Affine Option Pricing Model in Discrete Time

Eric Renault\textsuperscript{1}  Stanislav Khrapov\textsuperscript{2}

\textsuperscript{1}Brown University  
Providence, RI

\textsuperscript{2}New Economic School  
Moscow, Russia

Third International Moscow Finance Conference  
November 8, 2013
Introduction
Continuous vs Discrete

Continuous time affine models with stochastic volatility:
- Cox, Ingersoll, and Ross (1985, Econometrica)
- Heston (1993, RFS)
- Duffie, Pan, and Singleton (2000, Econometrica)
Continuous vs Discrete

Popular because of computational convenience ...

... with historical probability measure (P):

- Robustness to temporal aggregation: Meddahi and Renault (2004, JoE)
- Robustness to cross-sectional aggregation (portfolio)
Continuous vs Discrete

Popular because of computational convenience ...

... with historical probability measure (P):
- Robustness to temporal aggregation: Meddahi and Renault (2004, JoE)
- Robustness to cross-sectional aggregation (portfolio)

... with risk neutral probability measure (Q):
- Structure preserving change of measure (affine structure preserved)
- Analytical tractability of computing derivative prices (inverse Fourier transform)
Discrete Time Extension

Volatility model:
- Darolles, Gourieroux, and Jasiak (2006, JTSA)
- Gourieroux and Jasiak (2006, JoF)

Option pricing model with conditional skewness:
- Feunou and Tedongap (2012, JBES)
### Discrete Time Extension

**Advantages of discrete time:**

- Computational/Statistical tractability
- More flexibility for higher order moments
Discrete Time Extension

Advantages of discrete time:
- Computational/Statistical tractability
- More flexibility for higher order moments

Challenges of discrete time:
- Accommodating leverage effect
- Keeping the advantage of structure preserving change of measure historical/risk-neutral
Affine Stochastic Volatility Model
The Model

CAR Volatility: Darolles, Gourieroux, and Jasiak (2006, JTSA)

\[ E \left[ \exp \left\{ -u\sigma_{t+1}^2 \right\} \mid I_t \right] = \exp \left\{ -a(u)\sigma_t^2 - b(u) \right\} \]
The Model

CAR Volatility: Darolles, Gourieroux, and Jasiak (2006, JTSA)

\[
E \left[ \exp \{-u \sigma_{t+1}^2\} \mid I_t \right] = \exp \{-a(u) \sigma_t^2 - b(u)\}
\]

Log Excess Return

\[
E \left[ \exp \{-vr_{t+1}\} \mid I_t \cup \sigma_{t+1}^2 \right] = \exp \{-\alpha(v) \sigma_{t+1}^2 - \beta(v) \sigma_t^2 - \gamma(v)\}
\]
The Model

Joint Return and Volatility

\[ E \left[ \exp \left\{ -u \sigma^2_{t+1} - vr_{t+1} \right\} \mid l_t \right] = \exp \left\{ -l(u, v) \sigma^2_t - g(u, v) \right\} \]
The Model

Joint Return and Volatility

\[ E \left[ \exp \left\{ -u\sigma_{t+1}^2 - vr_{t+1} \right\} \mid l_t \right] = \exp \left\{ -l(u, v) \sigma_t^2 - g(u, v) \right\} \]

\[ l(u, v) = a[u + \alpha(v)] + \beta(v) \]
\[ g(u, v) = b[u + \alpha(v)] + \gamma(v) \]
The Model

Joint Return and Volatility

\[ E \left[ \exp \left\{ -u \sigma^2_{t+1} - v r_{t+1} \right\} \mid I_t \right] = \exp \left\{ -l(u, v) \sigma^2_t - g(u, v) \right\} \]

\[
\begin{align*}
  l(u, v) &= a[u + \alpha(v)] + \beta(v) \\
g(u, v) &= b[u + \alpha(v)] + \gamma(v)
\end{align*}
\]

\[ \alpha(v) \neq 0 \iff \text{leverage!} \]
The affine structure is kept from $P$ to $Q$ when

\[ \text{Pricing Kernel} = \text{Exponential Affine} \]
Risk-Neutral Distribution

The affine structure is kept from P to Q when
Pricing Kernel = Exponential Affine

Stochastic Discount Factor (SDF):

\[ M_{t,t+1} (\theta) = \exp (-r_{f,t}) \exp \left\{ m_0 (\theta) + m_1 (\theta) \sigma_t^2 - \theta_1 \sigma_{t+1}^2 - \theta_2 r_{t+1} \right\} \]

- risk prices \( \theta_1 \leq 0 \) and \( \theta_2 \geq 0 \)
- \( m_0 (\theta), m_1 (\theta) \): bonds and stocks are priced correctly
Risk-Neutral Distribution

Risk-neutral pricing:

\[ E^Q \left[ H \left( r_{t+1}, \sigma_{t+1}^2 \right) \middle| I_t \right] = \exp \left( r_{f,t} \right) E \left[ M_{t,t+1} \left( \theta \right) H \left( r_{t+1}, \sigma_{t+1}^2 \right) \middle| I_t \right] \]

for any function \( H \)
Risk-Neutral Pricing

Risk-neutral pricing:

\[ E^Q \left[ H \left( r_{t+1}, \sigma^2_{t+1} \right) \mid I_t \right] = \exp(r_{f,t}) E \left[ M_{t,t+1}(\theta) H \left( r_{t+1}, \sigma^2_{t+1} \right) \mid I_t \right] \]

for any function \( H \)

Risk-neutral distribution:

\[ E^Q \left[ \exp \left( -u \sigma^2_{t+1} - vr_{t+1} \right) \mid I_t \right] = \exp \left( -l^* (u, v) \sigma^2_t - g^* (u, v) \right) \]

with

\[ l^* (u, v) = l(\theta_1 + u, \theta_2 + v) - l(\theta_1, \theta_2) \]
\[ g^* (u, v) = g(\theta_1 + u, \theta_2 + v) - g(\theta_1, \theta_2) \]
Affine Moments

Volatility moments:

\[
E \left[ \sigma_{t+1}^2 \mid I_t \right] = a'(0) \sigma_t^2 + b'(0) \\
V \left[ \sigma_{t+1}^2 \mid I_t \right] = -a''(0) \sigma_t^2 - b''(0)
\]

\[E\left[I_{t+1}\mid I_t\right]=E[\sigma_{t+1}^2]\]
Affine Moments

Volatility moments:

\[
E \left[ \sigma^2_{t+1} \mid l_t \right] = a'(0) \sigma^2_t + b'(0)
\]

\[
V \left[ \sigma^2_{t+1} \mid l_t \right] = -a''(0) \sigma^2_t - b''(0)
\]

Return expectation:

\[
E \left[ r_{t+1} \mid l_t^\sigma \right] = \alpha'(0) \sigma^2_{t+1} + \beta'(0) \sigma^2_t + \gamma'(0)
\]
Affine Moments

Volatility moments:

\[
E \left[ \sigma_{t+1}^2 \mid I_t \right] = a'(0) \sigma_t^2 + b'(0)
\]

\[
V \left[ \sigma_{t+1}^2 \mid I_t \right] = -a''(0) \sigma_t^2 - b''(0)
\]

Return expectation:

\[
E \left[ r_{t+1} \mid l_t^\sigma \right] = \alpha'(0) \sigma_{t+1}^2 + \beta'(0) \sigma_t^2 + \gamma'(0)
\]

Leverage effect:

\[
\phi \approx Corr \left[ r_{t+1}, \sigma_{t+1}^2 \mid I_t \right] = \alpha'(0) \left( \frac{V \left[ \sigma_{t+1}^2 \mid I_t \right]}{V \left[ r_{t+1} \mid I_t \right]} \right)^{1/2}
\]

\[\alpha(v) \neq 0 \iff \text{leverage!}\]
Option Pricing
Generalized Black-Scholes

Assume $\alpha(v), \beta(v), \gamma(v)$ are quadratic, then

$$r_{t+1} \mid l_t^\sigma \sim N(E[r_{t+1} \mid l_t^\sigma], V[r_{t+1} \mid l_t^\sigma])$$
Generalized Black-Scholes

Assume $\alpha(v), \beta(v), \gamma(v)$ are quadratic, then

$$r_{t+1} | I_t^\sigma \sim N(E[r_{t+1} | I_t^\sigma], V[r_{t+1} | I_t^\sigma])$$

Option price:

$$C_t(x_t, \phi) = E_t^Q[BS(S_t \xi_{t,t+1}(\phi), (1 - \phi^2) \sigma_{t+1}^2, K)]$$

where $x_t = \log(K/S_t)$ is the moneyness and

$$\log \xi_{t,t+1}(\phi) = E^Q[r_{t+1} | I_t^\sigma] + \frac{1}{2} V^Q[r_{t+1} | I_t^\sigma]$$

is price distortion
Leverage and Volatility Smirk

Two effects of $\phi$:

- Price distortion $S_t \xi_{t,t+1}(\phi)$
- Volatility $(1 - \phi^2) \sigma_{t+1}^2$
Leverage and Volatility Smirk

Two effects of $\phi$:

- Price distortion $S_t \xi_{t,t+1}(\phi)$
- Volatility $(1 - \phi^2) \sigma_{t+1}^2$

Around $\phi = 0$ the first order effect is through volatility:

$$C_t(x_t, \phi) \approx C_t(x_t, 0) + k\phi \cdot \text{Cov}^Q \left[ \sigma_{t+1}^2, \Phi(d) \right| l_t]$$

with

$$d = \frac{1}{2} \frac{\sigma_{t+1}}{\sigma_{t+1}} - \frac{x_t}{\sigma_{t+1}}$$
Leverage and Volatility Smirk

Two effects of $\phi$:
- Price distortion $S_t \xi_{t,t+1}(\phi)$
- Volatility $(1 - \phi^2) \sigma_{t+1}^2$

Around $\phi = 0$ the first order effect is through volatility:

$$C_t(x_t, \phi) \approx C_t(x_t, 0) + k\phi \cdot \text{Cov}^Q[\sigma_{t+1}^2, \Phi(d) \mid l_t]$$

with

$$d = \frac{1}{2} \sigma_{t+1} - \frac{x_t}{\sigma_{t+1}}$$

Cov() more positive out of the money

$\implies$ the smile is pushed down on the out of the money side
Leverage and Volatility Smirk
Estimation
Maximum Likelihood

Joint likelihood

\[
f ( r_{t+1}, \sigma_{t+1}^2 \mid \sigma_t^2; c, \rho, \delta, \phi, \theta_2 ) = f ( r_{t+1} \mid \sigma_{t+1}^2, \sigma_t^2; \phi, \theta_2 ) \\
\times f ( \sigma_{t+1}^2 \mid \sigma_t^2; c, \rho, \delta )
\]
Maximum Likelihood

Joint likelihood

\[ f \left( r_{t+1}, \sigma^2_{t+1} \mid \sigma^2_t; c, \rho, \delta, \phi, \theta_2 \right) = f \left( r_{t+1} \mid \sigma^2_{t+1}, \sigma^2_t; \phi, \theta_2 \right) \times f \left( \sigma^2_{t+1} \mid \sigma^2_t; c, \rho, \delta \right) \]

where

\[ f \left( r_{t+1} \mid \sigma^2_{t+1}, \sigma^2_t; \phi, \theta_2 \right) \sim \text{Normal} \]

\[ f \left( \sigma^2_{t+1} \mid \sigma^2_t; c, \rho, \delta \right) \sim \text{nc} - \text{Gamma} \]

\( \sigma^2_{t+1} \) is ARG(1) from Gourieroux and Jasiak (2006, JoF)
Spectral GMM

Singleton (2001, JoE), Chacko and Viceira (2003, JoE)

Moment functions:

\[ g_t(u, \theta) = Z_t \cdot \left[ \exp \left\{ -u \sigma_{t+1}^2 \right\} - \exp \left\{ -a(u) \sigma_t^2 - b(u) \right\} \right. \]

\[ \left. \exp \left\{ -ur_{t+1} + 1 \right\} - \exp \left\{ -\alpha(u) \sigma_{t+1}^2 - \beta(u) \sigma_t^2 - \gamma(u) \right\} \right] \]
Spectral GMM

Singleton (2001, JoE), Chacko and Viceira (2003, JoE)

Moment functions:

\[ g_t(u, \theta) = Z_t \cdot \left[ \exp \left\{ -u \sigma_t^2 \right\} - \exp \left\{ -a(u) \sigma_t^2 - b(u) \right\} \right. \]

\[ \left. \exp \left\{ -ur_{t+1} \right\} - \exp \left\{ -\alpha(u) \sigma_{t+1}^2 - \beta(u) \sigma_t^2 - \gamma(u) \right\} \right] \]

Moments to match:

\[ E \left[ \begin{array}{c} \text{Re} \{g_t(u, \theta)\} \\ \text{Im} \{g_t(u, \theta)\} \end{array} \right] = 0 \]
Model Fit
## Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>MLE</th>
<th></th>
<th>GMM</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\theta}$</td>
<td>$t$</td>
<td>$\hat{\theta}$</td>
<td>$t$</td>
</tr>
<tr>
<td>$c$</td>
<td>2.4e-5</td>
<td>29.9</td>
<td>6.7e-6</td>
<td>4.5</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.66</td>
<td>39.9</td>
<td>0.91</td>
<td>28.5</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.45</td>
<td>29.5</td>
<td>1.18</td>
<td>6.4</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-0.21</td>
<td>-14.3</td>
<td>-0.22</td>
<td>-10.2</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.57</td>
<td>0.7</td>
<td>1.90</td>
<td>0.9</td>
</tr>
</tbody>
</table>
Vol Risk Price Calibration

\[ \hat{\theta}_1 = \arg\min_{\theta_1} \text{RMSE}_{IV} (\theta_1) = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \left( IV_{Market}^j - IV_{Model}^j (\theta_1) \right)^2} \]
Vol Risk Price Calibration

\[ \hat{\theta}_1 = \arg \min_{\theta_1} \text{RMSE}_{IV}(\theta_1) = \sqrt{\frac{1}{N} \sum_{j=1}^{N} (IV_{Market}^j - IV_{Model}^j(\theta_1))^2} \]
Conclusion

- We have been able to build a discrete time version of Heston’s model
We have been able to build a discrete time version of Heston’s model.

Advantages of discrete time:

- Easier theoretical derivations
  (impact of leverage on volatility smile, etc...)
- Easier for statistical inference
- More flexibility for higher order moments
Conclusion

- We have been able to build a discrete time version of Heston’s model

- Advantages of discrete time:
  - Easier theoretical derivations
    (impact of leverage on volatility smile, etc...)
  - Easier for statistical inference
  - More flexibility for higher order moments

- Work in progress: take advantage of this flexibility for empirical fit better than standard Heston:
  - Two volatility factors (slow and fast mean reverting)
  - Mixture component in return $r_{t+1}$ given $I_t^\sigma$ for more kurtosis
    (gamma mixture to keep the affine structure) reminiscent of jumps in continuous time
Thank you!


