(z^3)+z^3(z^6-6)$.

Problem 48. * **The Mittag–Leffler theorem.** Consider any sequence $z_n \to \infty$ and any sequence of polynomials P_n. There exists a meromorphic function f on $\mathbb{C}$ with poles only at the points z_n and such that

$$f(z) = P_n\left(\frac{1}{z-z_n}\right) + E_n(z),$$

where E_n is a holomorphic function in a neighborhood of z_n.

Problem 49. Prove or disprove: given two entire functions f, g without common zeros, there exist entire functions α, β such that

$$\alpha f + \beta g = 1$$

identically on $\mathbb{C}$.

Problem 50. Find the number of zeros of the polynomial $f(z) = z^7 - 5z^4 + z^2 - 2 = 0$ in the disk $|z| < 1$.

Problem 51. Prove that, for sufficiently large n, the equation

$$1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} = 0$$

has no solutions in the disk of radius 100 around the origin.

4.6. **Integration.**

Problem 52. Compute the following integrals:

$$(a) \quad \int_0^1 e^{it}\cos(at)dt; \qquad (b) \quad \int_{-1}^1 \frac{dt}{t^2+i}$$

Problem 53. Compute the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Answer: $\frac{\pi}{2}$.

Problem 54. Compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(bx)\,dx.$$

Problem 55. Compute the Fresnel integrals

$$\int_0^\infty \sin(x^2)\,dx, \qquad \int_0^\infty \cos(x^2)\,dx.$$

Problem 56. Compute the integral

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t}.$$

Answer: $\frac{2\pi}{ab}$.

Problem 57. Find the residue of the 1-form

$$\frac{z^{2n}dz}{(1+z)^n}$$

at $z = -1$. Here n is a positive integer.

Problem 58. Find the integral

$$\int_0^{2\pi} \frac{dt}{1 - 2a\cos t + a^2}, \qquad 0 < a < 1.$$

Problem 59. Find the integral

$$\int_0^{2\pi} \frac{dt}{a + b\cos t}, \qquad a > b > 0.$$

4.7. **Multivalued analytic functions.**

Problem 60. Find the branch points of the function $\sqrt{1-z^2}$.

Problem 61. Consider the integral

$$\int_0^z \frac{dx}{(x-1)(x-2)(x-3)},$$

where the integration is performed over a curve avoiding the points 1, 2, 3. How many different values does this integral have?

Problem 62. Examine which of the following multi-valued functions have branches in a neighborhood of the given point that can be represented by convergent Laurent series:

$$\begin{aligned}
&(a) \quad f(z) = \sqrt{z}, \quad z = 0,\\
&(b) \quad f(z) = \sqrt{z(z-1)}, \quad z = \infty,\\
&(c) \quad f(z) = \sqrt{1+\sqrt{z}}, \quad z = 1.
\end{aligned}$$

Rigorously justify your answer.

4.8. **Conformal mappings.**

Problem 63. Find the images of the horizontal and the vertical lines under the map $z \mapsto z^2$.

Problem 64. Find a conformal mapping that takes the region $\{|z| < 1, \Im(z) > 0\}$ onto the unit disk $\{|z| < 1\}$.

Problem 65. Find a conformal mapping that takes the solid angle $0 < \arg(z) < \pi/6$ onto the unit disk $\{|z| < 1\}$.

Problem 66. Find all conformal transformations of the unit disk onto itself.

Problem 67. Find all conformal transformations of the upper half-plane onto itself.

Problem 68. Prove that any rational map f of degree 2 has the form $z \mapsto \alpha(\beta(z)^2)$, where α and β are fractional linear transformations.