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Capital, Aggregate Fluctuations and Idiosyncratic Risk

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Abstract

Nowadays idiosyncratic productivity shocks have a significant consequences in an aggregate production. We consider a model of an economy, where firms experience i.i.d. productivity shocks. In such an economy some specific capital allocation rules can lead to minimization of aggregate fluctuations. We define three classes of investment rules, which are resulting in vanishing of the uncertainty in the aggregate economy. Our innovation is a proportional capital allocation, when each firm receives new capital according to some Borel measurable function of its productivity. After that we examine dynamic properties of the aggregate output, savings and capital holdings under different strategies of capital distribution and determine the most optimal one.

Key words: investment; capital allocation; idiosyncratic risk

Contents

1 Introduction

Macroeconomic models typically consider the effect of aggregate shocks on economic variables such as aggregate savings and output growth. The role that idiosyncratic risk may play in generating aggregate fluctuations is usually ignored. However, there are evidences that support that persistence of idiosyncratic shocks in the economy increases over time.

It is a widely known fact that distributions of the capital and production have positive skew. Thus in the economy most of the firms have small or intermediate size, however most of goods are produced by large ones. For example, in the U.S. in 2013 sales of top 10 companies exceeded 13% of total GDP, sales of the first 50 companies were over 35% of the GDP (fig. 1). Therefore productivity and, as a result, sales of individual firms from the top of the list play a considerable role in the aggregate behavior of the economy. Hence, idiosyncratic shocks to these companies can lead to a sufficient changes in the aggregate output.

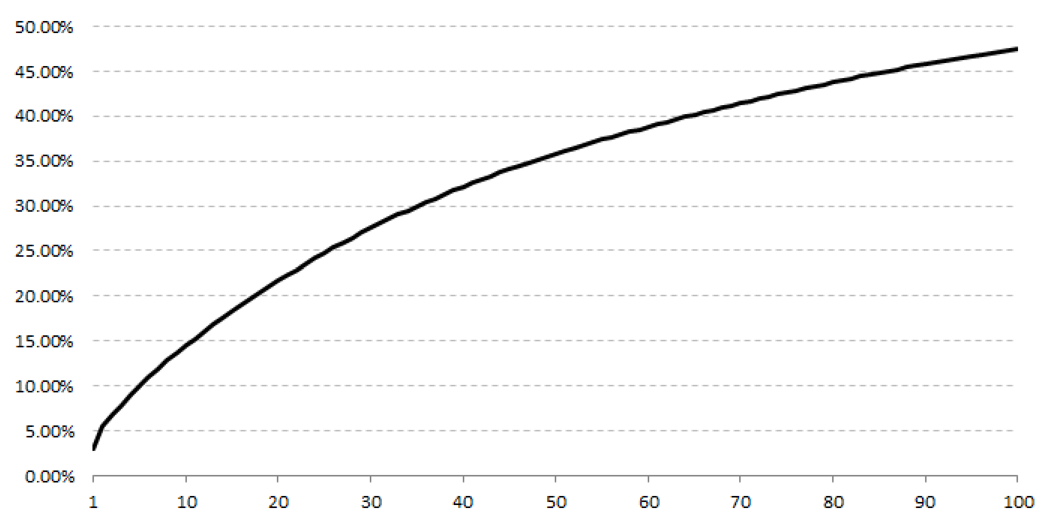

Figure 1: **Cumulative sales of top 100 U.S. companies as a percent of GDP in 2013.**

Irvine and Schuh (2002) consider empirical data for the U.S. economy and show that after 1984 the economy was subject to the following stylized facts:

- Firm-level production volatility has considerably risen.
- Volatility of aggregate output on contrary significantly decreased.
- Covariance between firms and between industries declined.

If we assume that the micro-structure of the economy (firm-level volatility and inter-firm correlations) is an exogenous parameter, arises a very interesting question concerning the mechanism of reduction of the aggregate fluctuations in such an environment. There are different views on this issue: Stock and Watson (2003) argue that this result can be achieved through improvement of the inventory management and the proper monetary policy. Dynan et al. (2006) show that the aggregate volatility reduction may be caused by the financial innovations (e.g. development of loan markets etc.). We examine how this result can be obtained through the regulation of the investment strategies. Specifically, we consider how at the economy, where firms experience independent and identically distributed productivity shocks, different investment rules can be used in order to minimize volatility of the aggregate production.

We want to determine long-term effects of a short-term investment decisions driving by the current firm productivities. In such an economy savings will be a function of only a current state of nature and rate of return, therefore each period investments would not depend on the expectations on the future path of the economy. For this purpose we choose an Overlapping Generations model as a baseline because it exhibits all the underline features. We consider the OLG growth model of Polemarchakis and Dutta (1992), which describes an economy with a population of firms that experience independent and identically distributed productivity shocks. If investment decisions are made before the realization of productivity, the limit economy converges to the risk-less one with constant aggregate production. However if new investments can be reallocated ex-post, aggregate productivity in general is not a constant because of the persistence of sunk production of less efficient firms that use the capital that was accumulated during previous periods. Their result leaves open the question of the stationarity of such an economy: in fact, in general it will not be stationary. Our innovation is consideration of the proportional capital allocation rules, when firms receive new capital according to some invertible continuous function of its productivity. This distribution of the capital leads to vanishing of the aggregate fluctuations, whereas the aggregate level of the output exceeds the one obtained under ex-ante capital allocation.

Our approach is consistent with the literature on Real Business Cycles, albeit preserving the role of idiosyncratic risk in generating aggregate fluctuations. Kydland and Prescott (1982) show that the delay of adjustment of the capital leads to aggregate fluctuations. They consider agents who make the optimal decisions on the level of consumption and investment subject to the expected future aggregate productivity level. The innovation of the model is that today's investment does not increase tomorrows capital. Because of the lag between the investment decision and the change in the capital stock, the model fits the cyclical variance of economic time series fairly well. We also consider the long term effect of the investment decisions, but instead of aggregate fluctuations we want to examine the effect of individual shocks.

There are many papers, which consider idiosyncratic risks and try to find the path of all economy due to such risks. One of classical models that study idiosyncratic risks is Aiyagari (1994).They consider an impact of stochastic labor income. Agents, who are faced with borrowing constraints, optimize their expected utility subject to idiosyncratic labor and deterministic income from risk-less bond. Accumulated asset holdings reflect the history of agent's individual shocks. In the equilibrium per capita asset holdings coincide per capita capital level. The model argues that under the idiosyncratic risk the capital holdings are higher than in the complete market equilibrium, whereas the interest rate is lower than its complete market analog. Even though this model is very good in explanation of the equilibrium path of the aggregate savings, this approach does not capture some important features that we wish to examine. Firstly it does not deal with capital-related risk. Secondly the behavior of the aggregate production does not studied properly. In our paper we want to study the risk that is determined by stochastic capital income and results of the aggregate output of the economy.

Angeletos (2007) considers the idiosyncratic risk of capital income. Households have access to production technology and to risk-less bonds. Production is stochastic and independent across agents. As a result markets are incomplete and the economy is risky. Angeletos receives similar to Aiyagaris result of the lower level of interest rate. But in his model this fact does not imply increase in the capital level. In general the space of all possible states of the world in such economy is infinite, nevertheless if it is used a Cobb-Douglas production function with linear return on capital and there is a homoscedasticity of preferences all optimal investment decisions are linear functions of the agents wealth. It gives an opportunity to reduce the number of dimensions, which are needed to describe the equilibrium, and finally to solve the model and to make some simulations. Angeletos and Panousi (2011) study a two-country extension of this model, that helps them to explain the cross-country differences in the interest rate and capital accumulation. But Angeletos (2007) and Angeletos and Panousi (2011) examine economies where capital depreciates fully: we do not. As a result every period each agent makes a decision on the level of capital that does not directly depend on the previous capital level, whereas in our approach the firm's capital holdings is a function of previous capital level and current productivity.

Even though Krusell and Smith (1999) consider a welfare effects of eliminating business cycles, their approach of modulating of fluctuations is close to ours. They study an economy with agents, who experiences idiosyncratic uncertainty in employment in the next period. Because of the fact that probability to be unemployed in the next period depends on the unemployment rate as well as on the current agent's status, this model represents a direct dependence of the state on the previous sequence of shocks. However Krusell and Smith consider infinite living agents, which may allow them to consider dynamic strategies, whereas our model represents a sequence of static decisions.

Carvajal and Polemarchakis (2011) examine the effect of the idiosyncratic endowments on the Pareto efficiency of the equilibrium allocations. They have found spaces of preferences that under additional constraints on the level of diversification of the commodities, which can be traded, guarantee an existence of finitely many Pareto efficient equilibria. However they consider two period model: we multi-period. As a result our model allows us to consider long-term effects of the individual shocks.

Gabaix (2005) considers questions that a very close to our. He shows that idiosyncratic shocks can help to explain aggregate fluctuations in the economy. Huggett and Ospina (1998) describe a steady-state equilibrium in the model with idiosyncratic labor income. Khan and Thomas (2011) study an (S,s) equilibrium in the economy with heterogeneous productivities, debt and capital holdings. Bianconi and Turnovsky (2003) show aggregate effects of individual shocks, but their main point is an individual gain from the regulation of the aggregate shock. Santos Monteiro (2008) study multiple equilibria that are generated by the idiosyncratic labor risk. Antunes et al. (2008) look for the stationary equilibrium of the general equilibrium models with independent exogenous abilities, endogenous bequest and financial market with deadweight costs. Hopenhayn and Prescott (1992) study the stability of invariant distributions of stochastically monotone processes.

The paper proceeds as follows. We present the model in Section 2, study aggregate outputs under different investment rules in Section 3. In section 4 we examine different classes of the investment rules and find optimal shapes of capital distribution functions. Finally, in section 5 we study the dynamic properties of the economy under different investment rules and discuss which ones are more preferable.

2 Model

In our paper we consider an Overlapping Generations model with agents living for two periods and infinitely living firms and financial intermediate. Agents has an endowment of labor when they are young. They supply their labor to competitive firm and use the wage for consumption and savings. Households' savings go to the financial intermediate that guaranties competitive fixed capital income and invests the capital in the firms operating in the economy.

The combination of the Overlapping Generations model with infinitely living financial intermediate was chosen because it allows us to concentrate on the long-term effects of the short-term investment decisions and individual shocks under different allocation strategies.

2.1 Households

Representative agent, who is young in period t chooses optimal levels of consumptions in both periods (C_t^t and C_{t+1}^t) and savings (S_t) taking wage (w_t) and interest rate (r_t) as given in order to maximize his intertemporal utility function $\overline{U}(C_t^t, C_{t+1}^t) = U(C_t^t) + \beta \cdot U(C_{t+1}^t)$, where $\beta \in (0,1)$:

$$max_{\{C_t^t, C_{t+1}^t, S_t\}}[U(C_t^t) + \beta \cdot U(C_{t+1}^t)] \tag{1}$$

Subject to:

$$\begin{aligned} C_t^t + S_t &\leq w_t \\ C_{t+1}^t &\leq (1 + r_t) \cdot S_t \end{aligned} \tag{2}$$

Optimization yields (3). An explicit solution can be obtained after specification of the utility function. We are going to use a standard notation - log-utility (i.e. $U(C_t) = log(C_t)$, which gives us (4).

$$
\begin{aligned}
&C_t^t = w_t - S_t \\
&C_{t+1}^t = (1+r_t)\cdot S_t \\
&\beta \cdot U^{'}((1+r_t)\cdot S_t) = U^{'}(w_t - S_t)
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
C_t^t &= \frac{1+r_t}{1+r_t+\beta}\cdot w_t \\
C_{t+1}^t &= \frac{(1+r_t)\cdot\beta}{1+r_t+\beta}\cdot w_t \\
S_t &= \frac{\beta}{1+r_t+\beta}\cdot w_t
\end{aligned}
\tag{4}
$$

The result (4) consistent with the intuition:

- As consumption during the first period is more valuable, the agent consumes more, when he is young ($C_t^t > C_{t+1}^t$).
- Sum of the consumption and savings during the first period equals to the total labor income of the household ($C_t^t + S_t = w_t$).
- When the wage is proportional to the income (which is true for example for Cobb-Douglas production function) savings of the households are proportional to the output at the correspondent period.

2.2 Firms

Every firm in the economy uses capital k_{nt}, labor l_{nt} and technology θ_{nt} to produce each period:

$$
x_{nt} = \theta_{nt}\cdot k_{nt}^{1-\alpha}\cdot l_{nt}^{\alpha} \qquad 0<\alpha<1 \tag{5}
$$

Firms can choose only labor, whereas capital is endogenously distributed between firms and productivity (θ_{nt}) is randomly assigned to each firm every period.

The capital of each firm changes by the following law:

$$k_{nt} = (1-\delta)\cdot k_{n\ t-1} + z_{n\ t-1} \qquad 0 \leq \delta \leq 1 \tag{6}$$

Given the capital holding, productivity and wage level each firm maximizes its profit, which will be paid to the stockholders as dividends:

$$max_{\{l_{nt}\}}[\pi_{nt}] = max_{\{l_{nt}\}}[x_{nt} - l_{nt}\cdot w_t] \tag{7}$$

First Order Condition:

$$\frac{d\pi_{nt}}{dl_{nt}} = \alpha\cdot\theta_{nt}\cdot\frac{k_{nt}^{1-\alpha}}{l_{nt}^{1-\alpha}} - w_t = 0 \tag{8}$$

So the optimal amount of labor used for production equals to :

$$l_{nt}^{*} = \left(\frac{\alpha\cdot\theta_{nt}}{w_t}\right)^{\frac{1}{1-\alpha}}\cdot k_{nt} \tag{9}$$

Using the labor market clearing condition and normalizing the total labor supply to one, we can get an expression for the equilibrium wage:

$$\sum_{n=1}^{N} l_{nt}^{*} = \sum_{n=1}^{N}\left(\frac{\alpha\cdot\theta_{nt}}{w_t}\right)^{\frac{1}{1-\alpha}}\cdot k_{nt} = L = 1 \tag{10}$$

$$w_t = \alpha\cdot\left(\sum_{n=1}^{N}\theta_{nt}{}^{\frac{1}{1-\alpha}}\cdot k_{nt}\right)^{1-\alpha} \tag{11}$$

Plugging in (9) and (11) into (5), we can obtain an equilibrium output of each firm in period t (12). As we can see after the optimization the output of any arbitrary firm is a function of not only endogenous for it parameters (capital, productivity) but also of the joint distribution of the capital and productivity of all the others firms.

$$
\begin{aligned}
x_{nt} &= \theta_{nt} \cdot k_{nt}^{1-\alpha} \cdot \left(\left(\frac{\alpha \cdot \theta_{nt}}{w_t} \right)^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{\alpha} \\
&= \theta_{nt} \cdot k_{nt}^{1-\alpha} \cdot \left(\frac{\theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt}}{\sum_{n=1}^{N} \theta_{nt}{}^{\frac{1}{1-\alpha}} \cdot k_{nt}} \right)^{\alpha} \\
&= \frac{\theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt}}{\left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{\alpha}}
\end{aligned}
\tag{12}
$$

The output of the whole economy can be obtained by the summarizing of the outputs of the individual firms (13). In general, the aggregate result can be calculated buy plugging in all the parameters (capital holdings and productivity) of each firm, but, as it will be shown below, under some capital-distribution rules the aggregate result can be simplified and, as a result, can be written using some aggregate terms regardless the micro-structure of the distribution.

$$
\begin{aligned}
X_t &= \sum_{n=1}^{N} x_{nt} = \sum_{n=1}^{N} \frac{\theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt}}{\left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{\alpha}} \\
&= \frac{\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt}}{\left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{\alpha}} = \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{1-\alpha}
\end{aligned}
\tag{13}
$$

2.3 Financial Intermediate

As it was mentioned above, financial intermediate operates infinitely, assuming deposits of the households and investing in firms. The net profit - firms' dividends (financial intermediate is assumed to be the only stockholder of all the firms) minus payment to households - is used for consumption. Therefore financial intermediate chooses the distribution of the new investments between the firms (z_{it}) in order to maximize the utility of its consumption in the current and all proceeding periods subject to three constraints. The first constraint is a pure budget constraint of the financial intermediate, the second one is the same as (6) - capital flow equation of the firm n and the last constraint yields that total investment cannot exceed households' deposits.

$$max_{\{z_{n\ t-1}, n\in[1,N]\}} \sum_{\tau=0}^{\infty} \beta_{FI}^{\tau} \cdot U^{FI}(C_{t+\tau}^{FI}) \tag{14}$$

Subject to:

$$\begin{aligned} C_t^{FI} \leq \sum_{n=1}^{N}(1+r_{nt}) \cdot k_{nt} - (1+r_{t-1}) \cdot S_{t-1} \\ k_{nt} = (1-\delta) \cdot k_{n\ t-1} + z_{n\ t-1} \\ \sum_{n=1}^{N} z_{n\ t-1} \leq S_{t-1} \end{aligned} \tag{15}$$

Where r_{nt} - return on capital of firm n at period t. From the equation (5) the marginal product of capital of firm n is:

$$\begin{aligned} 1+r_{nt} = MPK_{nt} & \\ &= \frac{dx_{nt}}{dk_{nt}} \\ &= (1-\alpha) \cdot \theta_{nt} \cdot \frac{l_{nt}^{\alpha}}{k_{nt}^{\alpha}} = (1-\alpha) \cdot \theta_{nt} \cdot \left(\frac{\alpha \cdot \theta_{nt}}{w_t}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \frac{(1-\alpha) \cdot \alpha^{\frac{\alpha}{1-\alpha}} \cdot \theta_{nt}^{\frac{1}{1-\alpha}}}{w_t^{\frac{\alpha}{1-\alpha}}} \\ &= \gamma_t \cdot \theta_{nt}^{\frac{1}{1-\alpha}} \end{aligned} \tag{16}$$

As we have agreed before households make their savings at the end of their first period, when the productivities of the firms during the next period are not known yet. Therefore the fair rate of return of their savings will be equal to the expected or just average marginal product of capital.

$$1 + r_{t-1} = E(1 + r_{nt}) \tag{17}$$

This optimization problem (14, 15) cannot be solved explicitly, however we can consider different classes of rules of capital allocation and compare the long term effects of these actions. Therefore we need to make some assumptions on the preferences of the financial intermediate.

- Financial intermediate has the time preference coefficient β_{FI} close to unite, so its valuation of the consumption today and tomorrow are practically identical.
- Financial intermediate has sufficient degree of risk-aversion such that it tends to make investment decisions in order to smooth its consumption as much as possible.

In the next section we'll consider three generalized types of the capital allocation: ex-ante allocation, when all the firms get the same amount of capital, and two specifications of the ex-post allocation: when only the most productive firm gets new capital and when each firm receives capital accordingly to its productivity. Under some additional conditions these allocation rules will give the absence of the uncertainty on the aggregate behavior of the economy.

3 Aggregate Economy

In this section we will consider aggregate behavior of the economy under different investment rules. As all individual firms uses the same production technology (Cobb-Douglas, which gives us the fact that cost of capital is a constant fraction of the output) and the return on households savings is predetermined each period, the optimization problem of the financial intermediate shrinks to the optimization of the aggregate output given risk-less output. Therefore we can skip optimization problems of firms, households, financial intermediate and concentrate on the aggregate effect of different investments.

3.1 Ex-ante Allocation

Firstly, let's consider the simplest variation of the model, when the investment decision is done before the realization of the productivities. In this case all firms are ex-ante identical, hence, the investment is equally distributed between all of them.

The timing of the model:

- Households save some fraction of its labor income.
- Financial Intermediate assumes savings and distributes capital between the firms.
- Realization of productivities of all firms.
- A decision of each firm on the optimal level of labor.
- Production.
- Payment of dividends, wages and deposits.

As it was assumed before the capital is equally allocated between all the firms ($k_{nt} = k_{mt} = \frac{\sum_{n=1}^{N} k_{nt}}{N} = \frac{K_t}{N}$). Hence, the aggregate output (13) can be rewritten as follows:

$$X_t = \sum_{n=1}^{N} x_{nt} = \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{1-\alpha} = v_t \cdot K_t^{1-\alpha} \tag{18}$$

From the equation (18) we can conclude that the output of the whole economy depends on the aggregate capital stock and aggregate productivity shock. As individual shocks are iid, the aggregate are independently distributed. Moreover when the number of firms goes up without bound the aggregate productivity v_t converges to a constant. The proof of this fact that uses Strong Law of Large Numbers can be found in Polemarchakis (1992). I would like just to discuss the intuition of this finding. The sum of the individual productivities to the power of $\frac{1}{1-\alpha}$ divided by the number of firms equals to the sample average of the productivity shock to the power of $\frac{1}{1-\alpha}$. Under the Law of Large Numbers the sample average converges to the expected value. Thereafter we will use this fact and associate the sum of functions of the individual shocks divided by the number of the firms with expectation of this function under the assumption of the infinitely large size of the economy.

Therefore aggregate output in the limit economy equals to the:

$$X_t = v_t \cdot K_t^{1-\alpha} = \left(E(\theta^{\frac{1}{1-\alpha}}) \right)^{1-\alpha} \cdot K_t^{1-\alpha} \tag{19}$$

3.2 Ex-post Allocation: Simple Case

Much more interesting result can be obtained if there is a possibility to allocate the new capital after the realization of the productivities. In this case financial intermediate will be able to allocate its investments taking the firms' productivities into account. As now the amount of information available for the investor before making the decision is higher than in the previous example, obviously there is such investment rule, that gives at least the same level of the aggregate output.

At first glance when all the productivities are realized investment in the most productive company seems rather rational. Obviously the short run effect of this decision will outperform the output in the case of the equal capital distribution. However the behavior of the economy in long run is not so clear. Therefore let's consider this case in details.

The continuation of the model in this case is as follows:

- Households save some fraction of its labor income.
- Financial Intermediate assumes savings.
- Realization of productivities of all firms.
- Financial Intermediate transfer all the new capital to the most productive firm.
- A decision of each firm on the optimal level of labor.
- Production of the most productive firm and the sunk production of the others.
- Payment of dividends, wages and deposits.

From the equation (16) the marginal product of capital of each firm doesn't depend on the initial capital holdings of the firm. It depends only on the productivity level and the wage that is set in the economy. Thus, only the most productive firms would increase their capital holdings using new investments.

Proposition 1.
If $X_1, X_2, , X_n$ are iid random numbers distributed with the function $F(X)$, than $max_{\{i\}} X_i$ is distributed with probability function $F_{max}(X) = F^n(X)$.

Proof.

$$
\begin{aligned}
F_{max}(X) &= Prob\left(\max_i X_i < X\right) \\
&= Prob(X_1 < X) \cup Prob(X_2 < X) \cup \cup Prob(X_n < X) \\
&= F_1(X) \cdot F_2(X) \cdot \cdot F_n(X) \\
&= F(X) \cdot F(X) \cdot \cdot F(X) \\
&= F^n(X)
\end{aligned}
\tag{20}
$$

□

Proposition 2.

If $X_1, X_2, , X_n$ are iid random numbers distributed with the function $F(X)$, than $max_{\{i\}} X_i$ converges to a constant with probability 1.

Proof.

Assume $Prob(X_i < X) = 1 - \frac{1}{N}$

From the proposition 1 $Prob(\max_i X_i < X) = \left(1 - \frac{1}{N}\right)^n$

As $f(x) = a^x$ is continuously decreasing positive function of x when, $a \in (0,1)$ for any $\varepsilon > 0$ there is such x that $f(x) < \varepsilon$. Therefore for any $N > 2$ and $\varepsilon > 0$ there is such n_0 that for all $n > n_0$ we have $\left(1 - \frac{1}{N}\right)^n < \varepsilon$. Hence, as number of iid random variables increases maximum of these numbers converges to the upper bound of their distribution with probability 1.

□

When the number of firms increases, the distribution of the maximum of productivities shrinks to the one number (see proposition 2). Therefore the new capital, as it was mentioned above, will get the firm with the maximal possible productivity. Without loss of generality we can assume that only one firm has the maximal productivity level, otherwise two or more firms will share the new capital, but as the MPK doesn‘t depend on the capital level in our set up, the aggregate production will be the same as if only one firm gets all new investments.

Let‘s denote the index of the most productive firm by m. Then the capital holdings of firms (6) in period t are

$$k_{nt} = (1-\delta)\cdot k_{n\ t-1}, \quad n \neq m \tag{21}$$

$$k_{mt} = (1-\delta)\cdot k_{m\ t-1} + Z_{t-1} \tag{22}$$

$$\sum_{n=1}^{N} k_{nt} \equiv K_t = (1-\delta)\cdot K_{t-1} + Z_{t-1} \tag{23}$$

Using equations (11) and (13) we can find the new equilibrium wage and the aggregate output in such an economy

$$w_t = \alpha \cdot \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot (1-\delta)\cdot k_{n\ t-1} + \theta_{mt}^{\frac{1}{1-\alpha}} \cdot Z_{t-1} \right)^{1-\alpha} \tag{24}$$

$$
\begin{aligned}
X_t &= \sum_{n=1}^{N} x_{nt} \\
&= \left(\sum_{n=1}^{N} \theta_{nt}{}^{\frac{1}{1-\alpha}} \cdot (1-\delta) \cdot k_{n\ t-1} + \ \theta_{mt}{}^{\frac{1}{1-\alpha}} \cdot Z_{t-1} \right)^{1-\alpha} \\
&= \left(\frac{\sum_{n=1}^{N} \theta_{nt}{}^{\frac{1}{1-\alpha}} \cdot (1-\delta) \cdot k_{nt-1} + \theta_{mt}{}^{\frac{1}{1-\alpha}} \cdot Z_{t-1}}{K_t} \right)^{1-\alpha} \\
&= \left(\frac{K_{t-1} \cdot \left((1-\delta) \cdot \mu_t + \ \theta_{mt}{}^{\frac{1}{1-\alpha}} \cdot \frac{Z_{t-1}}{K_{t-1}} \right)}{(1-\delta) \cdot K_{t-1} + Z_{t-1}} \right)^{1-\alpha} \cdot K_t^{1-\alpha} \\
&= \left(\frac{(1-\delta) \cdot \mu_t + \ \theta_{mt}{}^{\frac{1}{1-\alpha}} \cdot \frac{Z_{t-1}}{K_{t-1}}}{(1-\delta) + \frac{Z_{t-1}}{K_{t-1}}} \right)^{1-\alpha} \cdot K_t^{1-\alpha} \\
&= \tilde{v}_t \cdot K_t^{1-\alpha}
\end{aligned}
\tag{25}
$$

Where μ_t is a weighted average productivity shock. Note that in this setup $\tilde{v}_t$ depends not only on the maximal value of productivity among all firms, but on the market capitalization of firms in the previous period (as long as the depreciation is less that 1). The latter does not converge to a constant even if the number of firms is sufficiently large. The intuition of this conclusion can be explained using the following example. Assume that in period t al firms had the same amount of capital. In period t+1 firm m_1 was the most productive. Hence, in this period the distribution of capital was

$$
\begin{aligned}
k_{n\ t+1} &= (1-\delta) \cdot k_{n\ t} = (1-\delta) \cdot \frac{K_t}{N}, \quad n \neq m_1 \\
k_{m_1\ t+1} &= \ (1-\delta) \cdot k_{m_1\ t} + Z_t = (1-\delta) \cdot \frac{K_t}{N} + Z_t
\end{aligned}
\tag{26}
$$

In period t+2 the most productive firm is $m_2 \neq m_1$ and new distribution of capital is

$$
\begin{aligned}
k_{n\ t+2} &= (1-\delta)^2 \cdot \frac{K_t}{N}, \qquad n \neq m_1 \quad n \neq m_2 \\
k_{m_1\ t+2} &= \ (1-\delta)^2 \cdot \frac{K_t}{N} + (1-\delta) \cdot Z_t \\
k_{m_2\ t+2} &= \ (1-\delta)^2 \cdot \frac{K_t}{N} + Z_{t+1}
\end{aligned}
\tag{27}
$$

And so on. After T steps the distribution of capital will be

$$
k_{j\ t+T} = (1-\delta)^T \cdot \frac{K_t}{N} + \sum_{i=0}^{T-1} a_i^j \cdot (1-\delta)^j \cdot Z_{t+T-j-1} \tag{28}
$$

where a_i^j equals to one, if firm j was the most productive in period t+i, and to zero otherwise ($a_i^j \neq a_i^k$, if $j \neq k$).

So, the capital holdings generally are not the same, moreover the allocation of capital changes over time even if the aggregate capital remains constant. It leads to the uncertainty of the aggregate productivity of the economy ($\tilde{v}_t$), which depends on distribution of capital and on the iid shocks. As a result such an economy is risky. In Polemarchakis (1992) it is shown that there is a stationary equilibrium in such an economy only if there is full depreciation. Let's briefly consider this case.

From the equation (26) when $\delta = 1$

$$
\begin{aligned}
k_{nt} &= 0,\ n \neq m \\
k_{mt} &= Z_{t-1}
\end{aligned}
\tag{29}
$$

Using (25), (29) and the fact that the most productive firm has maximal possible productivity (θ_{max}) the aggregate output equals to

$$X_t = \left(\theta_{mt}^{\frac{1}{1-\alpha}} \cdot Z_{t-1}\right)^{1-\alpha} = \theta_m \cdot Z_{t-1}^{1-\alpha} \tag{30}$$

Now let's consider the aggregate output in the economy without ex-post reallocation and full depreciation, from (19) under the assumption that shocks are from the uniform distribution $U[\theta_{min}, \theta_{max}]$

$$\begin{aligned} X_t &= \left(E(\theta^{\frac{1}{1-\alpha}})\right)^{1-\alpha} \cdot K_t^{1-\alpha} \\ &= \left(\int_{\theta_{min}}^{\theta_{max}} \frac{\mathrm{x}^{\frac{1}{1-\alpha}}}{\theta_{\max} - \theta_{\min}} \mathrm{d}x\right)^{1-\alpha} \cdot Z_{t-1}^{1-\alpha} \\ &= \left(\frac{\theta_{max}^{\frac{1}{1-\alpha}+1} - \theta_{min}^{\frac{1}{1-\alpha}+1}}{(\theta_{max} - \theta_{min}) \cdot (\frac{1}{1-\alpha} + 1)}\right)^{1-\alpha} \cdot Z_{t-1}^{1-\alpha} \end{aligned} \tag{31}$$

It is easy to show that the aggregate output (30) in this economy exceeds the output with ex-ante allocation (31). Let's denote $\theta_{min} = A \cdot \theta_{max}$ with $A < 1$, than

$$
\begin{aligned}
X_t &= \left(\frac{\theta_{max}^{\frac{1}{1-\alpha}+1} - \theta_{min}^{\frac{1}{1-\alpha}+1}}{(\theta_{max}-\theta_{min}) \cdot (\frac{1}{1-\alpha}+1)} \right)^{1-\alpha} \cdot Z_{t-1}^{1-\alpha} \\
&= \left(\frac{\theta_{max}^{\frac{1}{1-\alpha}+1} \cdot (1 - A^{\frac{1}{1-\alpha}+1})}{\theta_{max} \cdot (1-A) \cdot (\frac{1}{1-\alpha}+1)} \right)^{1-\alpha} \cdot Z_{t-1}^{1-\alpha} \\
&= \left(\frac{1 - A^{\frac{1}{1-\alpha}+1}}{(1-A) \cdot (\frac{1}{1-\alpha}+1)} \right)^{1-\alpha} \cdot \theta_{max} \cdot Z_{t-1}^{1-\alpha} \qquad (32) \\
&= \left(\frac{1 - A}{(1-A) \cdot (\frac{1}{1-\alpha}+1)} \right)^{1-\alpha} \cdot \theta_{max} \cdot Z_{t-1}^{1-\alpha} \\
&= \left(\frac{1}{\frac{1}{1-\alpha}+1} \right)^{1-\alpha} \cdot \theta_{max} \cdot Z_{t-1}^{1-\alpha} \\
&< \theta_{max} \cdot Z_{t-1}^{1-\alpha}
\end{aligned}
$$

This simple case helped us to understand that the ex-post capital allocation can skew the equilibrium output of the economy. Moreover the simplest and the most obvious rule, when all the new capital goes to the most productive firm, can significantly increase the aggregate output of the economy under condition of full capital depreciation. Unfortunately, without the condition of the full depreciation the solution is not stable and can‘t be considered for the further research. In the next section we will consider other investment rules that can be used under the ex-post reallocation.

3.3 Ex-post Allocation: Proportional Allocation

Now let’s assume that after the realization of the productivity shocks capital is distributed proportionally to the shocks. Let $f(x)$ is some Borel measurable function[1],

[1] Let (X, F_x) and (Y, F_y) be measurable spaces. A function $f : X \to Y$ is said to be (F_x, F_y)-measurable if $f^{-1}(A) \in F_x$ for any $A \in F_y$.

we don't specify it now, further we will consider different variants of its realization. The firm that experiences productivity θ_i will obtain $\frac{f(\theta_i)}{\sum_{n=1}^{N} f(\theta_n)}$ fraction of the new capital. As we have a restriction on the short sell of the existing capital, the fraction that firm will obtain should be non-negative for any realization of the shock. To guarantee this result we need $f(x) \geq 0, \forall x \in D(\theta)$. Moreover as finally all the new capital should be distributed between the firms the function of the shock should have positive values for the non-empty set of the realizations of shocks. This approach is fairly reasonable, the sum of the fractions adds up to one, under the assumption of the increasing function $f(x)$ the more productive is the firm, the more capital it will receive.

Assume that initially (at the period $t-1$) all firms held the same amount the capital ($k_{i\ t-1} = \frac{K_{t-1}}{N}, \forall i \in [1,N]$). Therefore at the period t firm i will have:

$$k_{it} = (1-\delta) \cdot k_{i\ t-1} + z_{i\ t-1} = (1-\delta) \cdot k_{i\ t-1} + \frac{f(\theta_{it})}{\sum_{n=1}^{N} f(\theta_{nt})} \cdot Z_{t-1} \tag{33}$$

And so on. After p steps the capital holdings of the i'th firms are equal to:

$$\begin{aligned} k_{i\ t+p} = (1-\delta)^p \cdot k_{i\ t-1} + \sum_{g=1}^{p} (1-\delta)^g \cdot \frac{f(\theta_{i\ t+p-g})}{\sum_{n=1}^{N} f(\theta_{n\ t+p-g})} \cdot Z_{t+p-g-1} + \\ \frac{f(\theta_{i\ t+p})}{\sum_{n=1}^{N} f(\theta_{n\ t+p})} \cdot Z_{t+p-1} \end{aligned} \tag{34}$$

Now let's consider the aggregate output. Using equations (13), (33) and (34):

$$
\begin{aligned}
X_t &= \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt} \right)^{1-\alpha} \\
&= \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot \left[(1-\delta) \cdot k_{n\ t-1} + \frac{f(\theta_{nt})}{\sum_{i=1}^{N} f(\theta_{it})} \cdot Z_{t-1} \right] \right)^{1-\alpha} \\
&= \left((1-\delta) \cdot \frac{K_{t-1}}{N} \cdot \sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} + Z_{t-1} \cdot \frac{\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot f(\theta_{nt})}{\sum_{i=1}^{N} f(\theta_{it})} \right)^{1-\alpha} \\
&= \left((1-\delta) \cdot K_{t-1} \cdot E\left(\theta_{nt}^{\frac{1}{1-\alpha}} \right) + Z_{t-1} \cdot \frac{E\left(\theta_{nt}^{\frac{1}{1-\alpha}} \cdot f(\theta_{nt}) \right)}{E\left(f(\theta_{it}) \right)} \right)^{1-\alpha} \\
&= \left((1-\delta) \cdot K_{t-1} \cdot E\left(\theta^{\frac{1}{1-\alpha}} \right) + Z_{t-1} \cdot \frac{E\left(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta) \right)}{E\left(f(\theta) \right)} \right)^{1-\alpha}
\end{aligned}
\tag{35}
$$

Using the fact that individual shocks are identically distributed and that the distribution does not change over time we have skipped sub-indexes on the final step in the equation (35) under the expectations.

Much more interesting result we can get if we consider the output in p periods ahead (to simplify notations let's denote $\tau = t + p$):

$$
\begin{aligned}
X_\tau &= \left(\sum_{n=1}^{N} \theta_{n\tau}^{\frac{1}{1-\alpha}} \cdot k_{n\tau}\right)^{1-\alpha} \\
&= \left(\sum_{n=1}^{N} \theta_{n\tau}^{\frac{1}{1-\alpha}} \left((1-\delta)^{p+1} k_{n\ t-1} + \sum_{g=0}^{p} (1-\delta)^g \frac{f(\theta_{n\ \tau-g}) Z_{\tau-g-1}}{\sum_{i=1}^{N} f(\theta_{i\ \tau-g})}\right)\right)^{1-\alpha} \\
&= \left((1-\delta)^{p+1} K_{t-1} \frac{\sum_{n=1}^{N} \theta_{n\tau}^{\frac{1}{1-\alpha}}}{N} + \sum_{g=0}^{p} (1-\delta)^g \frac{\sum_{n=1}^{N} \theta_{n\tau}^{\frac{1}{1-\alpha}} f(\theta_{n\ \tau-g}) Z_{\tau-g-1}}{\sum_{i=1}^{N} f(\theta_{i\ \tau-g})}\right)^{1-\alpha} \\
&= \left((1-\delta)^{p+1} \cdot K_{t-1} \cdot E(\theta_{n\tau}^{\frac{1}{1-\alpha}}) + \sum_{g=0}^{p} (1-\delta)^g \cdot \frac{E(\theta_{n\tau}^{\frac{1}{1-\alpha}} f(\theta_{n\ \tau-g}))}{E(f(\theta_{i\ \tau-g}))} Z_{\tau-g-1}\right)^{1-\alpha} \\
&= \left((1-\delta)^{p+1} \cdot K_{t-1} \cdot E(\theta^{\frac{1}{1-\alpha}}) + \sum_{g=0}^{p} (1-\delta)^g \cdot \frac{E(\theta_{\tau}^{\frac{1}{1-\alpha}} \cdot f(\theta_{\tau-g}))}{E(f(\theta))} Z_{\tau-g-1}\right)^{1-\alpha}
\end{aligned} \tag{36}
$$

Note that now on the last step we cannot skip all the time indexes under the expectations. We are not able to do this because the Borel measurable functions of the shocks taken with different lags are independently distributed (proposition 4).

Proposition 3.

Sigma-algebra generated by any Borel measurable function $f(x)$ is not bigger than the sigma-algebra generated by the random variable x (i.e. $\sigma(f(x)) \subseteq \sigma(x)$).

Proof.

This fact follows from the definition of the Borel measurable function. □

Proposition 4.

For any Borel measurable functions $f(x)$ and $g(x)$ if x_1 and x_2 are independent random variables than $f(x_1)$ and $f(x_2)$ are also independent.

Proof.

This fact follows from the proposition 3. □

Therefore using proposition 4 we can rewrite the sum in the brackets from the equation (36) as follows:

$$
\begin{aligned}
&\sum_{g=0}^{p}(1-\delta)^g \cdot \frac{E(\theta_{t+p}^{\frac{1}{1-\alpha}} \cdot f(\theta_{t+p-g}))}{E(f(\theta))} \cdot Z_{t+p-g-1} \\
&= \sum_{g=1}^{p}(1-\delta)^g \cdot \frac{E(\theta^{\frac{1}{1-\alpha}}) \cdot E(f(\theta)))}{E(f(\theta))} \cdot Z_{t+p-g-1} + \frac{E(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta))}{E(f(\theta))} \cdot Z_{t+p-1} \\
&= \sum_{g=1}^{p}(1-\delta)^g \cdot E(\theta^{\frac{1}{1-\alpha}}) \cdot Z_{t+p-g-1} + \frac{E(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta))}{E(f(\theta))} \cdot Z_{t+p-1} \\
&= E(\theta^{\frac{1}{1-\alpha}}) \cdot \sum_{g=1}^{p}(1-\delta)^g \cdot Z_{t+p-g-1} + \frac{E(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta))}{E(f(\theta))} \cdot Z_{t+p-1}
\end{aligned} \tag{37}
$$

Not that as aggregate capital does not depend on the allocation of new investment (as all firms experience the same depreciation rate).

$$
K_{t+p} = (1-\delta)^p \cdot K_t + \sum_{g=1}^{p}(1-\delta)^{g-1} \cdot Z_{t+p-g} \tag{38}
$$

Now let's plug in results of the (37) and (38) into the expression in the brackets from (36):

$$
\begin{aligned}
&(1-\delta)^{p+1} \cdot K_{t-1} \cdot E(\theta^{\frac{1}{1-\alpha}}) + \sum_{g=0}^{p}(1-\delta)^g \cdot \frac{E(\theta_{t+p}^{\frac{1}{1-\alpha}} f(\theta_{t+p-g}))}{E(f(\theta))} \cdot Z_{t+p-g-1} \\
&= E(\theta^{\frac{1}{1-\alpha}}) \cdot \left[(1-\delta)^{p+1} \cdot K_{t-1} + \sum_{g=1}^{p}(1-\delta)^g \cdot E(f(\theta_{t+p-g})) \cdot Z_{t+p-g-1}\right] \\
&+ Z_{t+p-1} \cdot \frac{E\left(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta)\right)}{E\left(f(\theta)\right)} \\
&= (1-\delta) \cdot K_{t+p-1} \cdot E(\theta^{\frac{1}{1-\alpha}}) + Z_{t+p-1} \cdot \frac{E\left(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta)\right)}{E\left(f(\theta)\right)}
\end{aligned} \tag{39}
$$

Finally aggregate output at the period $t+p$ equals to

$$X_{t+p} = \left((1-\delta) \cdot K_{t+p-1} \cdot E(\theta^{\frac{1}{1-\alpha}}) + Z_{t+p-1} \cdot \frac{E\left(\theta^{\frac{1}{1-\alpha}} \cdot f(\theta)\right)}{E\left(f(\theta)\right)} \right)^{1-\alpha} \tag{40}$$

Comparison of the results of calculation of the output 1 and p periods ahead ((35) and (40)) shows that under the proportional reallocation the economy becomes risk-less if at some arbitrary period of time all firms held the same amount of the capital. Moreover if the capital depreciation is greater than zero and the economy uses the proportional reallocation rule for sufficiently long time, than the effect of the initial unequal capital allocation can be neglected.

In this case the distribution of the capital holdings of each individual firm becomes stationary, that is does not change over the time (see Appendix A). As individual capital holdings represent a whole history of individual productivity shocks and follows the same stochastic stationary process, *"long-run distributions for an individual coincide with cross-section distributions for the population"* Aiyagari (1994).

To make a thumbnail estimation of the goodness of the effect of the proportional ex-post reallocation, let's compare the aggregate output of the economy with such type of the capital reallocation (40) and without it (19). For this purpose let‘s recall that expectation of the product equals to the product of expectations plus the covariance. Therefore it is convenient to rewrite the equation (40):

$$
\begin{aligned}
X_{t+p} = \Bigg(& (1-\delta)\cdot K_{t+p-1}\cdot E(\theta^{\frac{1}{1-\alpha}}) \\
& + Z_{t+p-1}\cdot \frac{E\left(\theta^{\frac{1}{1-\alpha}}\right)\cdot E\left(f(\theta)\right) + cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)}\Bigg)^{1-\alpha} \\
= & \left(\left[(1-\delta)\cdot K_{t+p-1} + Z_{t+p-1}\right]\cdot E(\theta^{\frac{1}{1-\alpha}}) + Z_{t+p-1}\cdot \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)}\right)^{1-\alpha} \\
= & \left(K_{t+p}\cdot E(\theta^{\frac{1}{1-\alpha}}) + Z_{t+p-1}\cdot \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)}\right)^{1-\alpha}
\end{aligned}
\tag{41}
$$

As a result when $f(\theta)$ is positively correlated with $\theta^{\frac{1}{1-\alpha}}$ and has positive mean[2], the aggregate output of the economy with ex-post proportional reallocation outperforms the output without re-allocation. The intuition of this result is very straightforward, when the ex-post capital reallocation is done to increase the aggregate output of the economy, it is quite obvious that more productive firm should gain more new capital rather than the less productive. The latter gives us positive correlation of the capital distribution and the productivity level.

In the next section we will try to find the specification of the function that would maximize the aggregate output. Obviously this specification will depend on the distribution function of the shocks. To simplify calculations we will assume that the shocks are drawn from the uniform distribution with positive values.

[2] As it was discussed above for any realization of the shock the $f(\theta)$ should be non-negative and should be positive for the non-empty set of shocks. Obviously the expected value of this function of the shock is positive.

4 Proportional Allocation: Optimal Function

As we have shown in the previous part the aggregate output in the economy with proportional ex-post reallocation is stable and exceeds the output in the case of the absence of the reallocation. Now it makes sense to investigate how the exact specification of the capital distribution function changes the aggregate output of the economy. From now assume that the productivity shocks are distributed with uniform distribution $U[\theta_{min}, \theta_{max}]$ with $\theta_{min} > 0$.

4.1 Linear Function

The simplest and very common group of functions as linear functions, i.e. functions that can be written as $f(x) = a \cdot x + b$ with $a \neq 0$. We will start our analysis with type of functions because even though this functions are very simple in use, the linearity of the expectation can lead to interesting results.

Firstly let‘s consider the fraction from the brackets in the equation of the aggregate output (41).

$$\begin{aligned}
&\frac{cov\left(\theta^{\frac{1}{1-\alpha}};f(\theta)\right)}{E\left(f(\theta)\right)}=\frac{cov\left(\theta^{\frac{1}{1-\alpha}};a\cdot\theta+b\right)}{E\left(a\cdot\theta+b\right)}=\frac{a\cdot cov\left(\theta^{\frac{1}{1-\alpha}};\theta\right)}{a\cdot E\left(\theta\right)+b}\\
&=\frac{a\cdot\left(E(\theta^{1+\frac{1}{1-\alpha}})-E(\theta^{\frac{1}{1-\alpha}})\cdot E(\theta)\right)}{a\cdot E\left(\theta\right)+b}\\
&=\frac{a\cdot\left(\int_{\theta_{min}}^{\theta_{max}}\frac{\mathrm{x}^{1+\frac{1}{1-\alpha}}}{\theta_{\max}-\theta_{\min}}\mathrm{d}x-\int_{\theta_{min}}^{\theta_{max}}\frac{\mathrm{x}^{\frac{1}{1-\alpha}}}{\theta_{\max}-\theta_{\min}}\mathrm{d}x\cdot\int_{\theta_{min}}^{\theta_{max}}\frac{\mathrm{x}}{\theta_{\max}-\theta_{\min}}\mathrm{d}x\right)}{a\cdot\int_{\theta_{min}}^{\theta_{max}}\frac{\mathrm{x}}{\theta_{\max}-\theta_{\min}}\mathrm{d}x+b}\\
&=\frac{a\cdot\left(\frac{\theta_{max}^{2+\frac{1}{1-\alpha}}-\theta_{min}^{2+\frac{1}{1-\alpha}}}{(2+\frac{1}{1-\alpha})\cdot(\theta_{max}-\theta_{min})}-\frac{\theta_{max}^{1+\frac{1}{1-\alpha}}-\theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha})\cdot(\theta_{max}-\theta_{min})}\cdot\frac{\theta_{max}+\theta_{min}}{2}\right)}{a\cdot\frac{\theta_{max}+\theta_{min}}{2}+b}=\cdots\\
&=a\cdot\frac{\frac{1}{1-\alpha}\cdot\left(\theta_{max}^{2+\frac{1}{1-\alpha}}-\theta_{min}^{2+\frac{1}{1-\alpha}}\right)-(2+\frac{1}{1-\alpha})\cdot\theta_{max}\cdot\theta_{min}\left(\theta_{max}^{\frac{1}{1-\alpha}}-\theta_{min}^{\frac{1}{1-\alpha}}\right)}{2\cdot(2+\frac{1}{1-\alpha})\cdot(1+\frac{1}{1-\alpha})\cdot(\theta_{max}-\theta_{min})\cdot(a\cdot\frac{\theta_{max}+\theta_{min}}{2}+b)}
\end{aligned}\tag{42}$$

Now let's try to find the values of a and b that would maximize this expression. As b is left only in the denominator, the whole expression increases when b decreases. Recall that $f(\theta)>0$ for all possible values of θ, therefore b has a lower bound ($a\cdot\theta_{min}+b\geq 0 \Rightarrow b\geq -a\cdot\theta_{min}$). Combining the fact that the expression increases when b decreases and the presence of the restriction we have that the inequality for b binds. Thus $b=-a\cdot\theta_{min}$. After plugging in the optimal value of b to the equation (43) we will get that the expression does not depend on the value of a.

$$
\begin{aligned}
& a \cdot \frac{\frac{1}{1-\alpha} \cdot \left(\theta_{max}^{2+\frac{1}{1-\alpha}} - \theta_{min}^{2+\frac{1}{1-\alpha}} \right) - (2+\frac{1}{1-\alpha}) \cdot \theta_{max} \cdot \theta_{min} \left(\theta_{max}^{\frac{1}{1-\alpha}} - \theta_{min}^{\frac{1}{1-\alpha}} \right)}{2 \cdot (2+\frac{1}{1-\alpha}) \cdot (1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min}) \cdot (a \cdot \frac{\theta_{max}+\theta_{min}}{2} + b)} \\
&= a \cdot \frac{\frac{1}{1-\alpha} \cdot \left(\theta_{max}^{2+\frac{1}{1-\alpha}} - \theta_{min}^{2+\frac{1}{1-\alpha}} \right) - (2+\frac{1}{1-\alpha}) \cdot \theta_{max} \cdot \theta_{min} \left(\theta_{max}^{\frac{1}{1-\alpha}} - \theta_{min}^{\frac{1}{1-\alpha}} \right)}{2 \cdot (2+\frac{1}{1-\alpha}) \cdot (1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min}) \cdot (a \cdot \frac{\theta_{max}+\theta_{min}}{2} - a \cdot \theta_{min})} \\
&= \frac{\frac{1}{1-\alpha} \cdot \left(\theta_{max}^{2+\frac{1}{1-\alpha}} - \theta_{min}^{2+\frac{1}{1-\alpha}} \right) - (2+\frac{1}{1-\alpha}) \cdot \theta_{max} \cdot \theta_{min} \left(\theta_{max}^{\frac{1}{1-\alpha}} - \theta_{min}^{\frac{1}{1-\alpha}} \right)}{(2+\frac{1}{1-\alpha}) \cdot (1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})^2}
\end{aligned} \tag{43}
$$

Assume $\alpha = \frac{1}{2}$:

$$
\begin{aligned}
\frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} &= \frac{2 \cdot \left(\theta_{max}^{2+2} - \theta_{min}^{2+2}\right) - (2+2) \cdot \theta_{max} \cdot \theta_{min} \left(\theta_{max}^{2} - \theta_{min}^{2}\right)}{(2+2) \cdot (1+2) \cdot (\theta_{max} - \theta_{min})^2} \\
&= \cdots = \frac{(\theta_{max}^2 - \theta_{min}^2)}{6}
\end{aligned} \tag{44}
$$

Combining result of (44) with (41) we can get the aggregate output in such an economy with $\alpha = \frac{1}{2}$:

$$
\begin{aligned}
X_{t+p} &= \left(K_{t+p} \cdot E(\theta^{\frac{1}{1-\alpha}}) + Z_{t+p-1} \cdot \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} \right)^{0.5} \\
&= \left(K_{t+p} \cdot \frac{\theta_{max}^2 + \theta_{max} \cdot \theta_{min} + \theta_{min}^2}{3} + Z_{t+p-1} \cdot \frac{\theta_{max}^2 - \theta_{min}^2}{6} \right)^{0.5}
\end{aligned} \tag{45}
$$

Hence, the optimal shape of the linear function is $f(x) = a \cdot (x - \theta_{min})$ with $a > 0$ (note that we eliminate negative values of a because $x - \theta_{min}$ is not negative for any realization of the shock and the value of the function of the shock should be non-negative, as we have discussed before). The intuition that in the class of linear function optimal solution gives only this shape is rather clear:

- It is optimal to allocate a zero fraction of the capital to the less productive firm $\Rightarrow f(\theta_{min}) = a \cdot (\theta_{min} - \theta_{min}) = 0$;

- Let's consider two different realizations of the function: $f_1(x) = a_1 \cdot (x - \theta_{min})$ and $f_2(x) = a_2 \cdot (x - \theta_{min})$ $\forall a_1 > 0, a_2 > 0, a_1 \neq a_2$. The fraction of the capital that will receive the firm with productivity θ does not depend on the choice of the function: $\frac{f_1(\theta)}{\sum_{n=1}^{N} f_1(\theta_n)} = \frac{a_1 \cdot (\theta - \theta_{min})}{a_1 \cdot \sum_{n=1}^{N} (\theta_n - \theta_{min})} = \frac{\theta - \theta_{min}}{\sum_{n=1}^{N} (\theta_n - \theta_{min})} = \frac{a_2 \cdot (\theta - \theta_{min})}{a_2 \cdot \sum_{n=1}^{N} (\theta_n - \theta_{min})} = \frac{f_2(\theta)}{\sum_{n=1}^{N} f_2(\theta_n)}$.

4.2 Power Function

The next type of functions that we will consider is the power functions, i.e. functions of the form $f(x) = x^{\gamma}$. The power functions are fairly convenient to use in our context, because we can always find an integral of the power function, therefore the explicit solution of the equation of the aggregate output exists.

Similarly to the equation (43):

$$
\begin{aligned}
\frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} &= \frac{\left(E(\theta^{\gamma+\frac{1}{1-\alpha}}) - E(\theta^{\frac{1}{1-\alpha}}) \cdot E(\theta^{\gamma})\right)}{E\left(\theta^{\gamma}\right)} \\
&= \frac{\left(\frac{\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}} - \theta_{min}^{\gamma+1+\frac{1}{1-\alpha}}}{(\gamma+1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})} - \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})} \cdot \frac{\theta_{max}^{\gamma+1} - \theta_{min}^{\gamma+1}}{(\gamma+1) \cdot (\theta_{max} - \theta_{min})}\right)}{\frac{\theta_{max}^{\gamma+1} - \theta_{min}^{\gamma+1}}{(\gamma+1) \cdot (\theta_{max} - \theta_{min})}} \\
&= \frac{\gamma+1}{\gamma+1+\frac{1}{1-\alpha}} \cdot \frac{\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}} - \theta_{min}^{\gamma+1+\frac{1}{1-\alpha}}}{\theta_{max}^{\gamma+1} - \theta_{min}^{\gamma+1}} - \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})}
\end{aligned}
\tag{46}
$$

As the second part of the expression does not depend on the γ the maximization of the whole expression equivalent to the maximization of its first part:

$$max_{\gamma}\left[\frac{\gamma+1}{\gamma+1+\frac{1}{1-\alpha}}\cdot\frac{\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}}-\theta_{min}^{\gamma+1+\frac{1}{1-\alpha}}}{\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1}}\right]$$

$$\begin{aligned}
&FOC: \\
&\frac{\frac{1}{1-\alpha}}{(\gamma+1+\frac{1}{1-\alpha})^2}\cdot\frac{\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}}-\theta_{min}^{\gamma+1+\frac{1}{1-\alpha}}}{\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1}} \\
&+\frac{\gamma+1}{\gamma+1+\frac{1}{1-\alpha}}\cdot\frac{(\theta_{max}^{\gamma+1+\frac{1}{1+\alpha}}\cdot log(\theta_{max})-\theta_{min}^{\gamma+1+\frac{1}{1+\alpha}}\cdot log(\theta_{min}))}{\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1}} \\
&-\frac{\gamma+1}{\gamma+1+\frac{1}{1-\alpha}}\cdot\frac{(\theta_{max}^{\gamma+1}\cdot log(\theta_{max})-\theta_{min}^{\gamma+1}\cdot log(\theta_{min}))\cdot(\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}}-\theta_{min}^{\gamma+1+\frac{1}{1-\alpha}})}{(\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1})^2} \\
&=\cdots=\frac{\frac{1}{1-\alpha}}{(\gamma+1+\frac{1}{1-\alpha})^2}\cdot\frac{\theta_{max}^{\gamma+1+\frac{1}{1-\alpha}}-\theta_{min}^{\gamma+1+\frac{1}{1-\alpha}}}{\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1}} \\
&-\frac{\gamma+1}{\gamma+1+\frac{1}{1-\alpha}}\cdot\frac{\theta_{max}^{\gamma+1}\cdot\theta_{min}^{\gamma+1}\cdot(\theta_{max}^{\frac{1}{1-\alpha}}-\theta_{min}^{\frac{1}{1-\alpha}})\cdot log(\frac{\theta_{max}}{\theta_{min}})}{(\theta_{max}^{\gamma+1}-\theta_{min}^{\gamma+1})^2}=0
\end{aligned} \tag{47}$$

Even though we cannot find the explicit solution for the optimal level of γ, the result can be obtained either numerically after determination of all the parameters of the economy $(\theta_{min},\theta_{max},\alpha)$ or using approximation under the assumption that $\frac{\theta_{max}-\theta_{min}}{\theta_{max}}\equiv\frac{\triangle}{\theta_{max}}\ll 1$. Under this assumption we can rewrite equation using the linear approximation: $\theta_{max}^{\beta}-\theta_{min}^{\beta}=\theta_{max}^{\beta}\cdot\left(1-(\frac{\theta_{min}}{\theta_{max}})^{\beta}\right)=\theta_{max}^{\beta}\cdot\left(1-(1-\frac{\triangle}{\theta_{max}})^{\beta}\right)=\theta_{max}^{\beta}\cdot\left(1-(1-\beta\cdot\frac{\triangle}{\theta_{max}})\right)=\theta_{max}^{\beta}\cdot\beta\cdot\frac{\triangle}{\theta_{max}}=\theta_{max}^{\beta-1}\cdot\beta\cdot\triangle$. Using this linearization equation (47) can be rewritten as:

$$\gamma=\frac{2\cdot\theta_{min}-\theta_{max}}{\theta_{max}-\theta_{min}} \tag{48}$$

Now let's discuss the intuition of the result. From the assumption that $\frac{\triangle}{\theta_{max}} \ll 1$ we can conclude that $\theta_{min} \approx \theta_{max} = \theta$, therefore $2 \cdot \theta_{min} - \theta_{max} \approx \theta$. Thus $\gamma \approx \frac{\theta}{\theta_{max} - \theta_{min}} \gg \theta$. As a result the power of the function is sufficiently high, i.e. the firm with higher productivity will obtain more capital rather than the one with lower productivity, which is consistent with our previous analysis.

4.3 Exponential Function

The third specification of the function is an exponential, i.e. $f(x) = e^{\phi \cdot x}$. In this case the part of the expression for the aggregate output with the covariance term will be as follows:

$$\frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} = \frac{\left(E(\theta^{\frac{1}{1-\alpha}} \cdot e^{\phi \cdot \theta}) - E(\theta^{\frac{1}{1-\alpha}}) \cdot E(e^{\phi \cdot \theta})\right)}{E(e^{\phi \cdot \theta})}$$
$$= \frac{E(\theta^{\frac{1}{1-\alpha}} \cdot e^{\phi \cdot \theta})}{E(e^{\phi \cdot \theta})} - E(\theta^{\frac{1}{1-\alpha}}) = \frac{\int_{\theta_{min}}^{\theta_{max}} \mathrm{x}^{\frac{1}{1-\alpha}} \cdot \mathrm{e}^{\phi \cdot \mathrm{x}} \mathrm{d}x}{\frac{e^{\phi \cdot \theta_{max}} - e^{\phi \cdot \theta_{min}}}{\phi}} - \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1 + \frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})} \tag{49}$$

In general, the integral in the nominator can be written using the gamma-function, but if we specify the the value of the α the result will be much easier. For example, when $\alpha = 0.5$ as we have considered before, equation (49) can be simplified:

$$
\begin{aligned}
&\frac{\int_{\theta_{min}}^{\theta_{max}} \mathrm{x}^{\frac{1}{1-\alpha}} \cdot \mathrm{e}^{\phi \cdot \mathrm{x}} \mathrm{d}x}{\frac{e^{\phi \cdot \theta_{max}} - e^{\phi \cdot \theta_{min}}}{\phi}} - \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})} \\
&= \frac{\frac{e^{\phi \cdot \theta_{max}} \cdot (\phi^2 \cdot \theta_{max}^2 - 2 \cdot \phi \cdot \theta_{max} + 2) - e^{\phi \cdot \theta_{min}} \cdot (\phi^2 \cdot \theta_{min}^2 - 2 \cdot \phi \cdot \theta_{min} + 2)}{\phi^3}}{\frac{e^{\phi \cdot \theta_{max}} - e^{\phi \cdot \theta_{min}}}{\phi}} - \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})} \\
&= \frac{e^{\phi \cdot (\theta_{max} - \theta_{min})} \cdot (\phi^2 \cdot \theta_{max}^2 - 2 \cdot \phi \cdot \theta_{max} + 2) - (\phi^2 \cdot \theta_{min}^2 - 2 \cdot \phi \cdot \theta_{min} + 2)}{\phi^2 \cdot (e^{\phi \cdot (\theta_{max} - \theta_{min})} - 1)} \\
&- \frac{\theta_{max}^{1+\frac{1}{1-\alpha}} - \theta_{min}^{1+\frac{1}{1-\alpha}}}{(1+\frac{1}{1-\alpha}) \cdot (\theta_{max} - \theta_{min})}
\end{aligned}
\tag{50}
$$

Optimization yields:

$$
\begin{aligned}
&e^{\phi \cdot (\theta_{max} - \theta_{min})} \cdot (\phi^2 \cdot \theta_{max}^2 \cdot (\theta_{max} - \theta_{min}) + 2 \cdot \phi \cdot \theta_{max} \cdot \theta_{min} - 2 \cdot \theta_{min}) \\
&\qquad - (2 \cdot \phi \cdot \theta_{min}^2 - 2 \cdot \theta_{min}) \\
&= (e^{\phi \cdot (\theta_{max} - \theta_{min})} \cdot (\phi^2 \cdot \theta_{max}^2 - 2 \cdot \phi \cdot \theta_{max} + 2) - (\phi^2 \cdot \theta_{min}^2 - 2 \cdot \phi \cdot \theta_{min} + 2)) \cdot \\
&\qquad \cdot \frac{(2 \cdot (e^{\phi \cdot (\theta_{max} - \theta_{min})} - 1) + (\theta_{max} - \theta_{min}) \cdot \phi e^{\phi \cdot (\theta_{max} - \theta_{min})}}{\phi \cdot (e^{\phi \cdot (\theta_{max} - \theta_{min})} - 1)}
\end{aligned}
\tag{51}
$$

Again this equation can be solved numerically, when all the parameters of the economy are known, or under the assumption of the relatively small distance between θ_{max} and θ_{min}. We will use the latter technique. After some calculations:

$$
\begin{aligned}
&\phi^2 \theta_{max} \cdot (\theta_{max}^2 + \theta_{max}\theta_{min} - 2 \cdot \theta_{min}^2) \\
&= \phi \cdot (\theta_{max} - \theta_{min}) \cdot (\phi \cdot \theta_{max}^2 - \theta_{max} + \theta_{min}) \cdot (3 + \phi \cdot (\theta_{max} - \theta_{min}))
\end{aligned}
\tag{52}
$$

This equation can be approximated after eliminating of the members that are proportional to the $\theta_{max} - \theta_{min}$ to the power larger than 1:

$$\phi \approx \frac{1}{\theta_{max}} \; or \; \phi \approx -\frac{1}{3 \cdot \theta_{max}} \tag{53}$$

Consistently to the ours previous intuition firm with higher productivity level should obtain more new capital rather than less productive one. Therefore the power of the exponent should be positive, so the only optimal solution is $\phi \approx \frac{1}{\theta_{max}}$.

4.4 Logarithmic Function

Finally let‘s consider logarithmic functions, the solution we will search in the form of $f(x) = log(c \cdot x) = log(c) + log(x)$.

$$\frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} = \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; log(c) + log(\theta)\right)}{E\left(log(c) + log(\theta)\right)} = \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; log(\theta)\right)}{log(c) + E\left(log(\theta)\right)} \tag{54}$$

As we can see the nominator of the equation (54) does not depend on the choice of the parameter c. Therefore the value of c that minimizes the denominator in the space of positive values (this condition we need because the nominator obviously is positive). Moreover similarly to the case of the linear function we have to remember that the value of the function should be non-negative for any realization of the shock (i.e.$\forall \theta : log(c) + log(\theta) \geq 0$ or just $c \geq \frac{1}{\theta_{min}}$). Combination of these two constraints yields that the latter one binds, so $c = \frac{1}{\theta_{min}}$. In this case equation (54) gives us:

$$
\begin{aligned}
\frac{cov\left(\theta^{\frac{1}{1-\alpha}};log(\theta)\right)}{log(c)+E\left(log(\theta)\right)} &= \frac{cov\left(\theta^{\frac{1}{1-\alpha}};log(\theta)\right)}{E\left(log(\theta)\right)-log(\theta_{min})} \\
&= \frac{cov\left(\theta^{\frac{1}{1-\alpha}};log(\theta)\right)}{\frac{\theta_{max}\cdot log(\theta_{max})-\theta_{max}-\theta_{min}\cdot log(\theta_{min})+\theta_{min}}{\theta_{max}-\theta_{min}}-log(\theta_{min})} \\
&= \cdots = \frac{\theta_{min}\cdot cov\left(\theta^{\frac{1}{1-\alpha}};log(\theta)\right)}{\theta_{max}-\theta_{min}}
\end{aligned}
\tag{55}
$$

Here we have used a linearization of the logarithm under condition of close θ_{min} and θ_{max}. Finally, to calculate the covariance we have to specify the value of α, as usual, let $\alpha = 0.5$.

$$
\begin{aligned}
&\frac{\theta_{min}\cdot cov\left(\theta^2;log(\theta)\right)}{\theta_{max}-\theta_{min}} = \cdots \\
&= \frac{\theta_{min}\cdot\left(\frac{2}{9}\cdot(\theta_{max}^3-\theta_{min}^3)-\frac{1}{3}\cdot\theta_{max}\cdot\theta_{min}\cdot(\theta_{max}+\theta_{min})\cdot log(\frac{\theta_{max}}{\theta_{min}})\right)}{(\theta_{max}-\theta_{min})^2} \\
&\approx \frac{(\theta_{max}-\theta_{min})\cdot(3\cdot\theta_{max}+4\cdot\theta_{min})}{18}
\end{aligned}
\tag{56}
$$

Again the last step in the equation (56) using the linearization under the assumption that θ_{max} and θ_{min} are close enough.

Now when we have found optimal shapes of the distribution functions from different main classes, we can proceed to the consideration of its dynamic properties. In the next section we will study impulse response functions of the aggregate output under different investment rules. As a result we will be able to compare resulting fluctuations and select optimal investment rule.

5 Impulse Response Functions

In the previous sections we have considered different types and functions of capital allocation. All this investment rules give us different resulting behavior of the economy. In this section firstly we will provide the classification of all studied investment rules with its properties. Then we will study the dynamic properties of these rules: amplitude and length of the shock persistence.

Investment rule			**Fraction that goes to n'th firm**
Ex-ante			$\frac{1}{N}$
Ex-post	Simple		$\begin{cases} 1 & \text{if } \theta_{nt} = \theta_{max} \\ 0 & \text{otherwise} \end{cases}$
	Proportional	Linear	$\frac{\theta_{nt}-\theta_{min}}{\sum_{i=1}^{N}\left(\theta_{it}-\theta_{min}\right)}$
		Power	$\frac{\theta_{nt}^{\gamma}}{\sum_{i=1}^{N}\theta_{it}^{\gamma}} \quad \gamma \approx \frac{2\cdot\theta_{min}-\theta_{max}}{\theta_{max}-\theta_{min}}$
		Exponential	$\frac{exp(\frac{\theta_{nt}}{\theta_{max}})}{\sum_{i=1}^{N}\exp(\frac{\theta_{it}}{\theta_{max}})}$
		Logarithmic	$\frac{log(\frac{\theta_{nt}}{\theta_{min}})}{\sum_{i=1}^{N}\log(\frac{\theta_{it}}{\theta_{min}})}$

Table 1: **Considered investment rules.**

To find impulse response functions first of all we need to calculate the steady state production level, capital holdings and other parameters of the economy. After that we will introduce a one period production shock, for instance, assume that in the period t production was 10% higher than the equilibrium level. Using equations for households' savings, capital flow and aggregate output we will be able to find the path of the economy after the shock.

5.1 Ex-ante Allocation

We will consider investment rules in the same order as we have in the previous sections. The simplest distribution was done in the case of ex-ante allocation. To find a steady state we need to combine equations for the aggregate output in this case (19), aggregate capital flow (38), household savings (4):

$$
\begin{aligned}
&X_t = \left(E\left(\theta^{\frac{1}{1-\alpha}}\right)\right)^{1-\alpha} \cdot K_t^{1-\alpha} \\
&K_t = (1-\delta) \cdot K_{t-1} + S_{t-1} \\
&S_{t-1} = \frac{\beta}{E\left(1+r_{nt}\right)+\beta} \cdot \alpha \cdot X_{t-1} \\
&1+r_{nt} = \frac{(1-\alpha) \cdot \alpha^{\frac{\alpha}{1-\alpha}} \cdot \theta_{nt}^{\frac{1}{1-\alpha}}}{\left(\alpha \cdot \left(\sum_{n=1}^{N} \theta_{nt}{}^{\frac{1}{1-\alpha}} \cdot k_{nt}\right)^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}}}
\end{aligned}
\tag{57}
$$

The steady state solution of the equation (57) yields:

$$
\begin{aligned}
&X = \left(\frac{\alpha \cdot \beta - (1-\alpha) \cdot \delta}{\beta \cdot \delta}\right)^{\frac{1-\alpha}{\alpha}} \cdot E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} \\
&K = \left(\frac{\alpha \cdot \beta - (1-\alpha) \cdot \delta}{\beta \cdot \delta}\right)^{\frac{1}{\alpha}} \cdot E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} \\
&S = \delta \cdot K
\end{aligned}
\tag{58}
$$

Let's now assume that $X_t = \phi_X \cdot X$. The shock in the output will shift savings. This change does not have to be proportional to the shock as the households' expectations on the marginal product of capital will change. The level of the aggregate capital does not change in this period, so $\phi_K = 1$. Values of the deviation of the capital and output at the next period we will denote with hats.

$$
\begin{aligned}
\widehat{\phi_X} &= \widehat{\phi_K}^{1-\alpha} \\
\widehat{\phi_K} &= (1-\delta)\cdot\phi_K + \delta\cdot\phi_S \\
\phi_S &= \phi_X \cdot \left(1 + \frac{(\widehat{\phi_K}^{\alpha} - 1)\cdot(1-\alpha)\cdot E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}}}{(1-\alpha)\cdot E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} + \beta K^{\alpha}\cdot\widehat{\phi_K}^{\alpha})}\right)
\end{aligned}
\tag{59}
$$

This recursive system of equations can be solved after determination of the initial parameters of the system. Assume $\phi_X = 1.1$ in this case the diagram of the impulse responses of the output (fig. 2) shows us that the latter gradually declines to its initial value. Moreover the rate of convergence decreases with depreciation rate. Capital holdings experience 1 period lag as the deviation of the capital occurs only after installation of the new capital (fig. 3).

The phenomenon that rate of convergence decreases with depreciation can be explained by the fact that when the fraction of the new capital in the equilibrium is not too high the deviation of the new investment does not shift aggregate parameters too much. Moreover the fact that all the firms receive the same amount of new capital whatever has happened leads to the lack of optimality of the capital allocation, therefore higher shocks decreases slower. The diagram of the response of the capital (fig. 3) can help to understand the previous explanation even better: from the graph we can see that the deviation of the capital, when the $\delta = 0.2$ is almost 10 times lower than when $\delta = 1.0$. Therefore when the value of depreciation is significantly high the response of the capital holdings is fairly high, it leads to rather high deviation of the output the next period as the only variable source of production - capital was shifted.

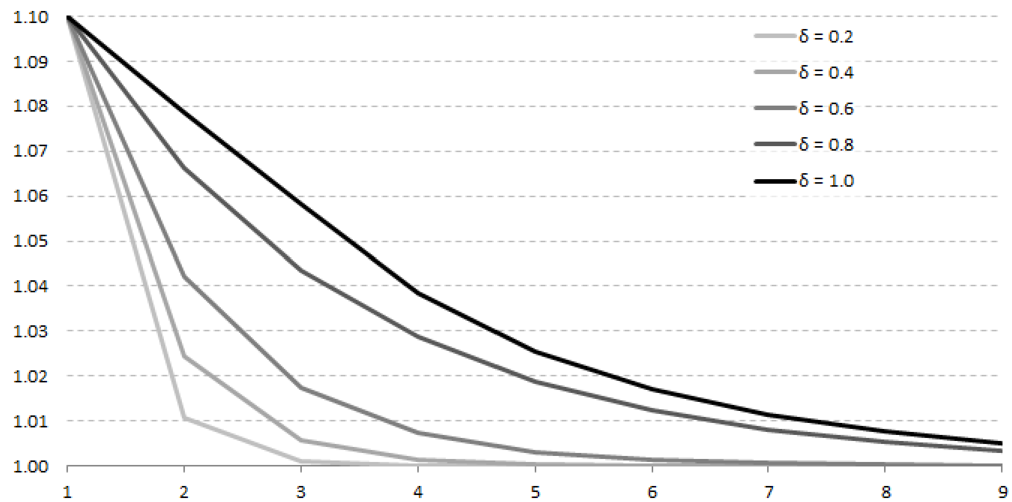

Figure 2: **Aggregate output in the economy with ex-ante capital allocation and with initial deviation of the output by 10%.**

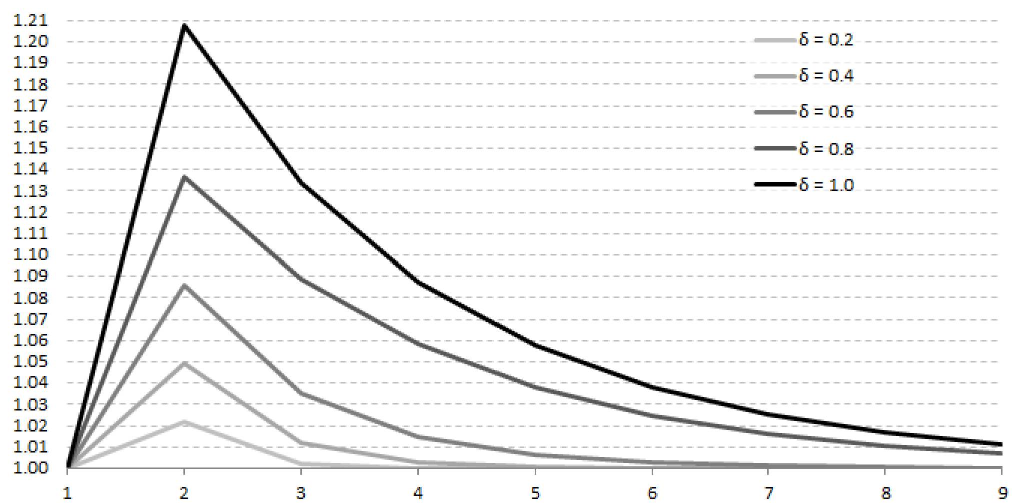

Figure 3: **Aggregate capital in the economy with ex-ante capital allocation and with initial deviation of the output by 10%.**

5.2 Ex-post Allocation: Simple Case

Now let's move to the case of the ex-post capital allocation. As we have agreed this case will be considered under condition of the full capital depreciation. Therefore the system of the equations that describes this economy is slightly different from (57).

$$
\begin{aligned}
&X_t = \theta_{max} \cdot K_t^{1-\alpha} \\
&K_t = S_{t-1} \\
&S_{t-1} = \frac{\beta}{E\big(1+r_{nt}\big)+\beta} \cdot \alpha \cdot X_{t-1} \\
&1+r_{nt} = \frac{(1-\alpha)\cdot\alpha^{\frac{\alpha}{1-\alpha}}\cdot\theta_{nt}^{\frac{1}{1-\alpha}}}{\Big(\alpha\cdot\Big(\sum_{n=1}^{N}\theta_{nt}{}^{\frac{1}{1-\alpha}}\cdot k_{nt}\Big)^{1-\alpha}\Big)^{\frac{\alpha}{1-\alpha}}}
\end{aligned}
\tag{60}
$$

Similarly, the steady state solution of this system is

$$
\begin{aligned}
&S = K = \Big(\frac{\alpha\cdot\beta-(1-\alpha)}{\beta}\cdot\theta_{max}\Big)^{\frac{1}{\alpha}} \\
&X = \theta_{max}\cdot K^{1-\alpha}
\end{aligned}
\tag{61}
$$

Using the same notations as in the previous subsection, we can get the responses on the shock in output:

$$
\begin{aligned}
&\widehat{\phi_X} = \phi_S^{1-\alpha} \\
&\widehat{\phi_K} = \phi_S \\
&\phi_S = \phi_X\cdot\Big(1+\frac{(\phi_S^{\alpha}-1)\cdot(1-\alpha)\cdot\theta_{max}}{(1-\alpha)\cdot\theta_{max}+\beta\cdot S^{\alpha}\cdot\phi_S^{\alpha}}\Big)
\end{aligned}
\tag{62}
$$

It is easy to mention that impulse response functions in this case (fig. 4) are pretty much similar to ones that we have obtained in the previous case.

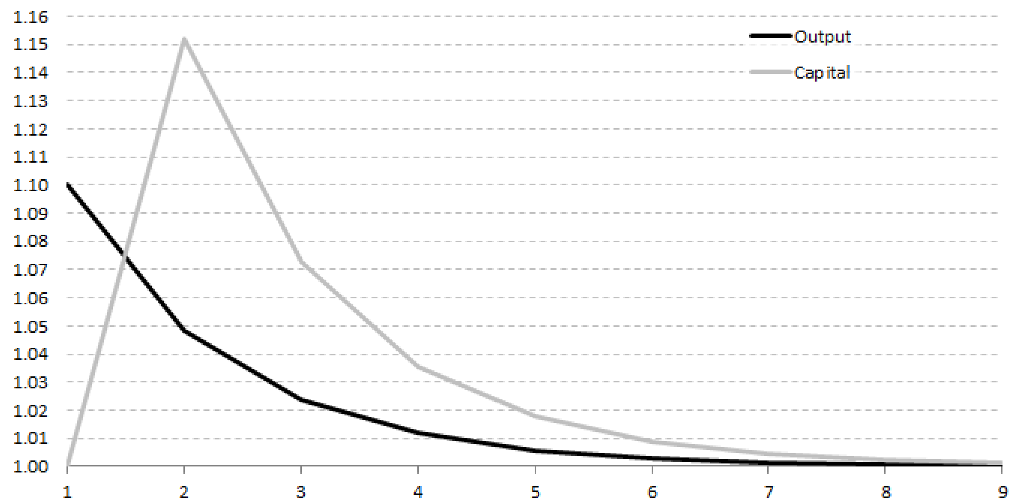

Figure 4: **Output, savings and capital holdings in the economy with ex-ante capital allocation and with initial deviation of the output by 10%.**

5.3 Ex-post Allocation: Proportional Allocation

The last step in our analysis is to study impulse responses in the case of the proportional capital allocation. This investment rule requires to solve the following system of equations:

$$\begin{aligned}
X_t &= \left(K_t \cdot E(\theta^{\frac{1}{1-\alpha}}) + S_{t-1} \cdot \frac{cov\left(\theta^{\frac{1}{1-\alpha}}; f(\theta)\right)}{E\left(f(\theta)\right)} \right)^{1-\alpha} \\
K_t &= (1-\delta) \cdot K_{t-1} + S_{t-1} \\
S_{t-1} &= \frac{\beta}{E\left(1+r_{nt}\right)+\beta} \cdot \alpha \cdot X_{t-1} \\
1+r_{nt} &= \frac{(1-\alpha) \cdot \alpha^{\frac{\alpha}{1-\alpha}} \cdot \theta_{nt}^{\frac{1}{1-\alpha}}}{\left(\alpha \cdot \left(\sum_{n=1}^{N} \theta_{nt}^{\frac{1}{1-\alpha}} \cdot k_{nt}\right)^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}}}
\end{aligned} \tag{63}$$

The steady state solution yields (64), equations for the responses are (65).

$$K=\left(\frac{\alpha\cdot\beta\cdot\left(E(\theta^{\frac{1}{1-\alpha}})+\delta\cdot\frac{cov\left(\theta^{\frac{1}{1-\alpha}};f(\theta)\right)}{E(f(\theta))}\right)-(1-\alpha)\cdot\delta E(\theta^{\frac{1}{1-\alpha}})}{\beta\cdot\delta\cdot\left(E(\theta^{\frac{1}{1-\alpha}})+\delta\cdot\frac{cov\left(\theta^{\frac{1}{1-\alpha}};f(\theta)\right)}{E(f(\theta))}\right)^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$S=\delta\cdot K \tag{64}$$

$$X=\left(E(\theta^{\frac{1}{1-\alpha}})+\delta\cdot\frac{cov\left(\theta^{\frac{1}{1-\alpha}};f(\theta)\right)}{E\left(f(\theta)\right)}\right)^{1-\alpha}\cdot K^{1-\alpha}$$

$$\widehat{\phi_X}=\widehat{\phi_K}^{1-\alpha}$$

$$\widehat{\phi_K}=(1-\delta)\cdot\phi_K+\delta\cdot\phi_S \tag{65}$$

$$\phi_S-\phi_X=\frac{\phi_X\cdot(\widehat{\phi_K}^{\alpha}-1)\cdot(1-\alpha)\cdot E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}}}{(1-\alpha)E\left(\theta^{\frac{1}{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}}+\beta K^{\alpha}\left(E(\theta^{\frac{1}{1-\alpha}})+\delta\frac{cov\left(\theta^{\frac{1}{1-\alpha}};f(\theta)\right)}{E(f(\theta))}\right)^{\alpha}\widehat{\phi_K}^{\alpha}}$$

We can notice that the system (65) is practically the same with (59) the only difference is that there are brackets with covariance in the denominator of the last equation.

Impulse response functions for all types of proportional allocation are very close to each other (up to the 6^{th} decimal)it was because covariance over the expectation are rather close for this functions too, moreover the fraction of these members in the final result is fairly small.

Interesting difference in the case of proportional capital allocation from the case of the ex-ante allocation is that even if the initial deviation from the steady state of the parameters increases with the depreciation rate, the rate of convergence also is higher when depreciation is higher. The possible explanation in this case is that more optimal capital allocation provides an opportunity to the economy to behave in better way. The result of this superior behavior is faster movement to the equilibrium path.

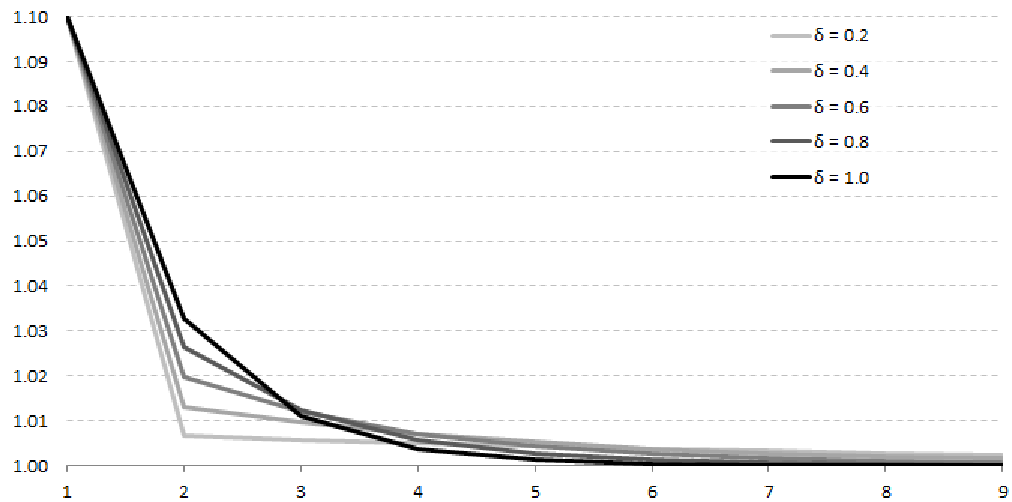

Figure 5: **Aggregate output in the economy with ex-post proportional (linear) capital allocation and with initial deviation of the output by 10%.**

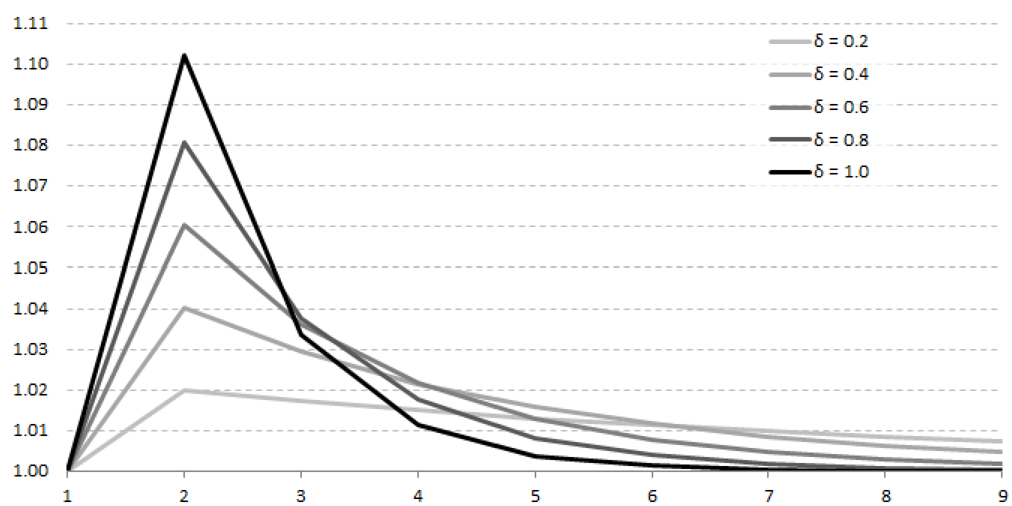

Figure 6: **Aggregate capital in the economy with ex-post proportional (linear) capital allocation and with initial deviation of the output by 10%.**

5.4 Comparison

To summarize the results let's consider all the investment rules and the level of the output that can be obtained under different rates of depreciation.

δ	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
Ex-ante	50.8	33.9	25.9	20.7	16.9
Simple ex-post	-	-	-	-	18.3
Linear ex-post	51.1	34.3	26.4	21.4	17.8
Power ex-post	50.9	34.0	26.1	21.0	17.3
Exponential ex-post	50.8	33.9	25.9	20.8	17.0
Logarithmic ex-post	51.1	34.3	26.4	21.4	17.8

Table 2: **Aggregate output level.**

As we can see all the investment rules leads to rather close level of the aggregate output. However proportional allocation outperforms ex-ante allocation under any values of depreciation rate. Among all types of the proportional capital allocation the best results have linear and logarithmic functions. This result intuitively is very clear: optimal parameters of the linear and logarithmic functions gives zero capital investment in the company with the lowest possible productivity, therefore capital was distributed more optimal.

There is no surprise that under full depreciation the best performance shows simple case of ex-post capital allocation. Obviously, we cannot do better than invest all the money to the most productive firm, if we 100% sure that the next period this firm would not use any capital in case if it is not the most productive once again.

Comparison of the impulse response functions of the aggregate outputs (7) gives us the result that proportional capital allocation has the highest convergence rate under full depreciation. From the previous discussion we can conclude that proportional allocation has grate convergence rate than the one in case of ex-ante allocation because of the more optimal capital investment. From this point of view it is not clear why the simple ex-post capital allocation has slower convergence rate than proportional allocation. But recall that simple ex-post allocation has the highest level of the aggregate parameters and aggregate productivity, therefore the same response in capital holdings has higher effect on the aggregate production.

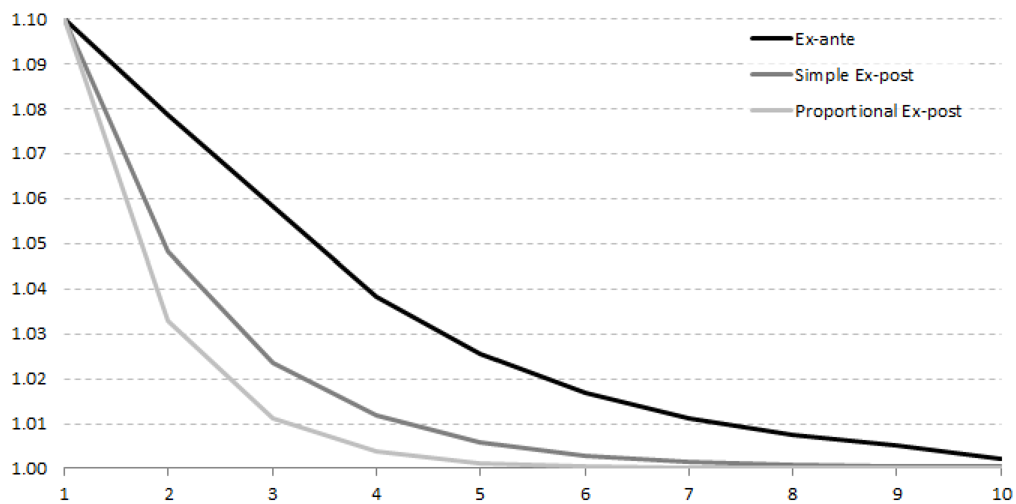

Figure 7: **Aggregate output in the economy with full depreciation and with initial deviation of the output by 10% under different investment rules.**

As a result we cannot give a general answer on the question, what type of capital allocation one should choose. When we deal with the depreciation rate smaller than one, there is no doubt that linear or logarithmic allocation should be selected, as both of them have the highest aggregate level of output and the rate of convergence. More tricky question occurs when we have full capital depreciation. On the one hand, simple ex-post allocation has higher level of production, on the other hand, proportional allocation has higher rate of convergence to the equilibrium path. The final decision on the allocation rule under full depreciation should be done with respect to the individual preferences and the risk tolerance.

6 Conclusion

There are different approaches of explanation of the stylized facts that describe recent behavior of the U.S. economy: decline of the aggregate fluctuations subject to increased level of the firm-specific shocks. In this paper we show that some classes of distribution of new investment among firms can diminish aggregate uncertainty in the economy. We have defined 3 classes of the capital investment rules that gives stationary distribution of the capital between the firms in the economy and under which aggregate fluctuations of production vanish. The first class is so called ex-ante capital allocation, when the distribution occurs before the realization of the individual productivities. The second one is a simple case of ex-post capital allocation, when only the most productive firm assumes new capital each period. Unfortunately, stationary equilibrium in such an economy exists only under full capital depreciation. The last studied class of distribution rules is our innovation and it deals with proportional ex-post capital allocation, when each firm receive new capital accordingly to some Borel measurable function of its productivity.

All classes of the capital allocations that we consider guarantee the persistence of the stationary capital distribution in the economy. As a result joint distribution of productivities and capital holdings of firms are stationary as well. Latter leads to the steady state equilibrium in the aggregate economy.

Among capital allocation rules that give stationarity without full capital depreciation best performance exhibit linear and logarithmic proportional capital allocations. Equilibrium production level and the convergence rate to the equilibrium path are the highest in this cases. This phenomenon can be explained by the diminished fraction of the new capital that goes to less productive firms.

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A Stationarity of the Capital Distribution

By the definition the distribution of capital is stationary if the joint distribution of capital holdings of individual firms is time invariant. We will proof it using the wide-sense stationarity, i.e. when the mean and auto-covariance functions do not depend on time.

Recall that capital holdings of firm i are described by the equation:

$$k_{iT} = \sum_{t=-\infty}^{T} (1-\delta)^{T-t} \cdot \frac{f\left(\theta_{it}\right) \cdot Z_t}{\sum_{n=1}^{N} f\left(\theta_{nt}\right)} \tag{A.1}$$

Expectation of the capital holding:

$$\begin{aligned}
E\left(k_{iT}\right) &= E\left(\sum_{t=-\infty}^{T} (1-\delta)^{T-t} \cdot \frac{f\left(\theta_{it}\right) \cdot Z_t}{\sum_{n=1}^{N} f\left(\theta_{nt}\right)}\right) \\
&= \frac{Z \cdot E\left(\sum_{t=-\infty}^{T} (1-\delta)^{T-t} \cdot f\left(\theta_{it}\right)\right)}{\sum_{n=1}^{N} f\left(\theta_n\right)} \\
&= \frac{Z \cdot \sum_{t=-\infty}^{T} (1-\delta)^{T-t} \cdot E\left(f\left(\theta_{it}\right)\right)}{\sum_{n=1}^{N} f\left(\theta_n\right)} \\
&= \frac{Z \cdot E\left(f\left(\theta_i\right)\right)}{\sum_{n=1}^{N} f\left(\theta_n\right)} \cdot \sum_{\tau=0}^{\infty} (1-\delta)^{\tau} \\
&= \frac{Z \cdot E\left(f\left(\theta_i\right)\right)}{\delta \cdot \sum_{n=1}^{N} f\left(\theta_n\right)}
\end{aligned} \tag{A.2}$$

Here we use explored above facts that in the equilibrium total investments are a constant fraction of the output, which is constant in equilibrium too. Therefore expected value of capital holdings does not depend on time, that is remains constant over time.

Auto-covariance of k_{iT_1} and k_{iT_2} ($T_2 > T_1$):

$$
\begin{aligned}
&COV(k_{iT_1}, k_{iT_2}) = E\left(\left(k_{iT_1} - E(k_{iT_1})\right) \cdot \left(k_{iT_2} - E(k_{iT_2})\right)\right) \\
&= \frac{Z^2 \cdot E\left(\left(\sum_{t=-\infty}^{T-1}(1-\delta)^{T_1-t}\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]\right)\left(\sum_{t=-\infty}^{T_2}(1-\delta)^{T_2-t}\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]\right)\right)}{\left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2} \\
&= \frac{Z^2 \cdot E\left(\left(\sum_{t=-\infty}^{T_1}(1-\delta)^{T_1-t}\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]\right)\left(\sum_{t=-\infty}^{T_2}(1-\delta)^{T_2-t}\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]\right)\right)}{\left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2} \\
&= \frac{Z^2 \cdot E\left(\sum_{t=-\infty}^{T_1}(1-\delta)^{T_1+T_2-2*t}\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]^2\right)}{\left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2} + \\
&+ \frac{Z^2 \cdot E\left(\sum_{t_1 \neq t_2}(1-\delta)^{t_1+t_2} \cdot \left[f\left(\theta_{it_1}\right) - E\left(f(\theta_i)\right)\right]\left[f\left(\theta_{it_2}\right) - E\left(f(\theta_i)\right)\right]\right)}{\left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2} \\
&= \frac{Z^2 \cdot (1-\delta)^{T_2-T_1} \cdot \sum_{t=-\infty}^{T_1}(1-\delta)^{2*(T_1-t)} E\left(\left[f\left(\theta_{it}\right) - E\left(f(\theta_i)\right)\right]^2\right)}{\left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2} \\
&= \frac{Z^2 \cdot (1-\delta)^{T_2-T_1} \cdot VAR\left(f\left(\theta_i\right)\right)}{2 \cdot \delta \cdot (1-\delta) \cdot \left(\sum_{n=1}^{N} f\left(\theta_n\right)\right)^2}
\end{aligned}
\tag{A.3}
$$

As we can see from (A.3) covariance does not depend on the values of T_1 and T_2, but only on its difference. Therefore, mean and covariance of the capital holdings does not vary over time. Consequently, we can conclude, that the distribution of the capital of any arbitrary firm is stationary.

B Simulation Results

For the simulation we use the following parameters of the economy:

Parameter	α	β	θ_{min}	θ_{max}
Value	0.67	0.95	9	10

Table B.1: **List of Parameters.**

δ	parameter	ex-ante	ex-post				
			simple	linear	power	exp	log
0.2	capital	153.4	-	153.4	152.8	152.4	153.4
	output	50.8	-	51.1	50.9	50.8	51.1
	productivity	9.509	-	9.542	9.523	9.510	9.541
0.4	capital	45.1	-	46.0	45.5	45.2	46.0
	output	33.9	-	34.3	34.0	33.9	34.3
	productivity	9.509	-	9.575	9.536	9.512	9.573
0.6	capital	20.1	-	20.8	20.4	20.1	20.8
	output	25.9	-	26.4	26.1	25.9	26.4
	productivity	9.509	-	9.608	9.550	9.514	9.606
0.8	capital	10.4	-	11.0	10.6	10.4	11.0
	output	20.7	-	21.4	21.0	20.8	21.4
	productivity	9.509	-	9.640	9.564	9.515	9.638
1.0	capital	5.6	6.1	6.2	5.9	5.7	6.2
	output	16.9	18.3	17.8	17.3	17.0	17.8
	productivity	9.509	10.000	9.672	9.578	9.517	9.670

Table B.2: **Simulation Results.**

Here we consider implied productivity, so such θ that solves $X = \theta \cdot K^{1-\alpha}$