1. Organizational and methodological issues

- *The aim of the course.* The course "Holomorphic Dynamics" is a graduate-level special topics course aimed at mastering concepts and tools of modern one-dimensional holomorphic dynamics, specifically topological dynamics of polynomials and rational functions on the Riemann sphere.
- *Objectives of the course.* We will discuss basic examples of holomorphic dynamical systems with some emphasis on 1-dimensional rational functions. The principal objective of the course is to give sufficiently many meaningful and interesting examples rather than to introduce a general theory.

  Upon successful completion of the course, the students will be able to understand contemporary research papers in the field of holomorphic dynamics as well as conduct their own research.

  A student will *know* the following concepts: topological dynamical properties, topological conjugations, classification of periodic points (attracting, repelling, neutral, parabolic, Siegel, Cremer), Poncelet's correspondence as a holomorphic dynamical system, Newton's method as a holomorphic dynamical system, Chebyshev polynomials, the Julia set and the Fatou set of a rational function, the classification of Fatou components (attracting or super-attracting basins, parabolic basins, Siegel disks, Herman rings), the Mandelbrot set, the main cardioid, hyperbolic components, dynamical rays, landing patterns, invariant geodesic laminations, leaves and gaps, critical leaf laminations, majors and minors, the quadratic minor lamination.

  A student will be *able to*: distinguish topological types of periodic points of rational functions, study landing patterns of external rays of polynomials using the language of invariant laminations, study orbit forcing, carry out computer experiments and produce pictures of Julia sets, parameter spaces, orbits, etc.
- *Original methodological approaches used in the course.* A dynamical system is any system that evolves in time according to a fixed, prescribed law. Planetary motion, pendulum mechanisms, nuclear chain reactions, reproduction of biological species — all these events are modeled by dynamical systems. A branch of the mathematical theory of dynamical systems, called holomorphic dynamics, deals with dynamical systems that are given by analytic formulas, and therefore can be "complexified". As often happens in mathematics, it is easier to work with complex numbers than with real numbers. Thus the study of holomorphic dynamical systems is important.

  The following will be the main example for us. A rational function of one complex variable can be regarded as a self-mapping of the Riemann sphere. Iterates of this mapping form a dynamical system (if every point z "evolves" into $f(z)$ in time 1, then, since the law of evolution is fixed, a point z will "evolve" into "f applied n times to z" in time n). Even simple and familiar from high-school quadratic polynomials reveal extremely delicate structure when regarded as dynamical systems, and give rise to beautiful fractal pictures.

  Our exposition of complex dynamics has a distinct geometric flavor. We use visualization tools (including computer graphics) systematically. A general teaching philosophy, according to which this course has been developed, is the following:

the exposition should be arranged as the way from particular examples to general theorems. We consistently consider particular examples as being more important than general results.

Every technique we use is first explained in the simplest partial case, which makes the presentation clearer at the cost of being less "efficient". We rely on the principle that it is better to know different proofs of the same theorem rather than the same proof of different theorems. Sometimes the students are requested to fill in details of proofs.

- *The place of the course in the system of innovative qualifications that are formed in the course of study.* The course is offered to the (first or second year) Master of Science students in Mathematics, as well as some highly motivated and well-prepared undergraduate students. This is a part of an international M.Sc. programme conducted by the department of Mathematics in English. We assume a strong background in one-dimensional complex analysis. The course discusses advanced material and provides a necessary background for conducting a front-end independent research in the area of complex dynamics.

2. The content of the course

What makes this course unique (description of scientific and methodological features, comparison with similar courses offered by the NRU HSE and other universities in Russia and worldwide). The author of the course has created an independent, original syllabus, and will be implementing it for the first time in Fall 2013. There are special topics courses in dynamical systems offered by the HSE, in particular, "Introduction to Ergodic Theory", that discuss different aspects of dynamics. The course "Quasi-conformal mappings" is a direct continuation of this course. Courses in holomorphic dynamics are taught by experts as advanced graduate-level courses. The author has studied the syllabi of courses in complex dynamics that are being taught in the Harvard University (USA), the University of Toronto (Canada), Stony Brook University (USA), Université de Toulouse (France), University of Indiana at Bloomington (USA), etc. Some introductory part is common in all these courses, and the same standard material is included into our course. However, all courses deviate in the choice of more advanced directions.

Our choice follows some ideas of Thurston expressed in his paper "On the geometry and dynamics of rational functions". In particular, we review the notion of Thurston's laminations introduced in that paper. These are combinatorial objects providing topological models for polynomials with locally connected Julia sets. Studying invariant laminations requires almost no background and is accessible to undergraduate students. In general, we tried to make it possible to combine participation in this holomorphic dynamics course with an advanced complex analysis course. To this end, we start with the part of the theory that uses only the most basic concepts and tools of complex analysis. As we proceed, more advanced analytic tools are gradually employed.

We often start with applications, motivate a certain concept of a tool, then introduce it in the simplest possible set-up, after which we either proceed with a general discussion or just hint at how a general theory may develop. A traditional for this course methodological trick is the following: many particular examples are worked out; sometimes the same theorem is proved in different ways; sometimes a particular case is proved before the general statement, etc.

An introduction to the course. One can find dynamical systems everywhere. A dynamical system is any system that evolves in time according to a fixed, predetermined law. Examples include planetary motion, animal population growth, flow of water, pendulum mechanisms, nuclear chain reactions, to name a few. Dynamical systems can be continuous or discrete. Continuous systems are usually described by differential equations. Discrete systems are described by iterations of the same map. Let X be a set that represents all possible positions or states of our system. The time is assumed to be discrete, so we can only talk about the position at time 0, at time 1, time 2 etc. If the position of the system at time 0 is x, then the position of the same system at time 1 will be $f(x)$, where $f : X \to X$ is some map. Suppose that the position at time t depends only on the position immediately before, i.e. at time $t-1$, rather than on several preceding positions. Then, knowing the initial position x (the position at time 0), we can find all subsequent positions. These will be

$$f(x),\ f(f(x)),\ f(f(f(x))),\ \dots$$

The maps f, $f^{\circ 2} = f \circ f$, $f^{\circ 3} = f \circ f \circ f$, $\dots$, are called *iterations* of the map f. Given an initial position x, subsequent positions can be obtained by evaluating the iterations of f on x. These positions form a sequence $x_0 = x$, $x_1 = f(x)$, $x_2 = f^{\circ 2}(x)$, $\dots$ called the *orbit* of x. The orbit of x describes the evolution of our system in time.

Example 1. Population growth of a certain animal species (say, rabbits) can be modeled by a discrete dynamical system. Here X is the set of all nonnegative real numbers (well, the number of rabbits is usually an integer but a very large one; if we divide the number of rabbits by sufficiently large integer, then the ratio can change almost continuously — it is convenient to represent it by a real number). We measure the population size every year (or every month). Suppose that the population size at time n is x_n. Then the population dynamics can often by approximated by a recurrence relation of the form

$$x_{n+1} = f(x_n),$$

where f is a suitably chosen function. E.g. for small population sizes, a linear function usually works:

$$f(x) = \lambda x.$$

For $\lambda > 1$, this means an exponential growth of population. Of course, at a certain moment, this model cease to be adequate. If there are too many rabbits, they may lack food, which prevents the further exponential growth of population. Another very popular model uses a quadratic function f:

$$f(x) = \lambda x(1 - x).$$

For small x, it has almost the same dynamics as the linear function $x \mapsto \lambda x$, but for larger x different (and very complicated) dynamics can show up.

Example 2. To experiment with dynamical systems, one can use a pocket calculator. Scientific calculator is preferable, with many function buttons, say, the cosine button. Enter some number, and press the button. You will get the value of the function (cosine in our example) at the number you have entered. Now, if you press the same button, again, you will see the second iteration of this function, and so on. You just need to keep pressing the same button to see the evolution of the dynamical system on the screen.

In this course, we will study one-dimensional holomorphic dynamical systems, which means that X will be a Riemann surface, and f will be a holomorphic map. Examples of such maps include all rational functions (i.e. ratios of two polynomials) of one variable regarded as maps from the Riemann sphere to itself. (The Riemann sphere $\overline{\mathbb{C}}$ is the compact Riemann surface obtained from $\overline{\mathbb{C}}$ by adding one point at infinity; it is the same as the 1-dimensional complex projective line $\overline{\mathbb{C}}\mathrm{P}^1$) In fact, dynamics of rational functions on the Riemann sphere will be our main subject. Even very simple rational functions can exhibit very complicated dynamical behavior. E.g. there are many open questions about dynamics of quadratic polynomials $z \mapsto z^2 + c$.

Plan of the course. Allocation of classroom hours among the topics and types of assignments.

No	Topic	weeks	Total hours	Lectures
1	Examples	2	10	4
2	Fatou and Julia sets	2	10	4
3	Böttcher's theorem and external rays	1	5	2
4	The Mandelbrot set and its local connectivity	2	15	4
5	Classification of periodic Fatou components	3	15	6
6	The measurable Riemann mapping theorem	2	15	4
7	Analytic parameterization of hyperbolic components	2	10	4
8	Invariant laminations and their properties	2	10	4
Total		16	90	32

The structure of the syllabus.

Section 1: Examples. These include fractional linear transformations (we give a complete classification of their topological dynamics), Newton's method for polynomial equations, small perturbations of the map $z \mapsto z^2$, the Chebyshev polynomials and the Lattés examples (these are examples of completely integrable holomorphic dynamical systems), the Poncelet correspondence as a holomorphic dynamical system.

References: [1, 2, 6]

Section 2: Fatou and Julia sets. We give definitions and discuss the simplest properties of the Fatou and Julia sets of rational functions. These properties illustrate the informal description of the Fatou set as the set, on which the dynamics is stable (and in general simple) and the informal description of the Julia set as the set, on which the dynamics is unstable and chaotic.

References: [1, 2, 3, 6]

Section 3: Böttcher's theorem and external rays. The Böttcher theorem provides a dynamically meaningful coordinate system near a fixed (or periodic) super-attracting point of a holomorphic map. Suppose that the leading term of the map near this fixed point is z^d. Then there is a local holomorphic semi-conjugacy of the map with $z \mapsto z^d$. In other words, there is a local polar coordinates system such that the map raises the radial coordinate to the dth power and multiplies the angular coordinate by d. Böttcher coordinates allow to define dynamical rays as loci given by fixing a value of the angular coordinate. The external rays of a polynomial, i.e., the dynamical rays associated with the super-attracting fixed point at infinity, play a crucial role in studying topological dynamics of the polynomial.

References: [1, 2, 3]

Section 4: The Mandelbrot set. The *Mandelbrot set* is is perhaps the most well-known mathematical set outside of the mathematical community. It can be defined as the set of all complex numbers c such that the sequence

$$c, \quad c^2+c, \quad (c^2+c)^2+c, \dots$$

is bounded (equivalently, does not tend to infinity; equivalently, is bounded by 2). The Mandelbrot set describes the family of quadratic polynomials $Q_c(z) = z^2 + c$. A point c belongs to the Mandelbrot set if and only if the forward orbit of 0 under Q_c escapes to infinity.an What is special about 0? It is the critical point of Q_c. Actually, the only finite critical point (the other critical point is at infinity). A general observation in complex dynamics is that the behavior of critical points is the most important, and determines to a large extent the dynamical behavior of all other points. The Mandelbrot set serves as a "table of contents" for quadratic polynomial dynamics.

References: [1, 3, 4, 5]

Section 5: Classification of Fatou components. Any periodic Fatou component of a rational function is an attracting basin, a super-attracting basin, a parabolic basin, a Siegel disk or a Herman ring. We discuss the structure of these components, including their dynamical parameterizations (the so called Fatou coordinates). A famous theorem of Sullivan, which will be proven later using the quasi-conformal techniques, states that there are no wandering components, i.e., every component is eventually periodic. This theorem completes the classification of Fatou components.

References: [1, 3, 6]

Section 6: Measurable Riemann mapping theorem. This theorem, due to Ahlfors and Bers, is of fundamental importance for holomorphic dynamics. An important viewpoint on rational functions is that they are branched covering preserving some conformal structure. The Ahlfors–Bers theorem states that, under fairly general assumptions on the conformal structure (which do not at all impose the smoothness of a conformal structure, not even the continuity), the conformal structure is integrable, i.e., there is an isomorphism between this conformal structure and the standard conformal structure on the Riemann sphere.

References: [3, 4, 5]

Section 7: Analytic parameterization of hyperbolic components The quasi-conformal methods allow to parameterize all hyperbolic components of the Mandelbrot set. If c is a hyperbolic parameter, then c, viewed as a point in the dynamical plane, is in some attracting basin. The dynamical coordinate of c in this basin defines the parameter coordinate of c. The surjectivity of the mapping thus obtained is based on the measurable Riemann mapping theorem.

References: [3, 5]

Section 8: Invariant laminations and their properties Invariant laminations were introduced by W. Thurston as models for the topological dynamics of polynomials on their Julia sets (provided that the latter are locally connected). A quadratic invariant lamination has one or two longest leaves, called *major*. The image of a longest leaf is called a *minor*. A crucial fact established by Thurston is that distinct minors do not cross inside the unit disk; this led to his construction of a combinatorial model of the Mandelbrot set.

Thurston's argument is based upon the fact that majors of a quadratic lamination never enter the region between them, a result that fails in the cubic case.

References:

[5, 8]

References

[1] J. Milnor, *Dynamics in one complex variable*, 3rd Ed., Annals of Mathematics Studies (Book 160), Princeton University Press 2006.

[2] H.-O. Peitgen, P. Richter, *The Beauty of Fractals: Images of Complex Dynamical Systems*, Springer 1986

[3] L. Carleson, T.W. Gamelin, *Complex dynamics*, Springer 1992

[4] T.W. Gamelin, *Complex Analysis*, Springer, 2001

[5] W. Thurston, *On the geometry and dynamics of iterated rational maps*, in: Complex Dynamics: Families and Friends, D. Schleicher (Ed.), A.K. Peters 2009

Additional references:

[6] A.F. Beardon, *Iteration of Rational Functions: Complex Analytic Dynamical Systems*, Springer 2000

[7] M.H.A. Newman, *Elements of the Topology of Plane Sets of Points*, 4th Ed., Campbridge, 1961

[8] D. Schleicher, *Appendix: Laminations, Julia sets, and the Mandelbrot set*, in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters (2009), 111–130.

Sample topics of course projects.

Topic 1: Holomorphic motion.
Topic 2: The Yoccoz puzzle.
Topic 3: Lattés examples.
Topic 4: Thurston's characterization theorem.
Topic 5: Matings and captures.
Topic 6: The Carathéodori theory.

3. Grading policy

We will use the following means of evaluation: homework assignments, quizzes, the written midterm test, the written final exam.

Quiz: 1–2 questions for 5 minutes,
Homework: 3–6 problems for one week,
Midterm and Final: 8 problems for 3 hours.

The intermediate grade is computed at the end of the first module by the formula "0.5 times the average homework grade in the first module plus 0.5 times the midterm grade". The final grade is computed at the end of the term by the formula "0.3 times the cumulated grade plus 0.3 times the intermediate grade plus 0.4 times the final exam grade", where the cumulated grade is the average homework grade in the second module. Quiz results and classroom participation are taken into account as bonus points when computing the cumulated grade.

4. Problems for quizzes, tests, homework assignments

The following is a pull of problems, from which homework assignments and questions for quizzes may be taken. This is just a sample. It can also be used by students as a list of practice problems while preparing for a midterm or the final exam.

Problem 1. Is the point $c = -2.1$ in the Mandelbrot set? Rigorously justify your answer.

Problem 2. Is the point $c = -1.1$ in the Mandelbrot set? Rigorously justify your answer.

Problem 3. Find the locus of parameter values c such that the quadratic polynomial $f_c(z) = z^2 + c$ has an attracting periodic cycle of period 2.

Problem 4. Prove that the Mandelbrot set is contained in the disk $|c| \leq 2$.

Problem 5. Prove that the Julia set of the polynomial $z \mapsto z^2 - 2$ is the interval $[-2, 2]$.

Problem 6. Suppose that the Julia set J_c of a polynomial $p_c(z) = z^2 + c$ is locally connected and that $c \in J_c$. Prove that K_c has no interior points.

Problem 7. Let d be the diameter of the unit disk, one of whose endpoints is $\overline{\frac{3}{12}}$. Prove that the critical leaf lamination $\mathcal{L}(d)$ has a finite invariant gap. Find this gap.

Problem 8. Find a diameter d of the unit disk (e.g. by providing a series expansion for one of its endpoints) with the following property: the critical leaf lamination $\mathcal{L}(d)$ has an infinite invariant gap G such that the restriction of σ_2 to the boundary of G is semi-conjugate to the rotation by angle $\frac{\sqrt{5}-1}{2} \in \mathbb{R}/\mathbb{Z}$.