CHEKANOV-TYPE THEOREM FOR SPHERIZED COTANGENT BUNDLES

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Abstract. We generalise a Chekanov theorem to the space of cooriented contact elements.

Introduction

1. Chekanov-type theorem for sperization of a cotangent bundle

1.1. Contact structure on $ST^*$. We recall standard notions from contact geometry [AG]. Let $B$ be a smooth manifold. Denote by $0_B$ the zero section of the cotangent bundle $T^*B$. The spherisation $ST^*B$ of its cotangent bundle $T^*B$ is a quotient space under the natural free action of a multiplicative group of positive real numbers $\mathbb{R}_+$ on $T^*B \setminus 0_B$, a positive number $a$ transform a pair $(q,p)$ ($q \in B, p \in T^*_q B$) into the pair $(q,ap)$. The space $ST^*B$ is a smooth manifold of the dimension $2 \dim B - 1$ and it carries a natural cooriented contact structure $\xi$ defined as follows. Consider the Liouville 1-form $\lambda = pdq$ on $T^*B$, it defines a cooriented (by $\lambda$ itself) hyperplane field $\lambda = 0$ on $T^*B \setminus 0_B$. That cooriented hyperplane field is invariant with respect to the action of $\mathbb{R}_+$ and it is tangent to orbits of the action. Hence the projection of that hyperplane field to $ST^*B$ is a cooriented hyperplane field on $ST^*B$ and it turns out to be a contact structure. We remark here, that there is no natural choice of a contact form on $ST^*B$.

1.2. Critical points and critical values of Legendrian manifolds in $J^1(B)$. We say that a point $x$ of a Legendrian manifold is a critical point of $\Lambda$ if it projects to a zero section $0_B$ under the natural projection $J^1(B) \rightarrow T^*B$. We say that a number $a$ is a critical value of $\Lambda$ if $a$ equals to the value of $u$-coordinate of a critical point of $\Lambda$.

1.3. Legendrian manifolds. The following notion generalise a notion of regular level set of a function on a manifold. Let $c \in \mathbb{R}$. Suppose that $\Lambda$ is transverse to $T^*B \times c \subset T^*B \times \mathbb{R} = J^1(B)$. Note that it implies that $c$ is not a critical value of $\Lambda$. Consider a manifold $L^c = \Lambda \cap T^*B \times c$. The intersection of $L^c$ with $0_B \times c$ is empty and the restriction of natural

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projection \((T^*B \setminus 0_B) \times c \to ST^*B\) to \(L^c\) is a legendrian immersion. We denote the image of \(L^c\) by \(\Lambda^c\) and we say in that situation that \(\Lambda^c\) is a c-reduction of \(\Lambda\).

1.4. Legendrian manifolds and generating families. Let us recall firstly the definition of generating family in the space of 1-jets of a function.

Let \(B\) be a manifold, consider the space \(J^1(B) = T^*B \times \mathbb{R}\) of one jets of functions on \(B\). The space \(J^1(B)\) is a contact manifold with the canonical contact structure given by the form \(du - \lambda\), where \(\lambda\) is (a lift of) Liouville form on \(T^*B\), \(u\) is the coordinate on the factor \(\mathbb{R}\). A smooth bundle \(E \to B\) and a generic function \(F: E \to \mathbb{R}\) generates a (immersed) Legendrian submanifold \(\Lambda_F \subset J^1(B)\) as follows. Consider a fiber of the bundle \(E \to \mathbb{R}\), and a critical point of the restriction of the function \(F\) to this fiber. For a sufficiently generic function \(F\) the set \(C_F\) consisting of all such points is a smooth submanifold of the total space \(E\) (the genericity condition is that the equation \(d_wF = 0\), where \(w\) is a local coordinate on a fiber of \(E \to B\), satisfies the condition of the implicit function theorem). At any point \(z\) of \(C_F\) the differential \(d_BF(z)\) of the function \(F\) along the base \(B\) is well defined. The rule \(z \mapsto (z, d_BF(z), F(z))\) defines an immersion \(l_F: C_F \to J^1(B)\) and its image is a Legendrian manifold \(\Lambda_F\) under definition. For a generating family \(F\) in a local trivialization \(B \times W\) of \(E\ \Lambda_F\) is given by the formula:

\[
\Lambda_F = \{(q, p, u) | \exists w_0 F_w(q, w_0) = 0, p = F_q(q, w_0), u = F(q, w_0)\},
\]

where \(q, p\) are coordinates on \(T^*B\).

Now we define a Legendrian (immersed) submanifold \(L_F\) in the space \(ST^*B\) starting from a smooth bundle \(\pi: E \to B\) and a generic function \(F: E \to \mathbb{R}\). For a point \(b \in B\) we denote by \(C^0_F(b)\) the zero level of the restriction of the function \(F\) to \(C_F\). Genericity conditions are the following - \(C_F\) is a manifold in a neighborhood of \(C^0_F\) and 0 is a regular value of \(C_F\). Consider a map \(C^0_F \to ST^*B\), \(z \mapsto (z, [d_BF(z)])\). That map is well defined since for \(z \in C^0_F\) \(d_BF(z) \neq 0\). Moreover, that map is a legendrian immersion and we denote its image by \(L_F\). We will be interested in embedded Legendrian manifolds only.

We remark here that if \(F\) is a generating function for a manifold \(\Lambda_F \subset J^1(B)\) then \(F\) is a generating function for a manifold \(L_F \subset ST^*B\) if and only if \(\Lambda_F\) is transversal to the hypersurface \(\{u = 0\}\) in \(J^1(B)\). The last condition is equivalent to the condition of emptiness ???.

Stabilization

Example

1.5. Legendrian isotopy lifting.

**Lemma 1.1.** Let \(B\) be a closed manifold and \(c \in \mathbb{R}\). Consider a compact Legendrian manifold \(\Lambda \subset J^1(B)\) such that its c-reduction \(\Lambda^c\) is well defined and \(\Lambda^c\) is an embedded manifold. Let \(L_{t\in[0,1]}\) be a legendrian isotopy of \(\Lambda^c = L_0\). Than there exists a legendrian isotopy
\( \Lambda_{t, \varepsilon \in [0,1]}, \Lambda_0 = \Lambda \) such that for any \( t \in [0,1] \) its \( c \)-reduction is defined and \( \Lambda_t^c = L_t \).

Proof. It is sufficient to prove the statement of lemma for \( c = 0 \). Consider the legendrian isotopy \( L_t \). By isotopy extension theorem there exists a contact flow \( \varphi_{t \in [0,1]} \), such that for any \( t \in [0,1] \) \( \varphi_t(L_0) = L_t \). Any contact isotopy of \( ST^*B \) lifts to a (homogeneous) Hamiltonian flow on \( T^*B \setminus 0_B \). More precisely – consider a Hamiltonian \( H_t: T^*B \setminus 0_B \to \mathbb{R} \) such that \( H_t(ap,q) = aH_t(p,q) \) for any positive number \( a \) (we will say that such a Hamiltonian is homogeneous). Then the flow of such a Hamiltonian function is well defined for all values of \( t \) and projects to a contact flow on \( ST^*B \). Moreover, any contact flow on \( ST^*B \) could be given as a projection of a unique Hamiltonian flow above.

We take a homogeneous Hamiltonian \( H_t \) corresponding to the flow \( \varphi_t \). Consider a function \( K_t(p,q,u) = H_t(p,q) \) on \( (T^*B \setminus 0_B) \times \mathbb{R} \subset J^1(B) \) as a contact Hamiltonian (see [AG]) with respect to the contact form \( du - \lambda \). Any set \( (T^*B \setminus 0_B) \times c \) is invariant under the flow of generated by \( K_t \) and coincides on it with the flow of \( H_t \) under the forgetful identifications \( T^*B \times c = T^*B \) that set with the flow of \( H_t \).

It follows from the explicit formula for the corresponding vector field: \( \dot{u} = K - pK_p, \dot{p} = K_q - pK_u, \dot{q} = -K_p \) (see [AG]). \( u \)-component of that contact vector field equals to zero since \( K \) is homogeneous. Hence the flow \( \psi_t \) generated by \( K \) satisfy \( \psi_t(\Lambda \cap (T^*B \setminus 0_B)) = L_t \).

But in general it is impossible to extend \( \psi_t \) to a flow on the whole space \( J^1(B) \) so we will change the function \( K_t \). Let us fix an arbitrary smooth function \( \tilde{H}_t : T^*B \to \mathbb{R} \) coinciding with \( H_t \) in a neighbourhood of infinity. Denote by \( P_t \) the function \( P_t(p,q,u) = \tilde{H}_t(p,q) \) and by \( P^C_t \) \( (C \in \mathbb{R}_+) \) the function \( P^C_t(p,q,u) = \frac{1}{C}P_t(Cp,q,u) \). We claim that for sufficiently big \( C \) the legendrian isotopy of \( \Lambda \) generated by the contact flow \( \Psi_t^C \) of \( P^C_t \) satisfies to the claim of lemma.

Let us fix a number \( a \) such that absolute value of any critical value of \( \Lambda \) is bigger then \( 2a \). Denote by \( X \subset \Lambda \) the subset formed by all points such that an absolute value of \( u \)-coordinate is at most \( a \), by \( Y \) we denote the closure of its complement \( \Lambda \setminus X \). The set \( X \) is a compact set and contained in \( (T^*B \setminus 0_B) \times \mathbb{R} \). Take a neighborhood \( U \subset T^*B \) of zero section, the support of \( P^C_t - K_t \) contains in \( U \times \mathbb{R} \) for sufficiently big \( C \). Hence, for sufficiently big \( C \) \( \Psi_t^C(X) = \psi_t(X) \) for all \( t \in [0,1] \). It remains to show that for sufficiently big \( C \) \( u \)-coordinate of any point in \( \Psi_t^C(Y) \) could not be zero and hence zero reduction of \( \Psi_t^C(\Lambda) \) is \( L_t \).

The coordinate \( u \) changes under an action of a contact Hamiltonian \( P \) according to the low: \( \dot{u} = P^C_t - p\frac{\partial P^C_t}{\partial p} \). So it is sufficient to show that the speed uniformly tends to zero. The following general consideration finishes the proof.

Consider a smooth vector bundle \( V \) over closed manifold \( M \). We denote by \( M(c) \) fiberwise multiplication by \( c \). We say that a smooth
function on $V$ is positively homogeneous of degree 1 at infinity if it coincides with a continuous positively homogeneous (i.e. $1/c(M(c))^*$-invariant for any positive $c$) of degree 1 function up to a sum with a compactly supported continuous function. Let $v$ be a vertical vector field coinciding with Euler vector field on each fiber. Consider an operator $D$ sending a function $g$ on $V$ to $g - L_v g$. For a positively homogeneous function $f$ the function $Df$ is a compactly supported function. Denote by $f^c$ the function $1/c(M(c))^*f$, i.e. for any $x \in V$ $f^c(x) = 1/cf(M(c)x)$.

**Lemma 1.2.** For any smooth positively homogeneous of degree 1 at infinity function $f$, $C^0$-norm of $D(1/c(M(c))^*f)$ tends to zero while $C \to +\infty$.

**Proof.** Indeed, $D(1/c(M(c))^*f) = \frac{1}{c}(M(c))^*Df$. Hence $C^0$-norm of $D(1/c(M(c))^*f)$ equals to $C^0$-norm of $f$ divided by $C$. □

1.6. Chekanov-type theorem. Consider the space $ST^*B$ of cooriented contact elements on a closed manifold $B$.

**Theorem 1.3.** Let $\{L_t\}_{t \in [0,1]}$ be a legendrian isotopy of a compact Legendrian manifold $L_0 \subset ST^*B$. Suppose $L_0$ is given by a generating family $F: E \to \mathbb{R}$, for a smooth compact fibration $E \to B$. Then there exists $N \in \mathbb{Z}_+$, such that $L_t$ is given by a generating family $G_t: E \times \mathbb{R}^N \to \mathbb{R}$ of the form:

$$G_t(e, q) = F(e) + Q(q) + f_t(e, q)$$

for a nondegenerate quadratic form $Q$ on $\mathbb{R}^N$ and compactly supported function $f_t$ such that $f_0 = 0$.

The (generalized) Chekanov theorem [Ch, P] has almost the same statement – $ST^*B$ is replaced by $J^1(B)$.

**Proof.** Proof is a reduction to (generalised) Chekanov theorem. By legendrian isotopy lifting lemma we reduce the problem to $J^1(B)$-case. □

**References**


[EG] Y. Eliashberg, M. Gromov