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NATIONAL RESEARCH UNIVERSITY

*Vladislav Podinovski, Olga Podinovskaya*

**CRITERIA IMPORTANCE THEORY  
FOR DECISION MAKING PROBLEMS  
WITH A HIERARCHICAL  
CRITERION STRUCTURE**

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Hierarchical structures are frequently used to formalize complex multicriteria decision analysis problems. The analytic hierarchy process (AHP) of Saaty (1980) is an established method used to solve practical multicriteria problems of the hierarchical nature. It is also well-known that AHP, and its modifications and generalizations poses a number of fundamental drawbacks that cannot in principle be overcome. The main drawbacks are the independence of the evaluation procedure of the criteria importance from the normalization of criterion values – this violates the requirement of the mathematical theory of measurement, and the lack of a formal definition of the notion of criteria importance. Furthermore, the use of ratio scales in AHP for the expression of importance of criteria is theoretically unsubstantiated. We suggest a new methodology suitable for the analysis of multicriteria problems with hierarchical structures. It is based on criteria importance theory.

Key words: multicriteria decision analysis, hierarchical criterion system, importance of criteria, criteria scales, criteria importance theory, Analytic Hierarchy Process (AHP)

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*Vladislav Podinovski* – Higher School of Economics, Moscow, Russia.

*Olga Podinovskaya* – LLC McKinsey & Company CIS, Moscow, Russia.

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## 1. Introduction

Hierarchical structures are frequently used to formalize complex multicriteria decision analysis problems. The analytic hierarchy process (AHP) of Saaty [1980] is an established method used to solve practical multicriteria problems of the hierarchical nature. It is also well-known that AHP, and its modifications and generalizations, including the analytic network process [Saaty, 1996], poses a number of fundamental drawbacks that cannot in principle be overcome (see, e.g., [Barzilai, 2001; Belton, Stewart, 2003]). The main drawbacks are the independence of the evaluation procedure of the criteria importance from the normalization of criterion values – this violates the requirement of the mathematical theory of measurement, and the lack of a formal definition of the notion of criteria importance. Furthermore, the use of ratio scales in AHP for the expression of importance of criteria is theoretically unsubstantiated [Barzilai, 2010].

In our paper we suggest a new methodology suitable for the analysis of multicriteria problems with hierarchical structures. It is based on criteria importance theory (CIT) developed by Podinovski [1976, 1993, 2002] and is free from the above drawbacks.

## 2. The hierarchical model

We use the mathematical model of individual decision making under certainty adopted in CIT [Podinovski, 1976, 2002] as the initial model:

$$\langle X, f, Z_0, R \rangle, \quad (1)$$

where  $X$  is the set of alternatives (in specific problems, these could be strategies, actions, variants, ...),  $f = (f_1, \dots, f_m)$ , where  $m \geq 2$ , is the vector criterion,  $f_i$  are particular criteria (attributes, indices, ...),  $Z_0$  is the common set of numerical values (also known as the range, or the common “scale”) of criteria  $f_i$ ,  $R$  is a non-strict preference relation as defined below.

The criterion  $f_i$  is the function defined on  $X$  and taking its values from  $Z_0$ . Thus, each alternative  $x$  is characterized by its vector estimate  $y = f(x) = (f_1(x), \dots, f_m(x))$ . Alternatives are compared by comparing their vector estimates. The set of all possible vector estimates (including the vector estimates of actual alternatives) is denoted  $Z = Z_0^m$ .

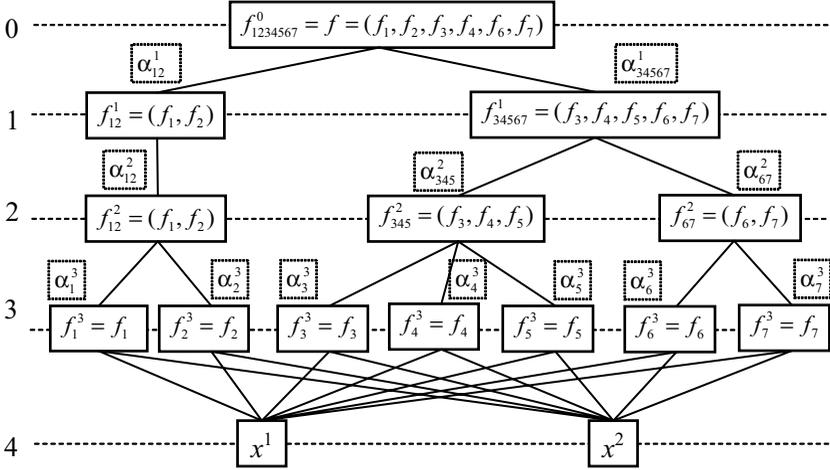
The preferences of the decision maker (DM) are modelled using the non-strict preference relation  $R$  on the set  $Z$ :  $yRz$  means that vector estimate  $y$  is not less preferable than  $z$ . The binary relation  $R$  is a (partial) quasi-order, that is  $R$  is reflexive and transitive:  $yRy$  holds, and also  $yRz$  and  $zRu$  implies  $yRu$  for any  $y, z, u \in Z$ . Relation  $R$  induces the (strict) preference relation  $P$  and indifference relation  $I$  on  $Z$ :  $yPz$  means  $yRz$  and not  $zRy$ , while  $yIz$  means  $yRz$  and  $zRy$ .

All individual criteria in model (1) are assumed homogeneous, i.e., measured on the same scale, and therefore have the same range  $Z_0$ . For example, in practical problems criteria may be assumed homogeneous if they are measured on the same point or linguistic scale. If criteria are measured on different scales, special methods may be used to convert them to the same scale [Podinovski, 1976].

The approach developed in this paper assumes that the individual criteria are a part of the hierarchical structure an example of which is shown in Figure 1. This structure represents 2 variants and 7 individual criteria arranged in 5 levels. The levels are denoted  $l$ , where  $l=0, \dots, 4$ .

At level  $l=2$  the (individual) criteria of the lower level are grouped into disjoint sets and form vector criteria. In Figure 1 there are three such vector criteria:  $f_{12}^2$ ,  $f_{345}^2$  and  $f_{67}^2$ . As above, the superscript shows that these criteria are of the level 2. The subscripts identify the individual criteria that constitute the corresponding vector criterion. For example,  $f_{345}^2 = (f_3, f_4, f_5)$ .

At level  $l=1$  the vector criteria of level 2 are grouped into “longer” vectors using the same principle. For example, the vector criterion  $f_{34567}^1$  is the “merger” of vector criteria  $f_{345}^2$  and  $f_{67}^2$  of the lower level 2. Note that the two criteria of level 1,  $f_{12}^1$  and  $f_{34567}^1$ , include all individual criteria and have no common individual criteria between them.



**Figure 1.** An example of a five-level hierarchy and the corresponding coefficients of importance  $\alpha$

At the top level  $l=0$ , the single vector criterion  $f^0_{1234567}$  is the original vector criterion  $f = (f_1, \dots, f_7)$ .

Figure 1 also shows the coefficients of importance  $\alpha$  that correspond to vector criteria of different levels. For example,  $\alpha^1_{12}$  is the coefficient of importance associated with the vector criterion  $f^1_{12}$ . The exact meaning of coefficients of importance is defined below.

In practical problems the hierarchical structure such as in Figure 1 may be constructed using either the “top-down” or “bottom-up” approach. It is important that each of the vector criteria at each level has a clear meaning. For example, in Figure 1 criterion  $f^1_{12}$  may represent the economic consequences of the decision option, and  $f^2_{34567}$  – its social consequences.

The suggested structure allows us to use methods of CIT for the analysis of complex hierarchical multicriteria problems. It also simplifies a theoretical justification of the known methods by which the coefficients of importance  $\alpha$  corresponding to criteria of lower levels can be calculated taking into account the coefficients of importance of the criteria of the higher levels.

It is worth noting that the suggested hierarchical structure is in principle different from the hierarchical structures used in AHP [Saaty, 1980] and other known approaches. All of the latter approaches, including AHP, while giving a verbal description of the different levels of the hierarchy, do not identify the values of such criteria, except for the individual criteria at the lowest level. The notion of importance for such verbal structure is difficult to define rigorously. Furthermore, the known methods of assessment of criteria importance based on the comparison of specially constructed criterion structures (see, e.g., [Edwards, Barron, 1994; Podinovski, 1976, 1993, 2002]) cannot be used either.

### 3. Some facts from criteria importance theory

For ease of reference, below we present the basic results from CIT developed for the conventional *non-hierarchical* decision analysis problems. Assume that the set  $Z_0$  is finite:  $Z_0 = \{1, \dots, k, \dots, q\}$ ,  $q \geq 2$ . We refer to  $Z_0$  as the set, or range, of *gradations*. Unless stated otherwise, the criterion scale is assumed to be ordinal (i.e., the DM prefers gradation  $k+1$  to  $k$  for all  $k$ , but the use of gradations  $k$  in any arithmetic operations is incorrect). We further define the Pareto relation  $R^\emptyset$ :  $yR^\emptyset z \Leftrightarrow y_i \geq z_i, i = 1, \dots, m$ . (The symbol of empty set  $\emptyset$  indicates that the relation  $R^\emptyset$  is not based on any additional information about the preferences of the DM).

To solve the multicriteria decision problem, the methodology of CIT aims at expanding the preference relation  $R^\emptyset$  by using additional information about the preferences of the DM. Such information may consist of the judgments about the relative importance of criteria and the differences between the gradations in  $Z_0$  [Podinovski, 2008].

Following Podinovski [1979], let  $A$  and  $B$  be two disjoint and non-empty subsets of the set of all indexes  $\{1, \dots, m\}$ . Let  $\{f_i\}_{i \in A}$  and  $\{f_i\}_{i \in B}$  be the corresponding sets of individual scalar criteria.

Let  $y = (y_1, \dots, y_m)$  be any vector of criteria values. Assume that all components  $y_i$  for  $i \in A$  are equal to the same value  $y_{\{A\}}$ , and all components  $y_{\{B\}}$  for  $i \in B$  are equal to the same value  $y_{\{B\}}$ . Let  $y^{AB}$  be the vector obtained from vector  $y$  as follows. All components  $y_i$  of vector  $y$  for  $i \in A$  are replaced by

the value  $y_{\{B\}}$ . Similarly, all components  $y_i$  for  $i \in B$  are replaced by the value  $y_{\{A\}}$ . The remaining components of vector  $y$  are unchanged. For example, let  $y = (2, 4, 4, 3, 2, 2)$ ,  $A = \{1, 5, 6\}$  and  $B = \{2, 3\}$ . Then  $y^{AB} = (4, 2, 2, 3, 4, 4)$ .

**Definition 1.** The groups of criteria  $\{f_i\}_{i \in A}$  and  $\{f_i\}_{i \in B}$  are *equally important* (such statement is denoted as  $A \approx B$ ) if the DM considers vectors  $y$  and  $y^{AB}$  as indifferent.

**Definition 2.** The group of criteria  $\{f_i\}_{i \in A}$  is *more important* than the group of criteria  $\{f_i\}_{i \in B}$  (such statement is denoted as  $A \succ B$ ) if the DM prefers  $y$  to  $y^{AB}$ , whenever  $y_{\{A\}} > y_{\{B\}}$ .

If both sets  $A$  and  $B$  have single elements, the above definitions coincide with the definition of importance of single criteria [Podinovski, 1976].

The quantitative information about criteria importance  $\Omega$  may consist of statements such as  $A \approx B$  or  $A \succ B$ . According to Definitions 1 and 2, each of such statements introduces the corresponding indifference relation  $I^{A \approx B}$  and strict preference relation  $P^{A \succ B}$  on the set  $Z$ . In particular,  $y I^{A \approx B} z$  iff (if and only if)  $z = y^{AB}$ , and  $y P^{A \succ B} z$  iff  $z = y^{AB}$ , provided  $y_{\{A\}} > y_{\{B\}}$ . The non-strict preference relation  $R^\Omega$  on  $Z$  (induced by information  $\Omega$ ) is defined as the transitive closure of the union of relations induced by all statement from  $\Omega$  and relation  $R^\emptyset$ :

$$R^\Omega = \text{TrCl}[(\bigcup_{\omega \in \Omega} R^\omega) \cup R^\emptyset]. \quad (2)$$

This means that  $y R^\Omega z$  is true if and only if there exists a sequence of vectors  $u^k \in Z$  such that

$$y R^1 u^1, u^1 R^2 u^2, \dots, u^{s-1} R^s z,$$

where  $R^j$  is either  $R^\omega$  ( $I^{A \approx B}$  if  $\omega = A \approx B$  and  $P^{A \succ B}$  if  $\omega = A \succ B$ ) or  $R^\emptyset$ . Existing methods of CIT allow for the verification of the consistency of information  $\Omega$  and the construction of the relation  $R^\Omega$ .

Let all criteria be of equal importance (such information is denoted by  $S$ ). For any  $k = 1, \dots, q-1$ , denote  $\sigma_k(y)$  the number of components  $y_i$  of vector  $y$  such that  $y_i \leq k$ , and let  $\sigma(y) = (\sigma_1(y), \dots, \sigma_{q-1}(y))$ , where  $q$  was defined above.

As proved in [Gaft, Podinovski, 1981], the decision rule for relation  $R^S$  is as follows:

$$yR^S z \Leftrightarrow \sigma(y) \leq \sigma(z), \quad (3)$$

where  $\leq$  is the component-wise non-strict vector inequality. Furthermore, if at least for one component the inequality on the right-hand side of (3) is strict (denoted  $\sigma(y) < \sigma(z)$ ), then  $yP^S z$ . Otherwise, if  $\sigma(y) = \sigma(z)$ , we have  $yI^S z$ . The same remark on  $\leq$  and  $=$  is true for decision rules that are introduced below and formulated using non-strict vector inequalities.

Assume that  $q > 2$  and suppose it is known that the increase of preferences along  $Z_0$  “slows down”. We denote this information  $\Delta\downarrow$ . This means that the increase of the DM’s preferences that corresponds to the transition from gradation  $k$  to gradation  $k+1$  is greater than the increase from gradation  $k+1$  to gradation  $k+2$ , for all  $k = 1, \dots, q-2$ . The statement  $i \approx j$ , together with the information  $\Delta\downarrow$ , induces the following preference relation  $P^{i \approx j \Delta\downarrow}$  on  $Z$ :

$$yP^{i \approx j \Delta\downarrow} z \Leftrightarrow y = [(z \parallel z_i + \delta, z_j - \delta), z_i + \delta \leq z_j - \delta] \vee \\ y = (z \parallel z_i - \delta, z_j + \delta), z_j + \delta \leq z_i - \delta]$$

where  $\delta$  is a natural number, and  $(z \parallel z_i + \delta, z_j - \delta)$  is the vector obtained from  $z$  by substituting its component  $z_i$  by  $z_i + \delta$  and  $z_j$  by  $z_j - \delta$ . As proved in [Podinovski, 2009], the decision rule for the relation  $R^{S\Delta\downarrow}$  induced by information  $S\&\Delta\downarrow$  on  $Z$  is as follows:

$$yR^{S\Delta\downarrow} z \Leftrightarrow \sum_{k=1}^l \sigma_k(y) \leq \sum_{k=1}^l \sigma_k(z), \quad l=1, \dots, q-1. \quad (4)$$

Now we define the notion of *degrees of importance superiority* [Podinovski, 2002]. It is based on the notion of  $N$ -model. Let  $N = (n_1, n_2, \dots, n_m)$  be a vector, all components of which are natural numbers, and let  $n = n_1 + \dots + n_m$ . Consider a decision model with  $m$  criteria and the set of vector estimates  $Z = Z_0^m$ . The corresponding  $N$ -model has the following vector of  $n$  scalar criteria:

$$\langle f_1^1, \dots, f_1^{n_1}, \dots, f_m^1, \dots, f_m^{n_m} \rangle, \quad (5)$$

where, for each  $i = 1, \dots, m$ , the set of equally important criteria  $f_i^1, \dots, f_i^{n_i}$  corresponds to the original scalar criterion  $f_i$ . We refer to the values of vector (5) as  $N$ -estimates. The set of all  $N$ -estimates is  $Z(N) = Z_0^n$ .

For any vector estimate  $y \in Y$  in the original model, the corresponding  $N$ -estimate  $y^N \in Z(N)$ . In particular, the first  $n_1$  components of vector  $y^N$  are equal to  $y_1$ , the next  $n_2$  components are equal to  $y_2$ , and so on. In other words,  $y^N$  is obtained by the cloning of components of  $y$ . Conversely, if  $y^N$  is an  $N$ -estimate such that  $y_1 = \dots = y_{n_1}$ , and so on, then the corresponding vector estimate  $y = (y_{n_1}, \dots, y_{n_m}) \in Z$ .

**Definition 3.** Criterion  $f_i$  is  $h_{ij}$  times as important as criterion  $f_j$  (such statement is denoted as  $i \succ^{h_{ij}} j$ ) if for any agreeing  $N$ -model (i.e. the  $N$ -model in which  $n_i/n_j = h_{ij}$ ) each of  $n_i$  criteria  $f_i^s$  and each of  $n_j$  criteria  $f_j^t$  are equally important.

Let  $\Theta$  be the quantitative information about the importance of criteria: such information consists of the statements in the form  $i \succ^{h_{ij}} j$ . We assume that the information  $\Theta$  is consistent and complete, i.e. it allows us to calculate the matrix  $H = (h_{ij})$  of degrees of importance superiority, whose positive elements satisfy equalities  $n_i/n_j = h_{ij}$ ,  $i, j, k \in \{1, \dots, m\}$ . Let  $R^\Theta$  be the non-strict preference relation induced by information  $\Theta$  on the set  $Z$ . It is defined by the following decision rule which is obtained on the basis of decision rule (3) for problems with equally important criteria (5) [Podinovski, 2002]:

$$yR^\Theta z \Leftrightarrow \sigma(y^N) \leq \sigma(z^N). \quad (6)$$

In (6), each component  $\sigma_k(y^N)$  of the vector  $\sigma(y^N) = (\sigma_1(y^N), \dots, \sigma_{q-1}(y^N))$  is defined as the number of components of vector  $y^N$  that are not greater than  $k$ .

The quantitative (or cardinal) coefficients of importance of criteria, induced by the information  $\Theta$ , are positive numbers  $\alpha_i$  such that  $\alpha_i/\alpha_j = h_{ij}$ ,  $i, j = 1, \dots, m$ , and such that the sum of all  $\alpha_i$  is equal to 1. The coefficients  $\alpha_i$  are unique.

Define

$$\alpha_{ik}(y) = \begin{cases} \alpha_i, y_i \leq k, \\ 0, y_i > k, \end{cases}, \quad i = 1, \dots, m; k = 1, \dots, q - 1; \quad (7)$$

$$\alpha_k(y) = \alpha_{1k}(y) + \dots + \alpha_{mk}(y), \quad k = 1, \dots, q - 1; \alpha(y) = (\alpha_1(y), \dots, \alpha_{q-1}(y)). \quad (8)$$

The decision rule using the coefficients of importance and based on (7) – (8) is as follows [Podinovski, 2002]:

$$yR^{\Theta}z \Leftrightarrow \alpha(y) \leq \alpha(z). \quad (9)$$

Let us introduce vectors  $\alpha^{[1,k]}(y) = \alpha_1(y) = \dots + \alpha_k(y)$ ,  $k = 1, \dots, q - 1$ . The decision rule describing relation  $yR^{\Theta\Delta\downarrow}$  specified by information  $\Theta\&\Delta\downarrow$  is as follows [Podinovski, 2009]:

$$yR^{\Theta\Delta\downarrow}z \Leftrightarrow \alpha^{[1,k]}(y) \leq \alpha^{[1,k]}(z), \quad k = 1, \dots, q - 1. \quad (10)$$

Some methods for the practical elicitation of different types of qualitative and quantitative information about the importance of criteria, and the corresponding decision rules were developed by Podinovski [2009] and Nelyubin and Podinovski [2011].

#### 4. Criterion groups of equal importance

A new decision rule is needed for multicriteria decision problems with a hierarchical structure. Assume that the set of  $m$  criteria is divided into  $s > 1$  disjoint and equally important groups of criteria  $\{f_i\}_{i \in A^1}, \dots, \{f_i\}_{i \in A^s}$ . We further assume that the criteria in each group are of equal importance, and at least one of the groups contains no less than two criteria. For simplicity and without loss of generality we assume that the first group includes the first  $m_1$  equally important criteria, the second group contains the next  $m_2$  equally important criteria, and so on. The last  $s$ -th group contains the last  $m_s$  equally important criteria. Therefore, the vector criterion  $f$  may be stated as

$$f = (\underbrace{f_1, \dots, f_{m_1}}_{m_1}, \underbrace{f_{m_1+1}, \dots, f_{m_1+m_2}}_{m_2}, \dots, \underbrace{f_{m_1+\dots+m_{s-1}+1}, \dots, f_m}_{m_s}). \quad (11)$$

The above information may be represented in the following form:  $S^* = \{A^1 \approx \dots \approx A^s; i \approx j, i, j \in A^t, t = 1, \dots, s\}$ . According to the general definition (2) this information induces the non-strict preference relation  $R^{S^*}$  on the set  $Z$ .

Let  $\mu$  be the least common multiple of the natural numbers  $m_1, \dots, m_s$ . Define  $n_1 = \mu / m_1, \dots, n_s = \mu / m_s$ , and consider the corresponding  $N$ -model. In the latter model, an initial vector estimate  $y = (y_1, \dots, y_m)$  corresponds to the  $N$ -estimate

$$y^N = \underbrace{(y_1, \dots, y_{m_1}; \dots; y_1, \dots, y_{m_1})}_{n_1}; \dots; \underbrace{(y_{m_1+\dots+m_{s-1}+1}, \dots, y_m; \dots; y_1, y_{m_1+\dots+m_{s-1}+1}, \dots, y_m)}_{n_s}. \quad (12)$$

The set of all  $N$ -estimates is  $Z_0^n$ . Let us note that number of components in  $y^N$  that were obtained from the  $t$ -th group of components of vector estimate  $y$  is equal to  $n_t m_t = \mu$ , and this number is the same for all groups. Therefore, in accordance with (6), the relation  $\bar{R}^{S^*}$  on  $Z$  we can define as follows:

$$y \bar{R}^{S^*} z \Leftrightarrow \sigma(y^N) \leq \sigma(z^N). \quad (13)$$

It can be shown that the relation  $\bar{R}^{S^*}$  expands the preference relation  $R^{S^*}$  in the following consistent way:  $\bar{R}^{S^*} \supset R^{S^*}$  (therefore,  $\bar{I}^{S^*} \supseteq I^{S^*}$ ) and  $\bar{P}^{S^*} \supset P^{S^*}$ .

## 5. Quantitative importance of group of criteria

For problems with hierarchical structure, we further need to define the degree of importance superiority applicable to groups of criteria. Let us first illustrate the idea of the definition given below by an example.

**Example 1.** Let the set of criteria in a five-criterion problem ( $m = 5$ ) be divided into two groups:  $\{f_1, f_2, f_3\}$  and  $\{f_4, f_5\}$ . Also let  $A = \{1, 2, 3\}$  and  $B = \{4, 5\}$  be the sets of criterion indexes (subscripts) from the first and the second groups, respectively. We now address the following question: how we could formalize, for example, the statement that the group of criteria  $\{f_1, f_2, f_3\}$  is twice as important as the group  $\{f_4, f_5\}$ ?

To answer this question, let us expand the vector estimate  $y = (y_1, y_2, y_3, y_4,$

$y_5$ ) to the vector  $(y_1, y_2, y_3, y_1, y_2, y_3, y_4, y_5)$  and consider the latter vector as a vector estimate in the eight-criterion problem, with three *equally important* groups. Each of the first two groups consists of three criteria, and the third group consists of two criteria. Furthermore, if there is information about the importance of the original scalar criteria  $y_i$ ,  $i=1,\dots,5$ , such information should also apply to the extended eight-criterion problem.

Let  $\{f_i\}_{i \in A}$  and  $\{f_i\}_{i \in B}$  be two disjoint groups of criteria. Consider any two integers  $n_a, n_b > 0$  and define  $h = n_a/n_b$ . Let us expand the vector  $y = (y_1, \dots, y_m)$  by replicating  $n_a - 1$  times its components corresponding to the group  $A$ . We refer to the set of such  $n_a$  identical groups as the clone of group  $A$ , or simply clone  $A$ . Similarly, we replicate  $n_b - 1$  times the components of vector  $y$  corresponding to the group  $B$ . We refer to the set of  $n_b$  identical groups as clone  $B$ .

The above cloning procedure generates a new expanded decision problem referred to as the  $(n_a; n_b)$ -problem. In this problem, any two groups from clone  $A$  are equally important and any two groups from clone  $B$  are equally important. Moreover, any information about the importance of the original scalar criteria  $y_i$  is also retained in the expanded problem.

**Definition 4.** The group of criteria  $\{f_i\}_{i \in A}$  is  $h = n_a/n_b$  times as important as the group of criteria  $\{f_i\}_{i \in B}$  (such statement is denoted as  $A \succ^h B$ ), if in the  $(n_a; n_b)$ -problem any group of criteria from clone  $A$  and any group of criteria from clone  $B$  are equally important.

In a practical application the information about degrees of importance superiority of one group of criteria over another may be elicited using methods similar to those developed for single criteria [Podinovski, 2002]. In such methods we should consider vector estimates in which components from the same group are all equal to each other.

Let the set of criteria be divided into  $s$  groups as in (11). Suppose that, using Definition 4, for any two groups  $\{f_i\}_{i \in A_r}$  and  $\{f_i\}_{i \in A_s}$  we have established that the former group is  $h_{rs}$  times as important as the latter:  $A_r \succ^{h_{rs}} A_s$ .

Let us further assume that for any two criteria  $f_i$  and  $f_j$  from the same group it is known that  $i \succ^{h_{ij}} j$ . Such information is denoted by the Greek letter  $T$ .

Based on information T, and using the process of cloning described above (applied to groups of criteria and individual scalar criteria), we can construct the T-model in which all groups of criteria are equally important, and all scalar criteria within any group are also equally important.

The described cloning process may be performed “bottom-up”, where we start by cloning groups of criteria, followed by the cloning of individual criteria within each group. Alternatively, the process can be performed “top-down”. In this case we first clone single criteria within groups, followed by the cloning of the resulting groups. The following example illustrates this procedure.

**Example 2.** In Example 1 we showed how the information that the group of criteria  $A$  is twice as important as group  $B$  ( $A \succ^2 B$ ) could be represented by creating an expanded vector of criteria in which all groups of criteria were equally important. Suppose we additionally have the following information concerning individual scalar criteria:  $1 \succ^3 2$ ,  $2 \approx 3$ ,  $4 \succ^{5/2} 5$ . Then all the information we have can be represented as  $T = \{A \succ^2 B; 1 \succ^3 2, 2 \succ^1 3; 4 \succ^{5/2} 5\}$ .

First, we construct the T-model using the “top-down” approach. As in Example 1, using the statement  $A \succ^2 B$ , we expand the original vector  $y = (y_1, y_2, y_3, y_4, y_5)$  to  $(y_1, y_2, y_3, y_1, y_2, y_3, y_4, y_5)$ . Next, we clone the single criteria using the information  $1 \succ^3 2$ ,  $2 \approx 3$  and  $4 \succ^{5/2} 5$ . The resulting vector, which we refer to as the T-estimate, is as follows:

$$y^T = (y_1, y_1, y_1, y_2, y_3; y_1, y_1, y_1, y_2, y_3; y_4, y_4, y_4, y_4, y_4, y_5, y_5). \quad (14)$$

Alternatively, we can use the “bottom-up” approach. Using the information  $1 \succ^3 2$ ,  $2 \approx 3$  and  $4 \succ^{5/2} 5$ , we first clone individual criteria to expand the original vector  $y = (y_1, y_2, y_3, y_4, y_5)$  to vector  $(y_1, y_1, y_1, y_2, y_3; y_4, y_4, y_4, y_4, y_4, y_5, y_5)$ . We then use the information  $A \succ^2 B$  to obtain (14).

Note that in the resulting T-model all groups of criteria are equally important, and all criteria in each group are also equally important. Therefore, using the approach described in Section 4, we can associate this model with the  $N$ -model in which all criteria are equally important. We refer to this model as the  $NT$ -model. The following example clarifies this correspondence.

**Example 3.** The T-model constructed in Example 2 includes three groups of criteria, and all three groups are equally important. The first two of these groups contain  $m_1 = m_2 = 5$  equally important criteria, and the third group contains  $m_3 = 7$  equally important criteria. Since the least common multiple of  $m_1$ ,  $m_2$ , and  $m_3$  is 35, and because  $n_1 = \mu / m_1 = 7$ ,  $n_2 = \mu / m_2 = 7$ ,  $n_3 = \mu / m_3 = 5$ , the relevant NT-model contains  $105 = 5 \times 7 \times 3$  equally important criteria. This model is obtained by repeating the criteria from the first and second groups 7 times, and repeating the criteria from the third group 5 times. For example, for  $y = (2, 4, 5, 1, 3)$ , its T-estimate is (13) and its NT-estimate is

$$y^{NT} = (2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; \\ 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; 2, 2, 2, 2, 4, 5; \\ 1, 1, 1, 1, 1, 3, 3; 1, 1, 1, 1, 1, 3, 3; 1, 1, 1, 1, 1, 3, 3; 1, 1, 1, 1, 1, 3, 3; 1, 1, 1, 1, 1, 3, 3).$$

Because all criteria in the NT-model are equally important, such a model can be stated in an equivalent compact form: we need to indicate only the components of the original vector  $y$  and to use the superscripts to specify the number of times each component occurs in the NT-estimate. In our example, we have  $y^{NT} = (1^{25}, 2^{42}, 3^{10}, 4^{14}, 5^{14})$ .

Because all criteria in the NT-model are of equal importance, for the comparison of two vector estimates  $y$  and  $z$  we can use any method of CIT suitable for such criteria. For example, adapting (13), we obtain the following decision rule:

$$y \bar{R}^T z \Leftrightarrow \sigma(y^{NT}) \leq \sigma(z^{NT}) \quad (15)$$

**Example 4.** Let the set of gradations in the problem from Example 3 be  $Z_0 = \{1, 2, 3, 4, 5\}$ . Let us compare the three vector estimates  $y = (2, 4, 5, 1, 3)$ ,  $z = (3, 5, 5, 4, 2)$ ,  $u = (1, 5, 5, 5, 4)$ . We first construct the three corresponding NT-estimates. Our task is simplified by noting that, as in Example 3, we need to replicate the first component 42 times, the second and third 14 times, the fourth 25 times and the fifth 10 times. Therefore,

$$y^{NT} = (1^{25}, 2^{42}, 3^{10}, 4^{14}, 5^{14}), \quad z^{NT} = (2^{10}, 3^{42}, 4^{25}, 5^{28}), \quad u^{NT} = (1^{42}, 4^{10}, 5^{53}).$$

Furthermore,

$$\begin{aligned}\sigma_1(y^{NT}) &= 25, \quad \sigma_2(y^{NT}) = 67, \quad \sigma_3(y^{NT}) = 77, \quad \sigma_4(y^{NT}) = 91; \\ \sigma_1(z^{NT}) &= 0, \quad \sigma_2(z^{NT}) = 10, \quad \sigma_3(z^{NT}) = 52, \quad \sigma_4(z^{NT}) = 77; \\ \sigma_1(u^{NT}) &= 42, \quad \sigma_2(u^{NT}) = 42, \quad \sigma_3(u^{NT}) = 42, \quad \sigma_4(u^{NT}) = 52.\end{aligned}$$

Note that the following inequalities are true

$$\sigma_1(z^{NT}) < \sigma_1(y^{NT}), \quad \sigma_2(z^{NT}) < \sigma_2(y^{NT}), \quad \sigma_3(z^{NT}) < \sigma_3(y^{NT}), \quad \sigma_4(z^{NT}) < \sigma_4(y^{NT}).$$

Then, according to (15),  $z\bar{R}^T y$ . For  $u$  and  $z$  we have

$$\sigma_1(u^{NT}) > \sigma_1(z^{NT}), \quad \sigma_2(u^{NT}) > \sigma_2(z^{NT}), \quad \sigma_3(u^{NT}) < \sigma_3(z^{NT}), \quad \sigma_4(u^{NT}) < \sigma_4(z^{NT}),$$

so that  $z\bar{R}^T u$  and  $u\bar{R}^T z$  are not true, i.e.  $u$  and  $z$  are non-comparable with respect to  $\bar{R}^T$ .

The above method required the construction of NT-models. We used this approach to illustrate the theoretical definition of the preference relation  $\bar{R}^T$ . There also exists a simpler alternative approach based on the notion of *coefficients of group importance*  $\alpha_{\{t\}}$ . These are positive numbers that correspond to groups of criteria. The sum of  $\alpha_{\{t\}}$  is equal to 1, and we require that  $\alpha_{\{t\}}/\alpha_{\{r\}} = h_{tr}$ , for all  $t, r = 1, \dots, s$ . It is straightforward to prove that such coefficients of importance are unique for T.

Let  $\alpha_1^t, \dots, \alpha_{m_t}^t$  be the coefficients of importance of individual scalar criteria from the group  $A_t$ . (Note that these coefficients are valid only within the group  $A_t$  and cannot be directly compared to the coefficients of importance from the other groups of criteria.)

Let  $\alpha_i, i = 1, \dots, m$  be *final coefficients of importance* of the individual scalar criteria. These can be calculated using the conventional approach: if criterion  $f_i$  belongs to group  $\{f_i\}_{i \in A_t}$  then  $\alpha_i = \alpha_{\{t\}} \alpha_i^t$ . The coefficients  $\alpha_i$  allow us to simplify the construction of the NT-model and introduce a new decision rule based on (9). Let all coefficients  $\alpha_i$  be proper fractions:  $\alpha_i = n'_i/n''_i$ , where  $n'_i$

and  $n_i''$  are relatively prime numbers, and  $n_i' < n_i''$ . Furthermore, let all these fractions be reduced to the lowest common denominator  $c$ . It is easy to see that the  $NT$ -model with  $N = (n_1, \dots, n_m)$  is appropriate. The  $NT$ -estimates constructed in this way have dimension equal to  $c$ .

**Example 5.** In the example 4 problem for the information  $T = \{A \succ^2 B; 1 \succ^3 2, 2 \approx 3; 4 \succ^{5/2} 5\}$  we have:  $\alpha_{(1)} = \frac{2}{3}$ ,  $\alpha_{(2)} = \frac{1}{3}$ ;  $\alpha_1^1 = \frac{3}{5}$ ,  $\alpha_2^1 = \alpha_3^1 = \frac{1}{5}$ ;  $\alpha_1^2 = \frac{5}{7}$ ,  $\alpha_2^2 = \frac{2}{7}$ . Therefore,

$$\alpha_1 = \frac{2}{5}, \alpha_2 = \alpha_3 = \frac{2}{15}, \alpha_4 = \frac{5}{21}, \alpha_5 = \frac{2}{21}. \quad (16)$$

The lowest common denominator  $c$  of numbers 5, 15 и 21 is equal to 105. Fractions (14) being reduced to lowest common denominator become

$$\alpha_1 = \frac{42}{105}, \alpha_2 = \alpha_3 = \frac{14}{105}, \alpha_4 = \frac{25}{105}, \alpha_5 = \frac{10}{105}. \quad (17)$$

Thus,  $N$  for  $NT$ -model is the vector (42, 14, 14, 25, 10), and the dimension of  $NT$ -estimates is equal to 105. The same established in another way in example 3.

The equivalence of two considered ways for constructing  $NT$ -models allows to use decision rules (9) and (10) because they were justified just with the help of  $N$ -model [Podinovski, 2002, 2009]. For this it is necessary to account for final importance coefficients. Let us illustrate the “work” of decision rule (9) with an example.

**Example 6.** Let us compare vector estimates in example 3 by preference using final importance coefficients (16) reduced to lowest common denominator (17). According to (7) and (8) we have for  $y, z$  and  $u$ :

$$\alpha(y) = \left( \frac{25}{105}, \frac{67}{105}, \frac{77}{105}, \frac{91}{105} \right), \alpha(z) = \left( 0, \frac{10}{105}, \frac{52}{105}, \frac{77}{105} \right), \alpha(u) = \left( \frac{42}{105}, \frac{42}{105}, \frac{42}{105}, \frac{52}{105} \right).$$

Since  $\alpha(z) \leq \alpha(y)$ , then, in accordance to (9),  $z \bar{P}^T y$  is true. But both  $\alpha(z) \leq \alpha(u)$  and  $\alpha(u) \leq \alpha(z)$  are not true and, therefore,  $z$  and  $u$  are incomparable by  $\bar{R}^T$ . It is useful to compare calculations that are executed here with calculations from example 4.

## 6. Decision rules for problems with a hierarchical structure

In problems with a hierarchical criteria structure similar to that depicted in Figure 1 we can treat vector criteria of different levels as groups of component scalar criteria. For example, the vector criterion  $f_{345}^2$  in Figure 1 is treated as the group of criteria  $\{f_3, f_4, f_5\}$ . Therefore, all of the approaches and methods considered above are applicable to such problems.

While analyzing practical problems with a hierarchical structure, the decision maker may be asked to compare (vector) criteria of the same level  $l$ , which are the subvectors of the same vector criterion of the higher level  $l - 1$ . (We refer to this higher-level vector criterion as the common parent of the two subvectors of level  $l$ . For example, in Figure 1, criterion  $f_{34567}^1$  is the parent of criteria  $f_{345}^2$  and  $f_{67}^2$ .)

The scale gradations of parent vectors are defined as follows. The scale gradation of a vector criterion is equal to  $k \in Z_0$  if the scale gradations of all particular criteria included in the vector criterion are equal to  $k$ . For example, if the scale  $Z$  is linguistic and the value  $k$  corresponds to gradation “good”, then the parent vector criterion is also assigned the common general value  $k$  (“good”).

The qualitative information  $\Omega$  about the importance of criteria can be obtained using Definition 1 and Definition 2. Existing methods of CIT allow us to obtain the non-strict preference relation on  $Z$  taking into account information  $\Omega$  and also statements about the rate at which the DM’s preferences change along the criterion scale. To construct such relations, one could use the general approach developed by Osipova et al. [1984]. However, this method assumes a matrix representation of the binary preference relation. Due to the very large dimensions of such a matrix even for a small number of criteria, the practical application of this method is questionable. Unfortunately, no analytical methods for the solution of such problems exist at the present time. However, under some additional assumptions a suitable optimization method can be developed (see Example 8).

To obtain quantitative information on criteria importance one can use methods described in Podinovski [2002]. This information consists of statements about the degree of importance superiority of some criteria over the other.

The use of such information for the construction of the non-strict preference relation on  $Z$  requires the construction of the corresponding  $NT$ -model as described in section 5. However, as shown in the example below, it may be easier to calculate the final coefficients of importance first, and then use a suitable decision rule.

**Example 7.** Let the set of gradations in the problem from Example 1 be  $Z_0 = \{1, 2, 3, 4, 5\}$ . Suppose we have the following quantitative information about the criteria importance

$$f_{12}^1 \succ^{3/2} f_{34567}^1; f_{345}^2 \succ^{7/3} f_{67}^2; f_1^3 \succ^1 f_2^3; f_3^3 \succ^1 f_4^3; f_4^3 \succ^2 f_5^3; f_6^3 \succ^1 f_7^3 \quad (18)$$

The above data allows us to calculate the coefficients of importance shown in Figure 1:

$$\alpha_{12}^1 = 0.6, \alpha_{34567}^1 = 0.4; \alpha_{12}^2 = 1, \alpha_{345}^2 = 0.7, \alpha_{67}^2 = 0.3; \alpha_1^3 = 0.5, \alpha_2^3 = 0.5, \\ \alpha_3^3 = 0.4, \alpha_4^3 = 0.4, \alpha_5^3 = 0.2, \alpha_6^3 = 0.5, \alpha_7^3 = 0.5.$$

(The information about the degree of superiority of one criterion over another may be elicited in the form of precise or interval statements. Such information may then be used to calculate the coefficients of criteria importance using the eigenvalue method of Saaty [1980] or some other methods [Podinovski, 2007]. The above coefficients of importance allow us to calculate the final importance coefficients  $\alpha_i$  of partial criteria  $f_i$ :

$$\alpha_1 = 0.3, \alpha_2 = 0.3, \alpha_3 = 0.112, \alpha_4 = 0.112, \alpha_5 = 0.056, \alpha_6 = 0.06, \alpha_7 = 0.06.$$

Suppose that the values of the vector criterion  $f = (f_1, f_2, f_3, f_4, f_6, f_7)$  for alternatives  $x^1$  and  $x^2$  are as follows:

$$y = f(x^1) = (4, 4, 3, 5, 3, 1, 2), \quad z = f(x^2) = (5, 3, 2, 4, 4, 3, 1). \quad (19)$$

If there is no information about the rate of increase of the preferences along the criterion scale, i.e. the scale is ordinal, then we can use decision rule (9) to compare alternatives with respect to the DM's preferences. Note that for the vectors

$$\alpha(y) = (0.06, 0.12, 0.288, 0.888), \quad \alpha(z) = (0.06, 0.172, 0.532, 0.7)$$

neither  $\alpha(y) \leq \alpha(z)$  nor  $\alpha(z) \leq \alpha(y)$  is true, the alternatives  $x^1$  and  $x^2$  are uncomparable. Now assume that it is known that the rate of increase of the preferences along the criterion scale “slows down”. According to the definition in Section 3, in this case we have

$$\alpha^{[1,1]}(y) = 0.06, \alpha^{[1,2]}(y) = 0.18, \alpha^{[1,3]}(y) = 0.468, \alpha^{[1,4]}(y) = 1.356, \\ \alpha^{[1,1]}(z) = 0.06, \alpha^{[1,2]}(z) = 0.232, \alpha^{[1,3]}(z) = 0.764, \alpha^{[1,4]}(z) = 1.464.$$

Then the non-strict inequalities from (10)

$$\alpha^{[1,1]}(y) \leq \alpha^{[1,1]}(z), \alpha^{[1,2]}(y) \leq \alpha^{[1,2]}(z), \alpha^{[1,3]}(y) \leq \alpha^{[1,3]}(z), \alpha^{[1,4]}(y) \leq \alpha^{[1,4]}(z)$$

are satisfied. Furthermore, we have the strict inequality  $\alpha^{[1,4]}(y) < \alpha^{[1,4]}(z)$ . Therefore, alternative  $x^1$  is preferred to  $x^2$ .

**Remark 1.** The method developed in Podinovski (2012) may be used for the sensitivity analysis of the optimal decision alternative with respect to the changes of the coefficients of importance.

Above we assumed that the coefficients of criteria importance were assessed exactly, as *point estimates*. However, quantitative information about importance may also be obtained in the form of *set estimates*, e.g. in the form of intervals (Podinovski, 2002). The point estimates usually require that we make certain additional assumptions about the preference structure of the DM. In the absence of such assumptions we may only be able specify a set  $A$  of feasible values of the coefficient of importance  $\alpha$ . In this case, we may subsequently use the common principle of decision theory (see, e.g., [Weber, 1987; Podinovski, 2008]) for the definition of the non-strict preference relation  $R(A)$ , induced on  $Z$  by the information about the importance in the form  $A$ .

Namely, we define

$$yR(A)z \Leftrightarrow yR(\alpha)z, \text{ for any } \alpha \in A, \quad (20)$$

where  $R(\alpha)$  is the non-strict preference relation defined by the relevant decision rule for the known value of  $\alpha$ . For example, if the criterion scale is ordinal, the decision rule is (9). The described approach extends to the case in which the information about the importance of criteria is qualitative.

**Example 8.** Assume that in Example 7 the quantitative information (18) is not available but instead we have the following qualitative information:

$$f_{12}^1 \succ f_{34567}^1; f_{345}^2 \succ f_{67}^2; f_1^3 \approx f_2^3; f_3^3 \approx f_4^3; f_4^3 \succ f_5^3; f_6^3 \succ f_7^3 \quad (21)$$

For the qualitative information (21) the set A is defined by the following system of equalities and inequalities:

$$\begin{aligned} \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 > 0, \alpha_5 > 0, \alpha_6 > 0, \alpha_7 > 0, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = 1, \alpha_1 + \alpha_2 > \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\ \alpha_3 + \alpha_4 + \alpha_5 > \alpha_6 + \alpha_7, \alpha_1 = \alpha_2, \alpha_3 = \alpha_4, \alpha_4 > \alpha_5, \alpha_6 = \alpha_7. \end{aligned} \quad (22)$$

If we the criterion scale is ordinal then, according to (9) and (20), the relation  $R(A)$  is defined as follows:

$$yR(A)z \Leftrightarrow \alpha(y) \leq \alpha(z), \text{ for any } \alpha \in A. \quad (23)$$

To illustrate the application of (23), consider alternatives  $x^1$  and  $x^2$  whose vector estimates are defined in (19). Taking into account (7) and (8), for  $y$  and  $z$  we have:

$$\begin{aligned} \alpha_1(y) = \alpha_6, \alpha_2(y) = \alpha_6 + \alpha_7, \alpha_3(y) = \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7, \\ \alpha_4(y) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7. \\ \alpha_1(z) = \alpha_7, \alpha_2(z) = \alpha_3 + \alpha_7, \alpha_3(z) = \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7, \\ \alpha_4(z) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7. \end{aligned}$$

We can now restate (23) in the extended form:  $yR(A)z$  is true if and only if, for any  $\alpha \in A$ , the following inequalities are true:

$$\begin{aligned} \alpha_6 \leq \alpha_7, \alpha_6 + \alpha_7 \leq \alpha_3 + \alpha_7, \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 \leq \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 \leq \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \end{aligned} \quad (24)$$

According to (21),  $\alpha_6 = \alpha_7$ . Therefore the inequalities (23) are equivalently restated as

$$\max_{\alpha \in \bar{A}} \alpha_6 - \alpha_3 \leq 0, \max_{\alpha \in \bar{A}} \alpha_5 - \alpha_2 \leq 0, \max_{\alpha \in \bar{A}} \alpha_1 + \alpha_5 - \alpha_4 \leq 0. \quad (25)$$

In (25),  $\bar{A}$  is the set of values of  $\alpha$  defined by (22) where all strict inequalities are replaced by epy non-strict inequalities (such substitution is acceptable [Podinovski, 2004]).

The verification of inequalities (25) requires solving three linear programming programs. For example, the first of these is:

$$\alpha_6 - \alpha_3 \rightarrow \max$$

subject to:

$$\begin{aligned} \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0, \alpha_5 \geq 0, \alpha_6 \geq 0, \alpha_7 \geq 0, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = 1, \alpha_1 - \alpha_2 = 0, \alpha_3 - \alpha_4 = 0, \alpha_6 - \alpha_7 = 0, \\ -\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \leq 0, \\ -\alpha_3 - \alpha_4 - \alpha_5 + \alpha_6 + \alpha_7 \leq 0, -\alpha_4 + \alpha_5 \leq 0. \end{aligned}$$

The above program may be solved by any linear optimizer. Its optimal value is 0.4166667, which does not satisfy the first inequality in (25). Therefore,  $zR(A)y$  is not true. Similarly, it can be established that  $zR(A)y$  is also not true. Therefore, alternatives  $x^1$  and  $x^2$  are incomparable by  $R(A)$ . This result was entirely expected because these alternatives were incomparable even in Example 7, in which the exact values of the coefficients of importance were known.

**Remark 2.** The decision rule considered in Example 8 is based on the assumption that quantitative coefficients of importance exist, and uses qualitative information  $\Omega$  about their values. It is worth noting that in case of an ordinal criterion scale the assumption that quantitative coefficients of importance exist does not lead to an extension of the preference relation  $R^\Omega$ . However, if the increase of preferences along the criterion scale is “slowing down”, the assumption that quantitative coefficients of importance exist does generally lead to the extension of relation  $R^{\Omega\Delta\downarrow}$  [Nelyubin, Podinovski, 2012].

## References

- Barzilai J.* Notes on the analytic hierarchy process // Proceedings of the NSF Design and Manufacturing Research Conference, Tampa, Florida, 2001. P. 1–6. <[http://scientificmetrics.com/downloads/publications/Barzilai\\_2001\\_Notes\\_on\\_the\\_Analytic\\_Hierarchy\\_Process.pdf](http://scientificmetrics.com/downloads/publications/Barzilai_2001_Notes_on_the_Analytic_Hierarchy_Process.pdf)>
- Barzilai J.* Preference function modelling: The mathematical foundations of decision theory // Trends in Multiple Criteria Decision Analysis / M. Ehrgott, J.R. Figueira, S. Greco (eds.). N. Y.: Springer, 2010. P. 57–86.
- Belton V., Stewart T.J.* Multiple Criteria Decision Analysis. An Integrated Approach. Cluwer, Boston, 2003.
- Edwards W., Barron F.H.* 1994. SMARTS and SMARTER: improved simple methods for multiattribute utility measurement // Organization Behavior and Human Processes. 1994. Vol. 60. P. 306–325.
- Gaft M.G., Podinovski V.V.* Construction of decision rules in decision-making problems // Automation and Remote Control. 1981. Vol. 42. P. 806–814.
- Nelyubin A.P., Podinovski V.V.* Optimization methods in multi-criteria decision making analysis with interval information on the importance of criteria and values of scale gradations // Automatic Documentation and Mathematical Linguistics. 2011. Vol. 45. P. 202–210.
- Nelyubin A.P., Podinovski V.V.* On the relationship between qualitative and quantitative importance of criteria in multicriterial decision making problems // Several Problems of Applied Mathematics and Mechanics / I. Gorgidze, T. Lominadze (eds.). N. Y.: Nova Publishers, 2012. P. 43–55.
- Osipova V.A., Podinovski V.V., Yashina N.P.* On non-contradictory extension of preference relations in decision making problems // USSR Computational Mathematics and Mathematical Physics. 1984. Vol. 24. P. 128–134.
- Podinovski V.V.* Multicriterial problems with importance-ordered criteria // Automation and Remote Control. 1976. Vol. 37. P. 1728–1736.
- Podinovski V.V.* 1979. Axiomatic solution of the problem of evaluating the importance of criteria in multicriterial decision-making problems // State-of-the-Art of Operations Research Theory / N.N. Moiseev (ed.). Moscow: Nauka, 1979. P. 117–145. (In Russian.)

- Podinovski V.V.* Multicriteria optimization problems involving importance-ordered criteria // *Modern Mathematical Methods of Optimization* / K.-H. Elster (ed.). Berlin: Akademie Verlag, 1993. P. 254–267.
- Podinovski V.V.* Criteria importance theory. *Mathematical Social Sciences*. 1994. Vol. 27. P. 237–252.
- Podinovski V.V.* The quantitative importance of criteria for MCDA // *Journal of Multi-criteria Decision Analysis*. 2002. Vol. 11. P. 1–15.
- Podinovski V.V.* Decision under multiple estimates for the importance coefficients of criteria and probabilities of values of uncertain factors // *Automation and Remote Control*. 2004. Vol. 65. P. 1817–1833.
- Podinovski V.V.* Interval articulation of superiority and precise elicitation of priorities // *European Journal of Operational Research*. 2007. Vol. 180. P. 406–417.
- Podinovski V.V.* Analysis of multicriteria choice problems by methods of the theory of criteria importance, based on computer systems of decision making support // *Journal of Computer and System Sciences International*. 2008. Vol. 47. P. 221–225.
- Podinovski V.V.* On the use of importance information in MCDA problems with criteria measured on the first ordered metric scale // *Journal of Multi-Criteria Decision Analysis*. 2009. Vol. 15. P. 163–174.
- Podinovski V.V.* Sensitivity analysis for choice problems with partial preference relations // *European Journal of Operational Research*. 2012. Vol. 221. P. 198–204.
- Saaty T.L.* *The Analytic Hierarchy Process*. N. Y.: McGraw Hill, 1980.
- Saaty T.L.* *Decision Making with Dependence and Feedback: The Analytic Network Process*. Pittsburgh: RWS Publications, 1996.
- Weber M.* Decision making with incomplete information // *European Journal of Operational Research*. 1987. Vol. 28. P. 44–57.

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Иерархические структуры часто используются для формализации сложных многокритериальных задач принятия решений. Для анализа таких задач в 1980 г. Т. Saaty представил разработанный им метод анализа иерархий (“The Analytic Hierarchy Process” – АНР). Этот широко известный метод, как и его модификации, обладает рядом принципиальных, причем неустраняемых недостатков. К основным таким недостаткам относятся независимость процедур оценивания важности критериев и нормализации критериальных оценок альтернатив, что нарушает требование математической теории измерений, и отсутствие формального определения понятия важности критериев. Более того, в свете современных исследований по теории измерений возможность оценивания предпочтений в шкале отношений исключается. Мы предлагаем новую методологию для анализа многокритериальных задач с иерархической структурой. Она основана на теории важности критериев.

Ключевые слова: анализ многокритериальных решений, иерархическая критериальная система, важность критериев, шкалы критериев, теория важности критериев, метод анализа иерархий (АНР)

*Подinovский В.В.*, Национальный исследовательский университет «Высшая школа экономики», Москва.

*Подinovская О.В.*, Мак-Кинзи и Компания СиАйЭс, Москва.

Контактное лицо: Подinovский Владислав Владимирович, профессор кафедры высшей математики на факультете экономики ВШЭ; podinovski@hse.ru; тел. (495)6211342

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**Теория важности критериев для задач принятия решений  
с иерархической критериальной структурой**

*(на английском языке)*

Зав. редакцией оперативного выпуска *А.В. Заиченко*  
Технический редактор *Ю.Н. Петрина*

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