Singular Problems for Integro-differential Equations in Dynamic Insurance Models

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Abstract A second-order linear integro-differential equation with Volterra integral operator and strong singularities at the endpoints (zero and infinity) is considered. Under limit conditions at the singular points, and some natural assumptions, the problem is a singular initial problem with limit normalizing conditions at infinity. An existence and uniqueness theorem is proved and asymptotic representations of the solution are given. A numerical algorithm for evaluating the solution is proposed; calculations and their interpretation are discussed. The main singular problem under study describes the survival (non-ruin) probability of an insurance company on infinite time interval (as a function of initial surplus) in the Cramér–Lundberg dynamic insurance model with an exponential claim size distribution and certain company's strategy at the financial market assuming investment of a fixed part of the surplus (capital) into risky assets (shares) and the rest of it into a risk-free asset (bank deposit). Accompanying "degenerate" problems are also considered that have an independent meaning in risk theory.

1 Introduction

The important problem concerning the application of financial instruments in order to reduce insurance risks has been extensively studied in recent years (see, e.g., [1, 3, 4], and references therein). In particular in [3, 4] the optimal investing strategy is studied for risky and risk-free assets in Cramér–Lundberg (C.-L.) model with budget constraint, i.e., without borrowing.

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This paper complements and revises some results of [4]. The parametric singular initial problem (SIP) for an integro-differential equation (IDE) considered here is a part of the optimization problem stated and analyzed in [3, 4]: the solution of this SIP gives the survival probability corresponding to the optimal strategy when the initial surplus values are small enough. The singular problem under study is also interesting both as an independent mathematical problem and for the models in risk theory. We give more complete and rigorous analysis of this problem in comparison with [4] and add some new "degenerate" problems having independent meaning in risk theory. Some new numerical results are also discussed.

The paper is organized as follows. In Sect. 2 we set the main mathematical problem and formulate the main results concerning solvability of this problem and the solution behavior; we describe also two "degenerate" problems (when some parameters in the IDE are equal to zero) and discuss their exact solutions. In Sect. 3 we give a rather brief description of the mathematical model for which the problem in question arises (for detailed history, models' description and derivation of the IDE studied here, see [3, 4]). In Sect. 4 we describe our approach to the problem and give brief proofs of main results (for some assertions, we omit the proofs since they are given in [4]). In Sect. 5 we study an accompanying singular problem for capital stock model (the third "degenerate" problem); the results of this section are completely new. Numerical results and their interpretation are given in Sect. 6.

2 Singular Problems for IDEs and Their Solvability

2.1 Main Problem

The main singular problem under consideration has the form:

$$
\begin{aligned}
&(b^2/2)u^2\varphi''(u) + (au+c)\varphi'(u) - \lambda\varphi(u) \\
&\quad + (\lambda/m)\int_0^u \varphi(u-x)\exp(-x/m)dx = 0, \qquad 0 < u < \infty,
\end{aligned} \tag{1}
$$

$$
\{|\lim_{u\to+0}\varphi(u)|, |\lim_{u\to+0}\varphi'(u)|\} < \infty, \quad \lim_{u\to+0}[c\varphi'(u) - \lambda\varphi(u)] = 0, \tag{2}
$$

$$
0 \le \varphi(u) \le 1, \quad u \in \mathbf{R}_+, \tag{3}
$$

$$
\lim_{u\to\infty}\varphi(u) = 1, \quad \lim_{u\to\infty}\varphi'(u) = 0. \tag{4}
$$

Here in general all the parameters a, b, c, λ, m are real positive numbers.

The second limit condition at zero is a corollary of the first one and IDE (1) itself. For this IDE, conditions (2) imply $\lim_{u\to+0}[u^2\varphi''(u)] = 0$ providing a degeneracy of the IDE (1) as $u \to +0$: any solution $\varphi(u)$ to the singular problem without initial data (1), (2) must satisfy IDE (1) up to the singular point $u = 0$.

The "truncated" problem (1)–(3) (constrained singular problem) always has the trivial solution $\varphi(u) \equiv 0$. A nontrivial solution is singled out by the additional limit conditions at infinity (4).

In what follows we use notation

$$(J_m\varphi)(u) = \frac{1}{m}\int_0^u \varphi(u-x)\exp(-x/m)dx = \frac{1}{m}\int_0^u \varphi(s)\exp(-(u-s)/m)ds, \quad (5)$$

where J_m is a Volterra integral operator and $J_m : C[0,\infty) \to C[0,\infty)$, $C[0,\infty)$ is the linear space of continuous functions defined and bounded on $\mathbf{R}_+$.

For IDE (1), the entire singular problem on $\mathbf{R}_+$ was neither posed nor studied before [4] and the present paper.

2.2 Formulation of the Main Results

The problem (1)–(4) may be rewritten in the equivalent parametrized form:

$$(b^2/2)u^2\varphi''(u) + (au+c)\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbf{R}_+, \quad (6)$$

$$\lim_{u\to+0}\varphi(u) = C_0, \quad \lim_{u\to+0}\varphi'(u) = \lambda C_0/c, \quad (7)$$

$$0 \le \varphi(u) \le 1, \quad u \in \mathbf{R}_+, \quad (8)$$

$$\lim_{u\to\infty}\varphi(u) = 1, \quad \lim_{u\to\infty}\varphi'(u) = 0. \quad (9)$$

Here C_0 is an unknown parameter whose value must be defined.

Lemma 1. *For IDE (6), let the values a, b, c, λ, m be fixed with $b \neq 0$, $c > 0$, $\lambda \neq 0$, $m > 0$, $a \in \mathbf{R}$. Then for any fixed $C_0 \in \mathbf{R}$ the IDE SIP (6), (7) is equivalent to the following singular Cauchy problem (SCP) for ODE:*

$$(b^2/2)u^2\varphi'''(u) + \left[c + (b^2+a)u + b^2u^2/(2m)\right]\varphi''(u) + (a - \lambda + c/m + au/m)\varphi'(u) = 0, \quad 0 < u < \infty, \quad (10)$$

$$\lim_{u\to+0}\varphi(u) = C_0, \quad \lim_{u\to+0}\varphi'(u) = \lambda C_0/c, \quad \lim_{u\to+0}\varphi''(u) = (\lambda - a - c/m)\lambda C_0/c^2. \quad (11)$$

There exists a unique solution $\varphi(u, C_0)$ to SCP (10), (11) (therefore also to the equivalent IDE SIP (6), (7)); for small u, this solution is represented by the asymptotic power series

$$\varphi(u, C_0) \sim C_0 \left[1 + \frac{\lambda}{c} \left(u + \sum_{k=2}^{\infty} D_k u^k / k \right) \right], \qquad u \sim +0, \tag{12}$$

where coefficients D_k are independent of C_0 and may be found by formal substitution of series (12) *into ODE* (10)*, namely from the recurrence relations*

$$D_2 = -[(a-\lambda)/c + 1/m], \tag{13}$$

$$D_3 = -[D_2(b^2 + 2a - \lambda + c/m) + a/m]/(2c), \tag{14}$$

$$D_k = -\{D_{k-1}[(k-1)(k-2)b^2/2 + (k-1)a - \lambda + c/m] + D_{k-2}[(k-3)b^2/2 + a]/m\}/[c(k-1)], \quad k = 4, 5, \ldots. \tag{15}$$

Theorem 1. *For IDE* (1)*, let all the parameters a, b, c, λ, m be fixed positive numbers and let the inequality*

$$2a/b^2 > 1 \tag{16}$$

be fulfilled. Then the following statements are valid:

1. *There exists a unique solution $\varphi(u)$ of the input singular linear IDE problem* (1)–(4) *and it is a smooth (infinitely differentiable) monotone nondecreasing on $\mathbf{R}_+$ function.*
2. *The function $\varphi(u)$ can be obtained as the solution $\varphi(u, C_0)$ of IDE SIP* (6), (7)*, namely, by solving the equivalent ODE SCP* (10), (11) *where the value $C_0 = \tilde{C}_0$ must be chosen to satisfy conditions at infinity* (4) *(as the normalizing condition); for $\tilde{C}_0$ defined in this way, the restriction $0 < \varphi(u, \tilde{C}_0) < 1$ is valid for any finite $u \in \mathbf{R}_+$, i.e., for $\varphi(u) = \varphi(u, \tilde{C}_0)$, inequalities* (3) *are fulfilled tacitly.*
3. *If the inequality $m(a-\lambda) + c > 0$ is fulfilled, then the solution $\varphi(u)$ is concave on $\mathbf{R}_+$; in particular, this is true when*

$$c - \lambda m > 0. \tag{17}$$

4. *If the inequality $m(a-\lambda) + c \leq 0$ is true, then $\varphi(u)$ is convex on a certain interval $[0, \hat{u}]$ where $\hat{u}$ is an inflection point, $\hat{u} > 0$.*
5. *For small u, due to Lemma 1 above, the solution $\varphi(u)$ is represented by asymptotic power series* (12)–(15) *where $C_0 = \tilde{C}_0$, $0 < \tilde{C}_0 < 1$.*
6. *For large u, the asymptotic representation*

$$\varphi(u) = 1 - K u^{1-2a/b^2}[1 + o(1)], \qquad u \to \infty, \tag{18}$$

takes place with $K = \tilde{C}_0 \tilde{K} > 0$ where in general the value $\tilde{K} > 0$ (as well as the value $\tilde{C}_0$) cannot be determined using local analysis methods.

2.3 *The "Degenerate" Problems and Their Exact Solutions*

A particular case of IDE (1) is considered "degenerate" when some of its parameters are equal to zero.

2.3.1 The First "Degenerate" Case: $\mathbf{a = b = 0}$, $\lambda > \mathbf{0}$, $\mathbf{m > 0}$, $\mathbf{c} > \lambda \mathbf{m} > \mathbf{0}$

For this case, the "degenerate" IDE problem

$$c\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \qquad u \in \mathbf{R}_+, \tag{19}$$

$$c\varphi'(0) - \lambda\varphi(0) = 0, \quad \lim_{u\to\infty}\varphi(u) = 1, \tag{20}$$

is equivalent to the ODE problem with one parameter:

$$c\varphi''(u) + (c/m - \lambda)\,\varphi'(u) = 0, \qquad u \in \mathbf{R}_+, \tag{21}$$

$$\varphi(0) = C_0, \qquad \varphi'(0) = \lambda C_0/c, \quad \lim_{u\to\infty}\varphi(u) = 1. \tag{22}$$

Then we obtain $C_0 = \tilde{C}_0 = 1 - \lambda m/c,\ 0 < \tilde{C}_0 < 1$, and

$$\varphi(u) = \varphi(u, \tilde{C}_0) = 1 - \frac{\lambda m}{c}\exp\left(-\frac{c - \lambda m}{mc}u\right), \qquad u \in \mathbf{R}_+. \tag{23}$$

If inequality (17) is not valid, i.e., $c \le \lambda m$, then there is no solution to problem (19), (20) [resp., to problem (21), (22)].

In what follows, function (23) is well known in classical C.-L. risk theory and has an independent meaning (see further Sect. 3.1).

2.3.2 The Second "Degenerate" Case: $\mathbf{b = 0}$, $\mathbf{a > 0}$, $\mathbf{c \ge 0}$, $\lambda > \mathbf{0}$, $\mathbf{m > 0}$

For $c > 0$, the "degenerate" IDE problem

$$\begin{aligned} &(au + c)\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \qquad u \in \mathbf{R}_+, \\ &c\varphi'(0) - \lambda\varphi(0) = 0, \qquad \lim\nolimits_{u\to\infty}\varphi(u) = 1, \end{aligned} \tag{24}$$

is equivalent to the parametrized ODE problem:

$$(au+c)\varphi''(u)+(a-\lambda+c/m+au/m)\varphi'(u)=0, \quad u\in \mathbf{R}_+,$$
$$\varphi(0)=C_0, \quad \varphi'(0)=\lambda C_0/c, \quad \lim_{u\to\infty}\varphi(u)=1. \tag{25}$$

This implies $C_0=\tilde{C}_0=(a/\lambda)(c/a)^{\lambda/a}\left[(a/\lambda)(c/a)^{\lambda/a}+I_c(0)\right]^{-1}$, $0<\tilde{C}_0<1$,

$$\varphi(u)=\varphi(u,\tilde{C}_0)=1-I_c(u)\left[I_c(0)+(a/\lambda)(c/a)^{\lambda/a}\right]^{-1}, \quad u\in\mathbf{R}_+, \tag{26}$$

where, taking into account the notation $\Gamma(p,z)=\int_z^\infty x^{p-1}\exp(-x)dx, \quad p>0$, for incomplete gamma-function (see, e.g., [2]), we have

$$I_c(u)=\int_u^\infty (x+c/a)^{\lambda/a-1}\exp(-x/m)dx$$
$$=m^{\lambda/a}\exp\Big(c/(am)\Big)\Gamma\Big(\lambda/a,u/m+c/(am)\Big), \quad u\geq 0. \tag{27}$$

In particular we obtain the asymptotic representation when $u\to\infty$:

$$\varphi(u)=1-m\left[(a/\lambda)(c/a)^{\lambda/a}+I_c(0)\right]^{-1}u^{\lambda/a-1}\exp(-u/m)[1+o(1)]. \tag{28}$$

For $c=0$, the solution to the IDE problem on $\mathbf{R}_+$,

$$u\varphi'(u)-(\lambda/a)[\varphi(u)-(J_m\varphi)(u)]=0, \quad \lim_{u\to+0}\varphi(u)=0, \quad \lim_{u\to\infty}\varphi(u)=1, \tag{29}$$

can be found as a solution to the equivalent ODE problem:

$$u^2\varphi''(u)+(1-\lambda/a+u/m)u\varphi'(u)=0, \quad u\in\mathbf{R}_+,$$
$$\lim_{u\to+0}\varphi(u)=\lim_{u\to+0}[u\varphi'(u)]=0, \quad \lim_{u\to\infty}\varphi(u)=1. \tag{30}$$

This implies the same formulas (26)–(28) with $c=0$ where $\Gamma(p)=\Gamma(p,0)$ is the usual Euler gamma-function. In particular, using the formula

$$\varphi'(u)=[m^{\lambda/a}\Gamma(\lambda/a)]^{-1}u^{\lambda/a-1}\exp(-u/m), \quad u\geq 0,$$

we obtain here: if $a<\lambda$ then $\varphi'(0)=0$; if $a=\lambda$ then $\varphi'(0)=1/m$ and $\varphi(u)=1-\exp(-u/m)$; if $a>\lambda$ then the function $\varphi'(u)$ is unbounded as $u\to+0$ but integrable on $\mathbf{R}_+$.

This “degenerate” case has an independent meaning in risk theory (see further Sect. 3.2).

3 Origin of the Problem: The Cramér–Lundberg Dynamic Insurance Models

3.1 The Classical C.-L. Insurance Model

Consider the classical risk process: $R_t = u + ct - \sum_{k=1}^{N_t} Z_k,\ t \geq 0$. Here R_t is the surplus of an insurance company at time t, u is the initial surplus, c is the premium rate; $\{N_t\}$ is a Poisson process with parameter λ defining, for each t, the number of claims applied on the interval $(0,t]$; $Z_1, Z_2, \ldots$ is the series of independent identically distributed random values with some distribution $F(z)$ ($F(0) = 0$, $\mathbf{E}Z_1 = m < \infty$), describing the sequence of claims; these random values are also assumed to be independent of the process $\{N_t\}$. For this model, the positiveness condition for the net expected income ("safety loading") has the form (17).

Denote by $\tau = \inf\{t : R_t < 0\}$ the time of ruin, then $\mathbf{P}(\tau < \infty)$ is the probability of ruin at the infinite time interval.

A classical result in the C.-L. risk theory [8]: under condition (17) and assuming existence of a constant $R_L > 0$ ("the Lundberg coefficient") such that equality $\int_0^\infty [1 - F(x)] \exp(R_L x) dx = c/\lambda > 0$ holds, the probability of ruin $\xi(u)$ as a function of the initial surplus admits the estimate $\xi(u) = \mathbf{P}(\tau < \infty) \leq \exp(-R_L u)$, $u \geq 0$. Moreover, if the claims are exponentially distributed,

$$F(x) = 1 - \exp(-x/m), \quad m > 0, \quad x \geq 0, \tag{31}$$

then $R_L = (c - \lambda m)/(mc) > 0$, and the survival probability $\varphi(u) = 1 - \xi(u)$ is given by the exact formula (23), i.e., coincides with the exact solution of the first "degenerate" problem to which input singular problem (1)–(4) reduces formally as $a = b = 0$ (see Sect. 2.3.1).

For c as a bifurcation parameter, the value $c = \lambda m$ is critical: if $c \leq \lambda m$ then $\varphi(u) \equiv 0,\ u \in \mathbf{R}_+$.

3.2 The C.-L. Insurance Model with Investment into Risky Assets

Now consider the case where the surplus is invested continuously into shares with price dynamics described by geometric Brownian motion model:

$$dS_t = S_t(adt + bdw_t), \quad t \geq 0. \tag{32}$$

Here S_t is the share price at time t, a is the expected return on shares, $0 < b$ is the volatility, $\{w_t\}$ is a standard Wiener process.

Denoting by X_t the company's surplus at time t we get $X_t = \theta_t S_t$, where θ_t is the amount of shares in the portfolio. Then the surplus dynamics meets the relation $dX_t = \theta_t dS_t + dR_t$. Taking into account (32), we obtain:

$$dX_t = aX_t dt + bX_t dw_t + dR_t, \quad t \geq 0. \tag{33}$$

In contrast with the classical model, condition (17) (the positiveness of "safety loading") is not assumed here.

For the dynamical process (33), the survival probability $\varphi(u)$ satisfies on $\mathbf{R}_+$ the following linear IDE (see, e.g., [3, 7] and references therein):

$$\lambda \int_0^u \varphi(u-z)dF(z) - \lambda \varphi(u) + (au+c)\varphi'(u) + (b^2/2)u^2\varphi''(u) = 0. \tag{34}$$

From (34), assuming exponential distribution of claims (31) we get the initial IDE (1) under study.

Assuming that there exists the solution $\varphi(u)$ of IDE (1) representing the survival probability as a function of initial surplus, the following statement (further called FKP-theorem) was obtained in [7].

Theorem 2. *Suppose $b > 0$ and the claims are distributed exponentially, i.e., (31) is valid. Then:*

1. *If inequality (16) of "robustness of shares" is fulfilled, then the asymptotic representation (18) holds with a certain constant $K > 0$.*
2. *If $2a/b^2 < 1$, then $\varphi(u) \equiv 0$, $u \in \mathbf{R}_+$.*

3.3 The C.-L. Model with Investment into a Risk-Free Asset

The model under study comprises a more general case where only a constant part α $(0 < \alpha < 1)$ of the surplus is invested in shares (with the expected return μ and volatility σ) whereas remaining part $1-\alpha$ is invested into a risk free asset (bank deposit with constant interest rate $r > 0$): the case $0 < \alpha < 1$ may be reduced to the case $\alpha = 1$ by a simple change of the parameters (shares characteristics), namely $a = \alpha\mu + (1-\alpha)r$, $b = \alpha\sigma$.

Moreover, when the surplus is invested entirely into a risk free asset (bank deposit with constant interest rate), we obtain the second "degenerate" problem (with or without premiums) to which the input singular problem (1)–(4) reduces formally as $b = 0$. For $a > 0$, $\lambda > 0$, $m > 0$, $c \geq 0$, there exists the exact solution (26), (27) and the asymptotic representation (28) is valid (for details, see Sect. 2.3.2).

Thus when the surplus is entirely invested into a risk free asset then the survival probability is not equal to zero, for $u > 0$, even if premiums (insurance payments) are absent ($c = 0$) and has a good asymptotic behavior as $u \to \infty$.

As far as we know formulas (26)–(28) are new for risk theory.

4 On the Approach to Main Problem and Proofs of Main Results

4.1 The Singular Problem for IDE: Uniqueness of the Solution and Its Monotonic Behavior

As shown in Sect. 3, we can formulate the input singular IDE problem in the form (6), (7), (9), where operator J_m is defined by (5), C_0 is an unknown parameter whose value must be found, and, for the solution to the problem (6), (7), (9), the restrictions needed are (8).

Lemma 2. *For IDE (6), let the values a, b, c, λ and m be fixed with $c > 0$, $\lambda > 0$, $m > 0$ whereas a and b are any real numbers ($a, b \in \mathbf{R}$). Then the following assertions are valid:*

1. *If there exists a solution $\varphi_1(u) = \varphi_1(u, C_0)$ to problem (6), (7), (9) with some $C_0 > 0$, then it is a unique solution to this problem.*
2. *Such $\varphi_1(u)$ satisfies restrictions (8), $0 < C_0 < 1$ and $\varphi_1'(u) > 0$ for any finite $u \in \mathbf{R}_+$, i.e., $\varphi_1(u)$ is a monotone nondecreasing on $\mathbf{R}_+$ function.*

Proof.

1. Supposing the opposite, let $\varphi_2(u)$ be any other solution to problem (6), (7), (9), i.e., $\varphi_2(u) \not\equiv \varphi_1(u)$. Then two cases may occur: the first one with $\lim_{u\to+0} \varphi_2(u) = \lim_{u\to+0} \varphi_1(u)$ and the second one with $\lim_{u\to+0} \varphi_2(u) \neq \lim_{u\to+0} \varphi_1(u)$.

   For the first case, it follows that there exists a nontrivial solution $\tilde{\varphi}(u)$ of IDE (6) satisfying conditions $\lim_{u\to+0} \tilde{\varphi}(u) = \lim_{u\to\infty} \tilde{\varphi}(u) = 0$. Let $0 < \tilde{u}$ be its maximum point: $\tilde{\varphi}(\tilde{u}) = \max_{u\in[0,\infty)} \tilde{\varphi}(u) > 0$ (if $\tilde{\varphi}(u)$ takes only nonpositive values, then we consider the solution $-\tilde{\varphi}(u)$ instead). Then $\tilde{\varphi}'(\tilde{u}) = 0$, $\tilde{\varphi}''(\tilde{u}) \leq 0$. But from IDE (6) a contradiction follows:

$$(b^2/2)\tilde{u}^2\tilde{\varphi}''(\tilde{u}) = \lambda[\tilde{\varphi}(\tilde{u}) - m^{-1}\int_0^{\tilde{u}} \tilde{\varphi}(s)\exp\big(-(\tilde{u}-s)/m\big)ds]$$
$$\geq \lambda\tilde{\varphi}(\tilde{u})\left[1 - m^{-1}\int_0^{\tilde{u}} \exp\big(-(\tilde{u}-s)/m\big)ds\right]$$
$$= \lambda\tilde{\varphi}(\tilde{u})\exp(-\tilde{u}/m) > 0. \tag{35}$$

   For the second case, there exists a linear combination of solutions $\hat{\varphi}(u) = c_1\varphi_1(u) + c_2\varphi_2(u)$ such that $\hat{\varphi}(u) \not\equiv 1$ and satisfies conditions $\lim_{u\to+0} \hat{\varphi}(u) = \lim_{u\to\infty} \hat{\varphi}(u) = 1$. If there exists a value $\hat{u} > 0$ with $\hat{\varphi}(\hat{u}) > 1$, then the first case argument is valid. Otherwise, the inequality $\hat{\varphi}(u) \leq 1 \quad \forall u \in \mathbf{R}_+$ contradicts to $\lim_{u\to+0} \hat{\varphi}'(u) = \lambda/c > 0$ which follows from (7).
2. The other assertions are proved analogously. □

4.2 SCPs for Accompanying Linear ODEs

4.2.1 Reduction of the Second-Order IDE to a Third-Order ODE

The known possibility of reducing the second-order IDE (6) to a third-order ODE is important for further exposition. First, we note that

$$(J_m\varphi)'(u) = \frac{1}{m}\left(\exp(-u/m)\int_0^u \varphi(x)\exp(x/m)dx\right)' \\ = [\varphi(u) - (J_m\varphi)(u)]/m. \quad (36)$$

Then differentiating IDE (6) in view of (36) gives a linear third-order IDE

$$(b^2/2)u^2\varphi'''(u) + [(b^2+a)u+c]\varphi''(u) + (a-\lambda)\varphi'(u) \\ +(\lambda/m)[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbf{R}_+, \quad (37)$$

which also implies the limit condition

$$\lim_{u\to+0}[c\varphi''(u) + (a-\lambda)\varphi'(u) + (\lambda/m)\varphi(u)] = 0. \quad (38)$$

Together with the input limit condition (2), it implies the limit equality

$$\lim_{u\to+0}[c\varphi''(u) + (a-\lambda+c/m)\varphi'(u)] = 0. \quad (39)$$

In order to remove the integral term, we add IDE (37) and initial IDE (6) multiplied by $1/m$ and get the linear third-order ODE (10). Then the same limit condition (39) must be fulfilled to provide a degeneration of this ODE as $u \to +0$.

Suppose $\psi(u) = \varphi'(u)$ and rewrite ODE (10) in more canonical forms for ODEs with pole-type singularities at zero and infinity (for classification of isolated singularities of linear ODE systems and general theory of ODEs of this class, see, e.g., the monographs [5, 6, 10] complementing each other). Now, for $\psi(u)$, we have to study the following singular ODEs: for small u, we need to consider the equation

$$(b^2/2)u^3\psi''(u) + \left[c + (b^2+a)u + b^2u^2/(2m)\right]u\psi'(u) \\ + \left[(a-\lambda+c/m)u + au^2/m\right]\psi(u) = 0, \quad u > 0, \quad (40)$$

and for large u, we shall consider the same equation in the form

$$(b^2/2)\psi''(u) + \left[c/u^2 + (b^2+a)/u + b^2/(2m)\right]\psi'(u) \\ + \left[(a-\lambda+c/m)/u^2 + (a/m)/u\right]\psi(u) = 0, \quad u > 0. \quad (41)$$

We see that both ODE (40) and equivalent ODE (41) have irregular (strong) singularities of rank 1 as $u \to +0$ and as $u \to \infty$.

4.2.2 Singularity at Zero: Replacement of the SIP for IDE by an Equivalent SCP for ODE

Proof of Lemma 1

First, we must show that the previous transformations permit us to replace the input SIP (6), (7) for an IDE by the SCP (10), (11) for an ODE.

In the straight direction (from the IDE SIP to the ODE SCP), the statement is evident. Now let $\tilde{\varphi}(u) = \tilde{\varphi}(u, C_0)$ be a solution of ODE SCP (10), (11). We have to prove that $\tilde{\varphi}(u)$ satisfies IDE (6).

Denote the left part of IDE (6) with the function $\tilde{\varphi}(u)$ by $g(u)$. We have to prove that $g(u) \equiv 0$. Indeed, the way ODE (10) was derived means that $g(u)$ meets the first-order ODE

$$g'(u) + g(u)/m = 0, \quad 0 \leq u < \infty,$$

with the general solution of the form $g(u) = \tilde{C}\exp(-u/m)$ where $\tilde{C}$ is an arbitrary constant. Since $\tilde{\varphi}(u, C_0)$ meets conditions (11), it follows from IDE (6) that $g(0) = 0$. This implies $\tilde{C} = 0$, i.e., $g(u) \equiv 0$.

The other statements of Lemma 1 follow from the results of [9] (see [4] for details).

4.2.3 SCP at Infinity and Its Two-Parameter Family of Solutions

For $\psi(u) = \varphi'(u)$, we have an SCP at infinity for the second-order ODE (41) with the conditions

$$\lim_{u\to\infty} \psi(u) = \lim_{u\to\infty} \psi'(u) = 0. \tag{42}$$

Using the known results for linear ODEs with irregular singularities, we obtain the following assertions (more complete in comparison with FKP-theorem).

Lemma 3. *For ODE (41), suppose that $b \neq 0$, $a > 0$, $m > 0$ whereas λ and c are arbitrary real numbers ($\lambda, c \in \mathbf{R}$). Then:*

1. *Any solution to ODE (41) satisfies conditions (42) so that SCP (41), (42) at infinity has a two-parameter family of solutions $\psi(u, d_1, d_2)$ where d_1 and d_2 are arbitrary constants.*
2. *For this family, the following representation holds:*

$$\psi(u, d_1, d_2) = d_1 u^{-2a/b^2}[1 + \chi_1(u)/u] + d_2 u^{-2}\exp(-u/m)[1 + \chi_2(u)/u]; \tag{43}$$

here the functions $\chi_j(u)$ *have finite limits as* $u \to \infty$ *and, for large u, can be represented by asymptotic series in inverse integer powers of u,*

$$\chi_j(u) \sim \sum_{k=0}^{\infty} \chi_j^{(k)}/u^k, \quad j = 1,2, \tag{44}$$

where the coefficients $\chi_j^{(k)}$ *may be found by substitution of* (43), (44) *in ODE* (41) *(*$j = 1,2,\ \ k \geq 0$*).*

3. *All solutions of the family* (43) *are integrable at infinity iff inequality* (16) *is fulfilled.*

For a detailed proof of Lemma 3, see [4].

Corollary 1. *Under the assumptions of Lemma 3, all solutions of ODE* (10) *have finite limits as* $u \to \infty$ *iff condition* (16) *is fulfilled.*

Summarizing all results, we obtain the proof of Theorem 1.

5 The Accompanying Singular Problem for Capital Stock Model (The Third "Degenerate" Case: $\mathbf{c = 0, b \neq 0, a > 0, \lambda > 0, m > 0}$)

For this case, the input singular IDE problem has the form:

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbf{R}_+, \tag{45}$$

$$\lim_{u\to+0} \varphi(u) = \lim_{u\to+0} [u\varphi'(u)] = 0, \tag{46}$$

$$\lim_{u\to\infty} \varphi(u) = 1, \quad \lim_{u\to\infty} \varphi'(u) = 0, \tag{47}$$

and restrictions (3) are needed for the solution.

The following lemma is analogous to Lemma 2 (with a similar proof).

Lemma 4. *For IDE* (45)*, let the values a, b,* λ*, and m be fixed with* $\lambda > 0$*,* $m > 0$ *whereas a and b are any real numbers (*$a, b \in \mathbf{R}$*). Then the following assertions are valid:*

1. *If there exists a solution* $\varphi_1(u)$ *to the problem* (45)–(47)*, then it is a unique solution to this problem.*
2. *Such* $\varphi_1(u)$ *satisfies restrictions* (3) *and* $\varphi_1'(u) > 0$ *for any finite* $u > 0$*, i.e.,* $\varphi_1(u)$ *is a monotone nondecreasing on* $\mathbf{R}_+$ *function.*

Analogously to the previous approach, the singular IDE problem (45)–(47) is equivalent to the following singular ODE problem:

$$(b^2/2)u^3\varphi'''(u) + \left[b^2 + a + b^2u/(2m)\right]u^2\varphi''(u)$$
$$+ (a - \lambda + au/m)u\varphi'(u) = 0, \quad 0 < u < \infty, \tag{48}$$

$$\lim_{u\to+0}\varphi(u) = \lim_{u\to+0}[u\varphi'(u)] = \lim_{u\to+0}[u^2\varphi''(u)] = 0, \tag{49}$$

$$\lim_{u\to\infty}\varphi(u) = 1, \qquad \lim_{u\to\infty}\varphi'(u) = \lim_{u\to\infty}\varphi''(u) = 0. \tag{50}$$

First, consider SCP at regular (weak) singular point $u = 0$, i.e., SCP (48), (49) introducing notation

$$\mu_1 = 1/2 - a/b^2 + \sqrt{(1/2 - a/b^2)^2 + 2\lambda/b^2}, \tag{51}$$

$$d_1 = \mu_1 + a/b^2, \quad d_2 = \mu_1 + 2a/b^2 - 1. \tag{52}$$

The following lemma is analogous to Lemma 1.

Lemma 5. *For IDE (45), let the values a, b, λ, m be fixed with $b \neq 0$, $\lambda > 0$, $m > 0$, $a \in \mathbf{R}$. Then:*

1. *The IDE SIP (45), (46) is equivalent to the ODE SCP (48), (49).*
2. *There exists a one-parameter family of solutions $\varphi(u, P_1)$ to the ODE SCP (48), (49) (therefore also to the equivalent IDE SIP (45), (46)) and the following representation holds:*

$$\varphi(u, P_1) = P_1 \int_0^u s^{\mu_1 - 1}\eta(s)ds; \tag{53}$$

here P_1 is a parameter, $0 < \mu_1$ is defined by (51), and $\eta(u)$ is a solution to SCP

$$u^2\eta''(u) + (2d_1 + u/m)u\eta'(u) + (d_2u/m)\eta(u) = 0, \quad u > 0, \tag{54}$$

$$\lim_{u\to+0}\eta(u) = 1, \quad \lim_{u\to+0}[u\eta'(u)] = 0, \tag{55}$$

where d_1 and d_2 are defined by (52); there exists a unique solution $\eta(u)$ to the SCP (54), (55) and it is a holomorphic function at the point $u = 0$,

$$\eta(u) = 1 + \sum_{k=1}^{\infty} P_{k+1}u^k, \quad |u| \le u_0, \quad u_0 > 0, \tag{56}$$

where the coefficients P_{k+1} may be found by formal substitution of series (56) into ODE (54), namely, from the recurrence relations:

$$P_2 = -d_2/(2md_1), \tag{57}$$

$$P_{k+1} = -P_k(k - 1 + d_2)/[mk(k - 1 + 2d_1)], \quad k = 2, 3, \ldots; \tag{58}$$

moreover, if $D_1 = \lim_{u\to+0}\varphi'(u, P_1)$, then $D_1 = 0$ when $a < \lambda$; $D_1 = P_1$ when $a = \lambda$; and at last $|D_1| = \infty$ when $a > \lambda$ (but $\varphi'(u, P_1)$ is integrable as $u \to +0$).

Summarizing the results and taking into account that Lemma 3 and Corollary 1 are valid for any $c \in \mathbf{R}$, we obtain

Theorem 3. *For IDE (45), let all the parameters a, b, λ, m be fixed positive numbers and let inequality (16) of "robustness of shares" be fulfilled. Then the following assertions are valid:*

1. *There exists a unique solution $\varphi(u)$ of singular linear IDE problem (45)–(47); it satisfies restrictions (3) and, for $u > 0$, is a smooth monotone nondecreasing function.*
2. *Such $\varphi(u)$ can be obtained by the formula*

$$\varphi(u) = \int_0^u s^{\mu_1 - 1} \eta(s) ds \Big/ \int_0^\infty s^{\mu_1 - 1} \eta(s) ds, \qquad u \geq 0, \tag{59}$$

   where $\eta(u)$ is defined in Lemma 5.
3. *For finite $u > 0$, the solution $\varphi(u)$ is represented by a convergent series which can be obtained using formulas (59), (56)–(58).*
4. *If $a > \lambda$, then the solution $\varphi(u)$ is concave on $\mathbf{R}_+$ with $\lim_{u\to+0} \varphi'(u) = \infty$ but $\varphi'(u)$ is an integrable on $\mathbf{R}_+$ function.*
5. *If $a \leq \lambda$, then $\varphi(u)$ is convex on a certain interval $[0, \hat{u}]$ where $\hat{u}$ is an inflection point, $\hat{u} > 0$; moreover if $a < \lambda$, then $\lim_{u\to+0} \varphi'(u) = 0$ whereas if $a = \lambda$, then $\lim_{u\to+0} \varphi'(u) = 1/\int_0^\infty \eta(s)ds > 0$.*
6. *For large u, the asymptotic representation (18) holds with $K > 0$ where in general the value $K > 0$ cannot be determined using local analysis methods.*

6 Numerical Examples and Their Interpretation

For the main case $c > 0$, our study shows that the input singular IDE problem (1)–(4) may be reduced to the auxiliary ODE SCP (10), (11) with the parameter C_0 to be defined, $0 < C_0 < 1$. The asymptotic expansion of the solutions at zero (12) is used to transfer the limit initial conditions (11) from the singular point $u = 0$ to a nearby regular point $u_0 > 0$; the derivatives of the solution may be evaluated by formal differentiation of the representation (12). Consequently, a regular Cauchy problem is to be solved starting from the point $u_0 > 0$. The parameter C_0 in (12) is evaluated numerically to satisfy the condition $\lim_{u\to\infty} \varphi(u) = 1$.

For the additional case $c = 0$, the singular IDE problem (45)–(47) is equivalent to the singular ODE problem (48)–(50). To solve this problem we use formula (59) and the auxiliary SCP (54), (55). The convergent power series (56)–(58) is used to transfer limit initial conditions (55) from the singular point $u = 0$ to a regular point $u_0 > 0$, and then a regular Cauchy problem is to be solved starting from this point.

Maple programming package was used as a numerical tool.

For all examples, we put $m = 1$, $\lambda = 0.09$, and for $a > 0$, $b \neq 0$, the shares are "robust": $2a/b^2 > 1$ (Figs. 1–5).

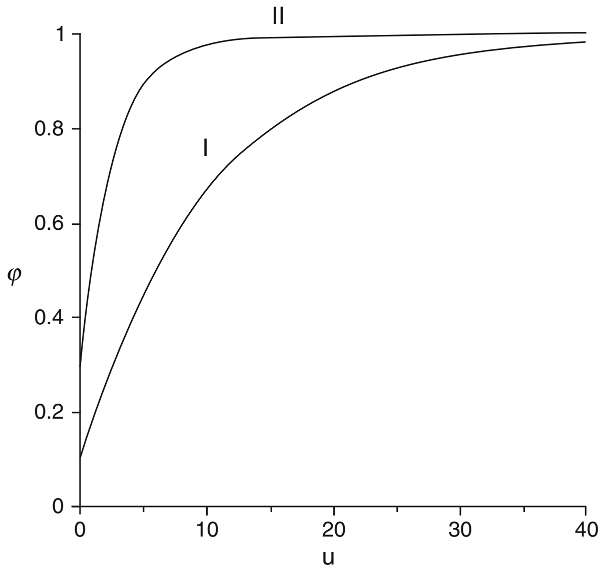

Fig. 1 The case $\mathbf{c} > \lambda\mathbf{m}$: $c = 0.1$; **I:** $\mathbf{a} = \mathbf{b} = \mathbf{0}$ (the first "degenerate" case with the exact solution); $C_0 = \varphi(0) = 0.1$, $D_1 = \varphi'(+0) = 0.09$; **II:** $\mathbf{a} = \mathbf{0.02}$, $\mathbf{b} = \mathbf{0.1}$; $C_0 = 0.295$, $D_1 = 0.265$

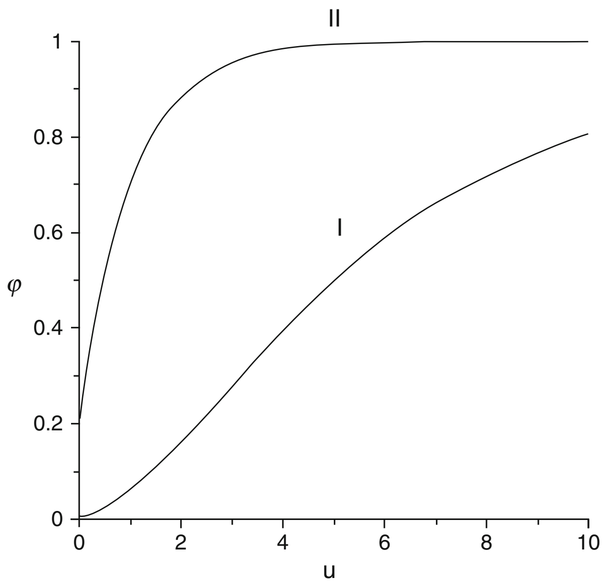

Fig. 2 The case $\mathbf{c} < \lambda\,\mathbf{m}$: $c = 0.02$, $\mathbf{b} = \mathbf{0.1}$; **I:** $\mathbf{a} = \mathbf{0.02}$ ($\mathbf{m}(\lambda - \mathbf{a}) > \mathbf{c}$: $\varphi(u)$ has an inflection); $C_0 = 0.00527$, $D_1 = 0.0237$; **II:** $\mathbf{a} = \mathbf{0.1}$ ($\mathbf{m}(\lambda - \mathbf{a}) < \mathbf{c}$: $\varphi(u)$ is concave); $C_0 = 0.194$, $D_1 = 0.872$

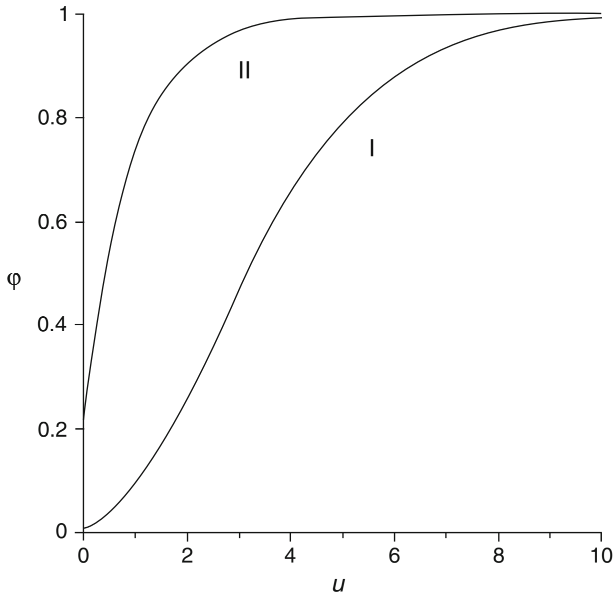

Fig. 3 The second "degenerate" case with premiums: $\mathbf{b} = \mathbf{0}$, $c = 0.02$ ($\mathbf{c} < \lambda\,\mathbf{m}$); **I:** $\mathbf{a} = \mathbf{0.02}$ ($\mathbf{m}(\lambda - \mathbf{a}) > \mathbf{c}$); $C_0 = 0.00704$, $D_1 = 0.0317$; **II:** $\mathbf{a} = \mathbf{0.1}$ ($\mathbf{m}(\lambda - \mathbf{a}) < \mathbf{c}$); $C_0 = 0.2046$, $D_1 = 0.9207$

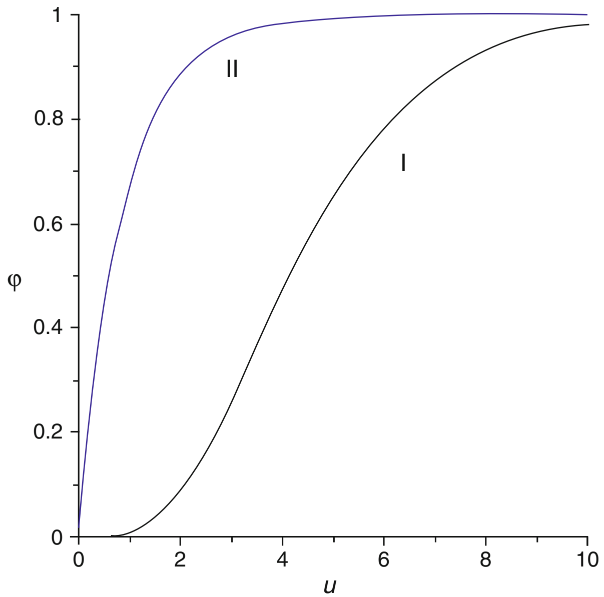

Fig. 4 The second "degenerate" case without premiums: $\mathbf{b} = \mathbf{0}, \mathbf{c} = \mathbf{0}$; **I:** $\mathbf{a} = \mathbf{0.02}$ ($\lambda > \mathbf{a}$); $\varphi(0) = \varphi'(0) = 0$; **II:** $\mathbf{a} = \mathbf{0.1}$ ($\lambda < \mathbf{a}$); $\varphi(0) = 0$, $\varphi'(+0) = \infty$

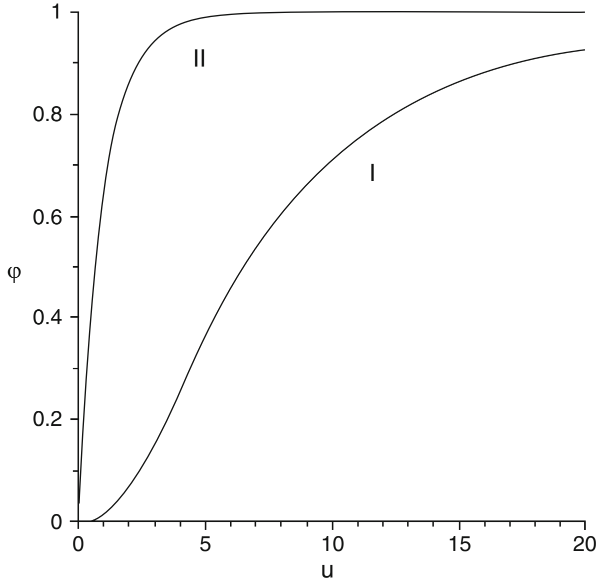

Fig. 5 The third "degenerate" case (the capital stock model): $\mathbf{c} = \mathbf{0}$, $\mathbf{b} = \mathbf{0.1}$; **I:** $\mathbf{a} = \mathbf{0.02}$ ($\lambda > \mathbf{a}$); $\varphi(0) = \varphi'(0) = 0$, $P_1 = 0.059587$; **II:** $\mathbf{a} = \mathbf{0.1}$ ($\lambda < \mathbf{a}$); $\varphi(0) = 0$, $\varphi'(+0) = \infty$, $P_1 = 0.861816$

7 Conclusions

The study shows that use of risky assets is not favorable for non-ruin with large initial surplus values and constant structure of the portfolio. However, the study of the cases when positiveness of the safety loading does not hold shows risky assets to be effective for small initial surplus values: while ruin is inevitable in the case without investing, the survival probability grows considerably as u grows in presence of investing even if the premiums are absent (moreover, the second derivative of the solution for small u is positive!). The study in [3, 4] of the optimal strategy for exponential distribution of claims shows that the part of risky investments should be $O(1/x)$ as present surplus x tends to infinity.

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