1 Introduction

Exponential Lévy models generalize the classical Black and Scholes setup by allowing the stock prices to jump while preserving the independence and stationarity of returns. There are ample reasons for introducing jumps in financial modeling. First of all, asset prices do jump, and some risks simply cannot be handled within continuous-path models. Second, the well-documented phenomenon of implied volatility smile in option markets shows that the risk-neutral returns are non-gaussian and leptokurtic. While the smile itself can be explained within a model with continuous paths, the fact that it becomes much more pronounced for short maturities is a clear indication of the presence of jumps. In continuous-path models, the law of returns for shorter maturities becomes closer to the Gaussian law, whereas in reality and in models with jumps returns actually become less Gaussian as the horizon becomes shorter. Finally, jump processes correspond to genuinely incomplete markets, whereas all continuous-path models are either complete or 'completable' with a small number of additional assets. This fundamental incompleteness makes it possible to carry out a rigorous analysis of the hedging error and find ways to improve the hedging performance using additional instruments such as liquid European options.

A great advantage of exponential Lévy models is their mathematical tractability, which makes it possible to perform many computations explicitly and to present deep results of modern mathematical finance in a simple manner. This has led to an explosion of the literature on option pricing
and hedging in exponential Lévy models in the late 90s and early 2000s, the literature which now contains hundreds of research papers and several monographs. However, some fundamental aspects such as asymptotic behavior of implied volatility or the computation of hedge ratios have only recently been given a rigorous treatment.

For background on exponential Lévy models, the reader may refer to textbooks such as [12, 43] for a more financial perspective or [2, 32] for a more mathematical perspective.

2 Lévy processes: basic facts

Lévy processes are a class of stochastic processes with discontinuous paths, which is at the same time simple enough to study and rich enough for applications, or at least to be used as building blocks of more realistic models.

Definition 1. A stochastic process $X$ is a Lévy process if it is càdlàg, satisfies $X_0 = 0$ and possesses the following properties:

- Independent increments;
- Stationary increments;

From these properties it follows that

- $X$ is continuous in probability: $\forall \varepsilon, \lim_{s \to 0} P[|X_{s+t} - X_t| > \varepsilon] = 0$;
- At any fixed time, the probability of having a jump is zero: $\forall t, P[X_{t-} = X_t] = 1$.

Lévy processes are essentially processes with jumps, because it can be shown that any Lévy process which has a.s.continuous trajectories is a Brownian motion with drift.

Proposition 1. Let $X$ be a continuous Lévy process. Then there exist $\gamma \in \mathbb{R}^d$ and a symmetric positive definite matrix $A$ such that

$$X_t = \gamma t + W_t,$$

where $W$ is a Brownian motion with covariance matrix $A$. 

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Proof. This result is a consequence of the Feller-Lévy central limit theorem, but since it is important for the understanding of Lévy processes, we give here a short proof (for the one-dimensional case).

It is enough to show that $X_1$ has Gaussian law, the rest will follow from the stationarity and independence of increments.

**Step 1** Let $\xi^k_n := X_k - X_{k-1}$. The continuity of $X$ implies that

$$\lim_{n} P[\sup_k |\xi^k_n| > \varepsilon] = 0,$$

for all $\varepsilon$. Let $a_n = P[|\xi^1_n| > \varepsilon]$. Since

$$P[\sup_k |\xi^k_n| > \varepsilon] = 1 - (1 - P[|\xi^1_n| > \varepsilon])^n,$$

we get that $\lim (1 - a_n)^n = 1$, from which it follows that $\lim n \log(1 - a_n) = 0$. But $n \log(1 - a_n) \leq -na_n \leq 0$. Therefore,

$$\lim_{n} n P[|X^1_n| > \varepsilon] = 0 \quad (1)$$

**Step 2** Using the independence and stationarity of increments, we can show that

$$\lim_{n} n E[\cos X^1_n - 1] = \frac{1}{2} \{\log Ee^{iX_1} + \log Ee^{-iX_1}\} := -A; \quad (2)$$

$$\lim_{n} n E[\sin X^1_n] = \frac{1}{2i} \{\log Ee^{iX_1} - \log Ee^{-iX_1}\} := \gamma. \quad (3)$$

**Step 3** The equations (1) and (2) allow to prove that for every function $f$ such that $f(x) = o(|x|^2)$ in a neighborhood of 0, $\lim n E[f(X^1_n)] = 0$ which implies that $\varepsilon > 0$

$$\lim n E[X^1_n 1_{|X^1_n| \leq \varepsilon}] = \gamma, \quad (4)$$

$$\lim n E[X^2_n 1_{|X^1_n| \leq \varepsilon}] = A, \quad (5)$$

$$\lim n E[|X^1_n|^3 1_{|X^1_n| \leq \varepsilon}] = 0. \quad (6)$$

$$\lim n E[|X^1_n|^4 1_{|X^1_n| \leq \varepsilon}] = 0. \quad (7)$$
Step 4  Assembling together the different equations, we finally get

\[ \log E[e^{iuX_1}] = n \log E[e^{iu X_{1n} 1_{X_{1n} \leq \varepsilon}}] + o(1) \]

\[ = n \log \left\{ 1 + iu E[X_{1n} 1_{X_{1n} \leq \varepsilon}] - \frac{u^2}{2} E[X_{1n}^2 1_{X_{1n} \leq \varepsilon}] + o(1/n) \right\} + o(1) \]

\[ = iu \gamma - \frac{Au^2}{2} + o(1) \xrightarrow{n \to \infty} iu \gamma - \frac{Au^2}{2} \]

where \( o(1) \) denotes a quantity which tends to 0 as \( n \to \infty \). □

The second fundamental example of Lévy process is the Poisson process.

2.1 The Poisson process

Definition 2. Let \((\tau_i)_{i \geq 1}\) be a sequence of exponential random variables with parameter \(\lambda\) and let \(T_n = \sum_{i=1}^n \tau_i\). Then the process

\[ N_t = \sum_{n \geq 1} 1_{t \geq T_n} \quad (8) \]

is called the Poisson process with parameter (or intensity) \(\lambda\)

Proposition 2 (Properties of the Poisson process).

1. For all \(t \geq 0\), the sum in (8) is finite a.s.
2. The trajectories of \(N\) are piecewise constant with jumps of size 1 only.
3. The trajectories are càdlàg.
4. \(\forall t > 0, N_{t-} = N_t\) with probability 1.
5. \(\forall t > 0, N_t\) follows the Poisson law with parameter \(\lambda t\):

\[ P[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \]

6. The characteristic function of the Poisson process is

\[ E[e^{iuN_t}] = \exp\{\lambda t(e^{iu} - 1)\} \]

7. The Poisson process is a Lévy process
The Poisson process counts the events with exponential interarrival times. In a more general setting, one speaks of a counting process.

**Definition 3.** Let \((T_n)\) be a sequence of times with \(T_n \to \infty\) a.s. Then the process
\[
N_t = \sum_{n \geq 1} 1_{t \geq T_n}
\]
is called a counting process.

In other words, a counting process is an increasing piecewise constant process with jumps of size 1 only and almost surely finite.

The first step towards the characterization of Lévy processes is to characterize Lévy processes which are counting processes.

**Proposition 3.** Let \((N_t)\) be a Lévy process and a counting process. Then \((N_t)\) is a Poisson process.

**Proof.** The proof uses the characterization of the exponential distribution by its memoryless property: if a random variable \(T\) satisfies
\[
P[T > t + s | T > t] = P[T > s]
\]
for all \(t, s > 0\) then \(T\) has exponential distribution.

Let \(T_1\) be the first jump time of the process \(N\). The independence and stationarity of increments give us:
\[
P[T_1 > t + s | T_1 > t] = P[N_{t+s} = 0 | N_t = 0]
= P[N_{t+s} - N_t = 0 | N_t = 0] = P[N_s = 0] = P[T_1 > s],
\]
which means that the first jump time \(T_1\) has exponential distribution.

Now, it suffices to show that the process \((X_{T_1+t} - X_{T_1})_{t \geq 0}\) is independent from \(T_1\) and has the same law as \((X_t)_{t \geq 0}\). Let \(f(t) := E[e^{iuX_t}]\). Then using once again the independence and stationarity of increments we get that \(f(t+s) = f(t)f(s)\) and \(M_t := \frac{e^{iuX_t}}{f(t)}\) is a martingale. Let \(T_1^n := n \wedge T_1\). Then by Doob’s optional sampling theorem,
\[
E[e^{iu(X_{T_1^n+t} - X_{T_1^n}) + ivT_1^n}] = E\left[\frac{f(T_1^n + t)}{f(T_1^n)} e^{ivT_1^n}\right] = E[e^{iuX_t}]E[e^{ivT_1^n}].
\]
The proof is finished with an application of the dominated convergence theorem.
Compound Poisson process  The Poisson process itself cannot be used to model asset prices because the condition that the jump size is always equal to 1 is too restrictive, but it can be used as building block to construct richer models.

Definition 4 (Compound Poisson process). The compound Poisson process with jump intensity $\lambda$ and jump size distribution $\mu$ is a stochastic process $(X_t)_{t \geq 0}$ defined by

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where $\{Y_i\}_{i \geq 1}$ is a sequence of independent random variables with law $\mu$ and $N$ is a Poisson process with intensity $\lambda$ independent from $\{Y_i\}_{i \geq 1}$.

In other words, a compound Poisson process is a piecewise constant process which jumps at jump times of a standard Poisson process and whose jump sizes are i.i.d. random variables with a given law.

Proposition 4 (Properties of the compound Poisson process). Let $(X_t)_{t \geq 0}$ be a compound Poisson process with jump intensity $\lambda$ and jump size distribution $\mu$. Then $X$ is a piecewise constant Lévy process and its characteristic function is given by

$$E[e^{iuX_t}] = \exp \left\{ t \lambda \int_{\mathbb{R}} (e^{iux} - 1) \mu(dx) \right\}.$$

Example 1 (Merton’s model). The Merton (1976) model is one of the first applications of jump processes in financial modeling. In this model, to take into account price discontinuities, one adds Gaussian jumps to the log-price.

$$S_t = S_0 e^{rt + X_t}, \quad X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad Y_i \sim N(\mu, \delta^2) \text{ independents.}$$

The advantage of this choice of jump size distribution is to have a series representation for the density of the log-price (as well as for the prices of European options).

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \exp \left\{ -\frac{(x-\gamma t - k\mu)^2}{2(\sigma^2 t + k\delta^2)} \right\} \sqrt{2\pi(\sigma^2 t + k\delta^2)}.$$
2.2 Poisson random measures

The notion of the Poisson random measure is central for the theory of Lévy processes: we shall use it in the next section to give a full characterization of their path structure.

**Definition 5** (Random measure). Let \((\Omega, P, \mathcal{F})\) be a probability space and \((E, \mathcal{E})\) a measurable space. Then \(M : \Omega \times \mathcal{E} \to \mathbb{R}\) is a random measure if

- For every \(\omega \in \Omega\), \(M(\omega, \cdot)\) is a measure on \(\mathcal{E}\).
- For every \(A \in \mathcal{E}\), \(M(\cdot, A)\) is measurable.

**Definition 6** (Poisson random measure). Let \((\Omega, P, \mathcal{F})\) be a probability space, \((E, \mathcal{E})\) a measurable space and \(\mu\) a measure on \((E, \mathcal{E})\). Then \(M : \Omega \times \mathcal{E} \to \mathbb{R}\) is a Poisson random measure with intensity \(\mu\) if

- For all \(A \in \mathcal{E}\) with \(\mu(A) < \infty\), \(M(A)\) follows the Poisson law with parameter \(E[M(A)] = \mu(A)\).
- For any disjoint sets \(A_1, \ldots, A_n\), \(M(A_1), \ldots, M(A_n)\) are independent.

In particular, the Poisson random measure is a positive integer-valued random measure. It can be constructed as the counting measure of randomly scattered points, as shown by the following proposition.

**Proposition 5.** Let \(\mu\) be a \(\sigma\)-finite measure on a measurable subset \(E\) of \(\mathbb{R}^d\). Then there exists a Poisson random measure on \(E\) with intensity \(\mu\).

**Proof.** Suppose first that \(\mu(E) < \infty\). Let \(\{X_i\}_{i \geq 1}\) be a sequence of independent random variables such that \(P[X_i \in A] = \frac{\mu(A)}{\mu(E)}, \forall i\) and \(\forall A \in \mathcal{B}(E)\), and let \(M(E)\) be a Poisson random variable with intensity \(\mu(E)\) independent from \(\{X_i\}_{i \geq 1}\). It is then easy to show that the random measure \(M\) defined by

\[
M(A) := \sum_{i=1}^{M(E)} 1_A(X_i), \quad \forall A \in \mathcal{B}(E),
\]

is a Poisson random measure on \(E\) with intensity \(\mu\).

Assume now that \(\mu(E) = \infty\), and choose a sequence of disjoint measurable sets \(\{E_i\}_{i \geq 1}\) such that \(\mu(E_i) < \infty, \forall i\) and \(\bigcup_i E_i = E\). We construct a Poisson random measure \(M_i\) on each \(E_i\) as described above and define

\[
M(A) := \sum_{i=1}^{\infty} M_i(A), \quad \forall A \in \mathcal{B}(E).
\]
Corollary 1 (Exponential formula). Let $M$ be a Poisson random measure on $(E, \mathcal{E})$ with intensity $\mu$, $B \in \mathcal{E}$ and let $f$ be a measurable function with $\int_B |e^{f(x)} - 1| \mu(dx) < \infty$. Then

$$E \left[ e^{\int_B f(x) M(dx)} \right] = \exp \left[ \int_B (e^{f(x)} - 1) \mu(dx) \right].$$

Definition 7 (Jump measure). Let $X$ be a $\mathbb{R}^d$-valued càdlàg process. The jump measure of $X$ is a random measure on $\mathcal{B}([0, \infty) \times \mathbb{R}^d)$ defined by

$$J_X(A) = \# \{ t : \Delta X_t \neq 0 \text{ and } (t, \Delta X_t) \in A \}.$$

The jump measure of a set of the form $[s, t] \times A$ counts the number of jumps of $X$ between $s$ and $t$ such that their sizes fall into $A$. For a counting process, since the jump size is always equal to 1, the jump measure can be seen as a random measure on $[0, \infty)$.

Proposition 6. Let $X$ be a Poisson process with intensity $\lambda$. Then $J_X$ is a Poisson random measure on $[0, \infty)$ with intensity $\lambda \times dt$.

Maybe the most important result of the theory of Lévy processes is that the jump measure of a general Lévy process is also a Poisson random measure.

Exercise 1. Let $X$ and $Y$ be two independent Lévy processes. Use the definition to show that $X + Y$ is also a Lévy process.

Exercise 2. Show that the memoryless property characterizes the exponential distribution: if a random variable $T$ satisfies

$$\forall t, s > 0, \quad P[T > t + s | T > t] = P[T > s]$$

then either $T \equiv 0$ or $T$ has exponential law.

Exercise 3. Prove that if $N$ is a Poisson process then it is a Lévy process.

Exercise 4. Prove that if $N$ and $N'$ are independent Poisson processes with parameters $\lambda$ and $\lambda'$ then $N + N'$ is a Poisson process with parameter $\lambda + \lambda'$.

Exercise 5. Let $X$ be a compound Poisson process with jump size distribution $\mu$. Establish that
• $E[|X_t|] < \infty$ if and only if $\int_{\mathbb{R}} |x| f(dx)$ and in this case
  $$E[X_t] = \lambda t \int_{\mathbb{R}} x f(dx).$$

• $E[|X_t|^2] < \infty$ if and only if $\int_{\mathbb{R}} x^2 f(dx)$ and in this case
  $$\text{Var}[X_t] = \lambda t \int_{\mathbb{R}} x^2 f(dx).$$

• $E[e^{X_t}] < \infty$ if and only if $\int_{\mathbb{R}} e^x f(dx)$ and in this case
  $$E[e^{X_t}] = \exp \left( \lambda t \int_{\mathbb{R}} (e^x - 1)f(dx) \right).$$

Exercise 6. The goal is to show that to construct a Poisson random measure on $\mathbb{R}$, one needs to take two Poisson processes and make the first one run towards $+\infty$ and the second one towards $-\infty$.

Let $N$ and $N'$ be two Poisson processes with intensity $\lambda$, and let $M$ be a random measure defined by

$$M(A) = \#\{t > 0 : t \in A, \Delta N_t = 1\} + \#\{t > 0 : -t \in A, \Delta N'_t = 1\}.$$

Show that $M$ is a Poisson random measure with intensity $\lambda$.

3 Path structure of a Lévy process

Definition 8 (Lévy measure). Let $X$ be an $\mathbb{R}^d$-valued Lévy process. The measure $\nu$ defined by

$$\nu(A) = E[\#\{t \in [0, 1] : \Delta X_t \neq 0 \text{ et } \Delta X_t \in A\}], \ A \in \mathcal{B}(\mathbb{R}^d)$$

is called the Lévy measure of $X$.

Theorem 1 (Lévy-Itô-decomposition). Let $X$ be an $\mathbb{R}^d$-valued Lévy process with Lévy measure $\nu$. Then

1. The jump measure $J_X$ of $X$ is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity $dt \times \nu$. 

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2. The Lévy measure $\nu$ satisfies 
\[ \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) \nu(dx) < \infty. \]

3. There exist $\gamma \in \mathbb{R}^d$ and a $d$-dimensional Brownian motion $B$ with covariance matrix $A$ such that
\[ X_t = \gamma t + B_t + N_t + M_t, \quad \text{where} \]
\begin{align*}
N_t &= \int_{|x| > 1, s \in [0, t]} x J_X(ds \times dx) \quad \text{and} \\
M_t &= \int_{0 <|x| \leq 1, s \in [0, t]} x \{ J_X(ds \times dx) - \nu(dx)ds \} \\
&= \int_{0 <|x| \leq 1, s \in [0, t]} x \tilde{J}_X(ds \times dx).
\end{align*}

The three terms are independent and the convergence in the last term is almost sure and uniform in $t$ on compacts.

The triple $(A, \nu, \gamma)$ is called the characteristic triple of $X$.

The proof is based on the following lemma.

**Lemma 1.** Let $(X, Y)$ be a 2-dimensional Lévy process such that $Y$ is piecewise constant and $\Delta X_t \Delta Y_t = 0$ for all $t$ a.s. Then $X$ and $Y$ are independent.

**Proof.** In view of the independence and stationarity of increments, it is enough to show that $X_1$ and $Y_1$ are independent. Let $M_t = \frac{e^{iuX_t}}{E[e^{iuX_1}]}$ and $N_t = \frac{e^{iuY_t}}{E[e^{iuY_1}]}$. Then $M$ and $N$ are martingales on $[0, 1]$. From the independence and stationarity of increments we deduce that for every Lévy process $Z$,
\[ E[e^{iuZ_t}] = E[e^{iuZ_1}]^t \quad \text{and} \quad E[e^{iuZ}] \neq 0, \forall u. \]

This means that $M$ is bounded. By Proposition 3, the number of jumps of $Y$ on $[0, 1]$ is a Poisson random variable. Therefore, $N$ has integrable variation on this interval. By the martingale property and dominated convergence we finally get
\[ E[M_1N_1] - 1 = E \left[ \sum_{i=1}^n (M_{i/n} - M_{(i-1)/n})(N_{i/n} - N_{(i-1)/n}) \right] \\
\rightarrow E \left[ \sum_{0 \leq t \leq 1} \Delta M_t \Delta N_t \right] = 0, \]
which implies $E[e^{iuX_t + ivY_t}] = E[e^{iuX_t}]E[e^{ivY_t}]$.

Proof of Theorem 1.

Part 1 Let $A \in B(\mathbb{R}^d)$ with $0 \notin \bar{A}$. Then $N^A_t = \#\{s \leq t : \Delta X_s \in A\}$ is a counting process and a Lévy process, hence, by Proposition 3, a Poisson process, which means that $J_X([t_1, t_2] \times A)$ follows the Poisson law with parameter $(t_2 - t_1)\nu(A)$ and that $J_X([t_1, t_2] \times A)$ is independent from $J_X([s_1, s_2] \times A)$ if $t_2 \leq s_1$. Let us now take two disjoint sets $A$ and $B$. By Lemma 1, $N^A$ and $N^B$ are independent, which proves that $J_X([s_1, s_2] \times A)$ and $J_X([t_1, t_2] \times B)$ are also independent, for all $s_1, s_2, t_1, t_2$.

Part 2 From the previous part, we deduce that $\nu(A) < \infty$ whenever $0 \notin \bar{A}$. It remains to show that

$$
\int_{\|x\| \leq \delta} \|x\|^2 \nu(dx) < \infty
$$

for some $\delta > 0$. Let

$$X^\varepsilon_t = \sum_{0 \leq s \leq t}^{1 > \|\Delta X_s\| > \varepsilon} \Delta X_s = \int_{\varepsilon < \|x\| \leq 1, s \in [0, t]} x J_X(ds \times dx)
$$

and $R^\varepsilon_t = X_t - X^\varepsilon_t$. Since $(X^\varepsilon_t, R^\varepsilon_t)$ is a Lévy process, Lemma 1 implies that $X^\varepsilon_t$ and $R^\varepsilon_t$ are independent. In addition, $|E[e^{iuX_t}]| > 0$ for all $t, u$. This means that

$$E[e^{iuX_t}] = E[e^{iuR^\varepsilon_t}]E[e^{iuX^\varepsilon_t}]$$

Therefore, $|E[e^{iuX_t}]|$ is bounded from below by a positive number which does not depend on $\varepsilon$. By the exponential formula, this is equivalent to

$$\left| \exp \left\{ t \int_{\|x\| \geq \varepsilon} (e^{iu}\nu(dx) - 1) \right\} \right| \geq C > 0,$$

which gives $\int_{\|x\| \geq \varepsilon} (1 - \cos(u\|x\|))\nu(dx) \leq \tilde{C} < \infty$. Since this result is true for all $u$, the proof of part 2 is completed.
Part 3 Observe first that the process $M$ is well defined due to the compensation of small jumps and the fact that the Lévy measure integrates $\|x\|^2$ near zero: introducing the process $$M^\varepsilon_t = \int_{\varepsilon \leq \|x\| < 1, s \in [0,t]} x \tilde{J}_X(ds \times dx),$$ we get that for $\varepsilon_1 < \varepsilon_2$,

$$E[(M^\varepsilon_1 - M^\varepsilon_2)^2] = t \int_{\varepsilon_2 \leq \|x\| < \varepsilon_1} x^2 \nu(dx)$$

and so, since the space $L^2$ is complete, for every $t$, $M^\varepsilon_t$ converges in $L^2$ as $\varepsilon \to 0$. Using Doob’s inequality we show that the convergence is uniform in $t$ on compact intervals. The process $X - N - M$ is then a continuous Lévy process independent from $N$ and $M$ in view of Lemma 1. We conclude with Proposition 1.

Corollary 2 (Lévy-Khintchine representation). Let $X$ be a Lévy process with characteristic triple $(A, \nu, \gamma)$. Its characteristic function is given by

$$E[e^{iuX_t}] = \exp \left\{ t \left( i\gamma u - \frac{A u^2}{2} + \int_{|x| \geq \varepsilon} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(dx) \right) \right\}, \quad (10)$$

Proof. Using the previous theorem and the exponential formula, we get

$$E[e^{iu(\gamma t + B_t + N_t + M^\varepsilon_t)}] = \exp \left\{ t \left( i\gamma u - \frac{A u^2}{2} + \int_{|x| \geq \varepsilon} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(dx) \right) \right\},$$

and we conclude using the dominated convergence theorem.

Example 2 (The variance gamma process). One of the simplest examples of Lévy processes with infinite intensity of jumps is the gamma process, a process with stationary independent increments and such that for all $t$, the law $p_t$ of $X_t$ is the gamma law with parameters $\lambda$ and $ct$:

$$p_t(x) = \frac{\lambda^ct}{\Gamma(ct)} x^{ct-1} e^{-\lambda x}.$$  

The gamma process is an increasing Lévy process whose characteristic function has a very simple form:

$$E[e^{iuX_t}] = (1 - iu/\lambda)^{-ct}.$$
One can easily show that the Lévy measure of the gamma process has a density given by

$$\nu(x) = \frac{ce^{-\lambda x}}{x}1_{x>0}. \quad (11)$$

Starting from the gamma process, we can construct a very popular jump model: the variance gamma process \([36, 35]\) which is obtained by changing the time scale of a Brownian motion with drift with a gamma process:

$$Y_t = \mu X_t + \sigma B_{X_t}.$$ 

Using $Y_t$ to model the logarithm of the stock price is usually justified by saying that the price follows a geometric Brownian motion on a stochastic time scale given by the gamma process \([23]\). The variance gamma process provides another example of a Lévy processes with infinite intensity of jumps, and its characteristic function is given by

$$E[e^{iuY_t}] = \left(1 + \frac{\kappa \sigma^2 u^2}{2} - i\mu \kappa u\right)^{-\kappa t}.$$ 

The parameters have the following intuitive interpretation: $\sigma$ is the scale parameter, $\mu$ is the parameter of asymmetry (skewness) and $\kappa$ is responsible for the kurtosis of the process (thickness of its tails).

**Exercise 7.** Let $X$ be a Lévy process with characteristic triple $(A, \nu, \gamma)$. Compute the probability that $X$ will have at least one negative jump of size bigger than $\varepsilon > 0$ on the interval $[0, T]$.

**Exercise 8.** Let $X$ be a Lévy process with Lévy measure $\nu(dx) = \lambda \nu_0(dx)$, where $\nu_0$ has no atom and satisfies $\nu_0(\mathbb{R}) = \infty$. For all $n \in \mathbb{N}$, let $k_n > 0$ be the solution of $\int_{k_n}^{\infty} \nu_0(dx) = n$. For a fixed $T$, give the law of the random variable

$$A_n = \#\{t \leq T : \Delta X_t \in [k_{n+1}, k_n]\}.$$ 

Use this result to suggest a method for estimating $\lambda$ from an observation of the trajectory of $X$ on $[0, T]$, supposing that $\nu_0$ is known.

**Exercise 9.** Let $X$ be a Lévy process with no diffusion component and a Lévy measure $\nu$ which satisfies $\int_{\mathbb{R}} |x| \nu(dx) < \infty$. Using the Levy-Ito decomposition, show that the trajectories of $X$ have a.s. finite variation (a function has finite variation if it can be represented as the difference of two increasing functions).
Exercise 10. Prove that the Lévy measure of the gamma process is given by equation (11). Show that the variance gamma process can be represented as a difference of two independent gamma process, and use this result to deduce the form of the Lévy measure of the variance gamma process.

4 Basic stochastic calculus for jump processes

4.1 Integrands and integrators

The main application of the stochastic integral in finance is the representation of self-financing portfolios: in the absence of interest rates, when the price of the risky asset is a continuous process $S_t$, and the quantity of the asset is denoted by $\phi$, the portfolio value is

$$V_T = \int_0^T \phi_t dS_t$$

We would like this relationship to hold in the presence of jumps as well, but what are the natural properties to impose on $S$ and $\phi$? The process $S$ must be right-continuous, since the price jumps arrive unexpectedly. On the other hand, the hedging strategy $\phi_t$ is based on the observations of the portfolio manager up to date $t$; it must therefore be left-continuous. The following example illustrates this: suppose that the asset price is given by $S_t = \lambda t - N_t$, where $N_t$ is a Poisson process with intensity $\lambda$, and let $T$ be the time of the first jump of $N$. If one could use the (càdlàg) strategy $\phi_t = 1_{[0,T]}(t)$ amounting to sell the asset just before the jump, there would clearly be an arbitrage opportunity, since

$$V_t = \int_0^t \phi_t dS_t = \lambda t \land T.$$ 

On the other hand, with the càglàd strategy $\phi_t = 1_{[0,T]}(t)$,

$$V_t = \int_0^t \phi_t dS_t = \lambda t \land T - N_t \land T,$$

which has zero expectation. It is therefore natural to consider adapted and left-continuous integrands.
The simplest (and the only one which can be realized in practice) form of a portfolio strategy is such where the portfolio is only rebalanced a finite number of times. We define a simple predictable process by

$$\phi_t = \phi_0 1_{t=0} + \sum_{i=0}^{n} \phi_i 1_{(T_i, T_{i+1}]}(t),$$  \hspace{1cm} (12)

where \(T_0 = 0, (T_i)_{i \geq 0}\) is a sequence of stopping times, and for each \(i\), \(\phi_i\) is \(\mathcal{F}_{T_i}\)-measurable and bounded. The space of simple predictable processes will be denoted by \(\mathcal{S}\).

For simple predictable processes, the stochastic integral is defined by

$$\int_0^t \phi_s dS_s := \sum_{i=0}^{n} \phi_i (S_{T_{i+1}} - S_{T_i})$$  \hspace{1cm} (13)

For a general adapted left-continuous process, the stochastic integral can then be defined as the continuous extension of the integral for simple predictable processes, using the topology of uniform convergence on compacts in probability (ucp).

The sequence of processes \((X^n)\) is said to converge ucp to the process \(X\) if for every \(t\), \((X^n - X)_t^*\) converges to 0 in probability, where \(Z_t^* := \sup_{0 \leq s \leq t} |Z_s|\). We denote by \(\mathcal{S}_{ucp}\) the space \(\mathcal{S}\) endowed with the topology of ucp convergence and by \(\mathbb{L}_{ucp}\) and \(\mathbb{D}_{ucp}\) the space of adapted and, respectively, left or right continuous processes, with the same topology. It is then possible to show that the space \(\mathcal{S}_{ucp}\) is dense in \(\mathbb{L}_{ucp}\), and to associate the ucp topology with a metric on \(\mathbb{D}_{ucp}\), for which this space will be complete. To extend the stochastic integration operator defined by (13) from \(\mathcal{S}_{ucp}\) to \(\mathbb{L}_{ucp}\), this operator must be continuous as a mapping from \(\mathcal{S}_{ucp}\) to \(\mathbb{D}_{ucp}\). Whether or not this is true, depends on the integrator \(S\), and we shall limit ourselves to the integrators for which this property holds.

**Definition 9.** The process \(S \in \mathbb{D}\) is a *semimartingale* if the stochastic integration operator defined by (13) is a continuous operator from \(\mathcal{S}_{ucp}\) to \(\mathbb{D}_{ucp}\).

Every adapted càdlàg process of finite variation on compacts is a semimartingale. This follows from

$$\sup_{0 \leq t \leq T} \left| \int_0^t \phi_s dS_s \right| \leq \text{Var}_0^T(S) \sup_{0 \leq t \leq T} \phi_t,$$
where $\text{Var}_T^T(S)$ denotes the total variation of $S$ on $[0, T]$. A square integrable càdlàg martingale is a semimartingale. For a simple predictable process $\phi$ of the form (12), and a square integrable martingale $M$, $\int_0^T \phi_t dM_t$ is also a martingale and

$$E \left( \int_0^T \phi_t dM_t \right)^2 \leq \sup_{0 \leq t \leq T, \omega \in \Omega} \phi_t^2 E[M_t^2].$$

Suppose now that $(\phi^n)$ is a sequence of simple predictable processes such that $\phi^n \to 0$ ucp. Then, using the Chebyshev’s inequality and Doob’s inequality, we get that

$$P \left( \left( \int_0^T \phi^n_t dM_t \right)^* > \varepsilon \right) \leq P \left( \left( \int_0^T \phi^n_t \mathbb{1}_{|\phi^n_t| \leq C} dM_t \right)^* > \varepsilon \right) + P((\phi^n)^* > C) \leq \frac{4C^2 E[M_T^2]}{\varepsilon^2} + P((\phi^n)^* > C) \to 0,$$

because the first term can be made arbitrarily small by choosing $C$ sufficiently small, and the second term can be made small by choosing $n$ sufficiently large.

Since the terms $\gamma_t$ and $N_t$ in the Lévy-Itô decomposition have finite variation and the terms $B_t$ and $M_t$ are square integrable martingales, every Lévy process is a semimartingale.

A deep result of the general theory of processes [38] is that every semimartingale is the sum of a finite variation process and a local martingale. The notion of local martingale extends that of the martingale: the process $(X_t)$ is a local martingale if there exists a sequence of stopping times $\{T_i\}_{i \geq 1}$ such that $T_i \to \infty$ when $i \to \infty$ and for each $i$, $(X_{T_i \wedge t})$ is a martingale.

### 4.2 Stochastic integral with respect to a Poisson random measure

In the Lévy-Itô decomposition, we have already encountered integrals of deterministic functions with respect to Poisson random measures and compensated Poisson random measures. In this section, our goal is to extend this notion of integral to stochastic integrands.

Let $M$ be a Poisson random measure of $[0, T] \times \mathbb{R}$ with intensity $\mu$. $\mu$ is supposed to be $\sigma$-finite: there exists a sequence $U_n \uparrow \mathbb{R}$ with $\mu([0, t] \times U_n) < \infty$ for all $t$. Typically, $M$ will be the jump measure of a Lévy process. We would
like to define the integral of $M$ or its compensated version with respect to a predictable function $\phi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, that is, a function which satisfies

(i) For all $t$, $(\omega, x) \mapsto \phi(\omega, t, x)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable.

(ii) For all $(\omega, x) \mapsto \phi(\omega, t, x)$ is left-continuous.

The stochastic integral of $\phi$ with respect to $M$ will be defined in two different settings:

Case 1: $\phi$ satisfies

$$\int_0^T \int_{\mathbb{R}} |\phi(t, y)| M(dt \times dy) < \infty \text{ p.s.}$$

In this case, the stochastic integral of $\phi$ with respect to $M = \sum \delta_{(T_i, y_i)}$ is defined as the absolutely convergent sum

$$\int_0^t \int_{\mathbb{R}} \phi(t, y) M(dt \times dy) := \sum_{i: T_i \leq t} \phi(T_i, y_i).$$

Case 2: $\phi$ is square integrable, that is, it satisfies

$$E \int_0^T \int_{\mathbb{R}} \phi^2(t, y) \mu(dt \times dy) < \infty$$

In this case, the construction is more involved, since we need to use the $L^2$ theory and continuous extension once again. We define simple predictable functions $\phi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ via

$$\phi(t, y) = \sum_{j=1}^m \phi_{0j} 1_{t=0} 1_{A_j}(y) + \sum_{i=1}^n \sum_{j=1}^m \phi_{ij} 1_{(T_i, T_{i+1})}(t) 1_{A_j}(y),$$

where $T_0 = 0$, $(T_i)_{i \geq 1}$ is a sequence of stopping times; for all $j$, $A_j \in \mathcal{B}(\mathbb{R})$ is such that $\mu([0, t] \times A_j) < \infty$ for all $t$; and for all $i$ and $j$, $\phi_{ij}$ is bounded and $\mathcal{F}_{T_i}$-measurable. The stochastic integral of a simple predictable function with respect to $M$ is defined by

$$\int_0^t \int_{\mathbb{R}} \phi(t, y) M(dt \times dy) := \sum_{i: T_i \leq t} \phi(T_i, y_i) \equiv \sum_{i,j=1}^{n,m} \phi_{ij} M((T_i \wedge t, T_{i+1} \wedge t] \times A_j).$$
In a similar fashion, we can define the integral with respect to the compensated measure $\tilde{M} = M - \mu$:

$$X_t = \int_0^t \int_\mathbb{R} \phi(t, y) \tilde{M}(dt \times dy) := \sum_{i,j=1}^{n,m} \phi_{ij} \{ M((T_i \land t, T_{i+1} \land t] \times A_j) - \mu((T_i \land t, T_{i+1} \land t] \times A_j) \}$$

This process is a martingale and satisfies the “isometry relation”:

$$E[X_T^2] = E \int_0^T \int_\mathbb{R} \phi^2(t, y) \mu(dt \times dy).$$

This isometry allows to extend the notion of stochastic integral with respect to a compensated Poisson random measure to square integrable predictable functions. Next, the localization procedure can be used to extend the definition to all functions $\phi$ adapted and left-continuous in $t$ and measurable in $y$, such that the process

$$A_t := \int_0^t \int_\mathbb{R} \phi^2(s, y) \mu(ds \times dy)$$

is locally integrable.

The stochastic integral with respect to a Poisson random measure is more general than that with respect to a Poisson process: if $S$ is a piecewise constant Lévy process, if $S$ is a piecewise constant Lévy process,

$$\int_0^T \phi_t dS_t = \sum \phi_t \Delta S_t = \int_0^T \int_\mathbb{R} \phi_t y J_S(dt \times dy),$$

that is, the integral with respect to a process can be written as the integral of a specific function with respect to the jump measure of the process.

The stochastic integral with respect to a Poisson random measure allows us to define a new class of processes, which extends the notion of the Lévy process, while still preserving an easy-to-understand mathematical structure: many authors call this class Lévy-Itô processes. Recall that a Lévy process satisfies (with a little change of notation)

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{|x|>1} x M(ds \times dx) + \int_0^t \int_{|x|\leq1} x \tilde{M}(ds \times dx),$$

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where $M$ is a Poisson random measure with intensity $dt \times \nu$. For a Lévy-Itô process, the coefficients can be non-constant and even random:

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|>1} \gamma_s(x) M(ds \times dx) + \int_0^t \int_{|x|\leq 1} \gamma_s(x) \tilde{M}(ds \times dx),$$

(14)

where $\mu$ and $\sigma$ are adapted locally bounded processes and $\gamma_t(x)$ is an adapted random function, left-continuous in $t$, measurable in $x$, such that the process:

$$\int_{|x|\leq 1} \gamma^2_t(x) \nu(dx)$$

is locally bounded.

The class of Lévy-Itô processes enjoys better stability properties than that of Lévy processes: if $(X_t)$ is a Lévy-Itô process then for every function $f \in C^2$, $(f(X_t))$ is also a Lévy-Itô process.

When $\int_{|x|>1} |\gamma_t(x)| \nu(dx)$ is also locally bounded, the process $X$ can be decomposed onto a ‘martingale part’ and a ‘drift part’:

$$X_t = \int_0^t (\mu_s + \int_{|x|>1} \gamma_t(x) \nu(dx)) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_R \gamma_s(x) \tilde{M}(ds \times dx),$$

and in the purely martingale case, often found in applications,

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t \int_R \gamma_s(x) \tilde{M}(ds \times dx),$$

(15)

and the isometry relation holds:

$$E[X_T^2] = E\left[\int_0^T \sigma_t^2 dt\right] + E\left[\int_0^T \int_R \gamma^2_t(x) \nu(dx) dt\right].$$

(16)

### 4.3 Change of variable formula for Lévy-Itô processes

In the absence of jumps, the change of variable formula (Itô formula) for a function $f \in C^2$ takes the form

$$f(X_T) = f(X_0) + \int_0^T f'(X_t) dX_t + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt.$$
When the process has a finite number of jumps on \([0,T]\), one can write
\[ X_t := X^c_t + \sum_{s \leq t} \Delta X_s \] and apply the same formula between the jump times:
\[ f(X_T) = f(X_0) + \int_0^T f'(X_t) dX^c_t + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt + \sum_{t \leq T: \Delta X_t \neq 0} \{ f(X_t) - f(X_{t-}) \}. \]

When the number of jumps is infinite, the latter sum may diverge, but we still have
\[ f(X_T) = f(X_0) + \int_0^T f'(X_t-) dX_t + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt + \sum_{t \leq T: \Delta X_t \neq 0} \{ f(X_t) - f(X_{t-}) - f'(X_{t-}) \Delta X_t \}. \tag{17} \]

To make the decomposition (14) appear and show that the class of Lévy-Itô processes is stable with respect to transformations with \(C^2\) functions, we rewrite the above expression as follows:
\[ f(X_T) = f(X_0) + \int_0^T \{ \mu_t f'(X_t) + \frac{1}{2} \sigma_t^2 f''(X_t) \}
+ \int_{|x| \leq 1} (f(X_t + \gamma_t(x)) - f(X_t) - \gamma_t(x) f'(X_t)) \nu(dx) dt
+ \int_0^T f'(X_t) \sigma_t dW_t + \int_0^T \int_{|x| \leq 1} (f(X_{t-} + \gamma_t(x)) - f(X_{t-})) \tilde{M}(dt \times dx)
+ \int_0^T \int_{|x| > 1} (f(X_{t-} + \gamma_t(x)) - f(X_{t-})) M(dt \times dx). \]

**Exercise 11.** Show that for a simple predictable process \(\phi\) of the form (12), and a square integrable martingale \(M\), \(\int_0^T \phi_t dM_t\) is also a martingale and
\[ E \left( \int_0^T \phi_t dM_t \right)^2 \leq \sup_{0 \leq t \leq T, \omega \in \Omega} \phi_t^2 E[M_t^2]. \]

**Hint:** Use Doob’s optional sampling theorem. Let \((X_t)\) be an \((\mathcal{F}_t)\)-martingale and let \(S\) and \(T\) be bounded stopping times with \(S \leq T\) a.s. Then,
\[ E[X_T|\mathcal{F}_S] = X_S, \quad p.s. \]
Exercise 12. The quadratic variation or ‘square bracket’ of a semimartingale can be defined by

\[ [X]_t := X_t^2 - X_0^2 - 2 \int_0^t X_s dX_s. \]

- Compute the quadratic variation for a general Lévy-Itô process, a Lévy process and a Poisson process.

- Show that if \( X \) is a Lévy-Itô process such that \( [X]_t \equiv t \) and \( X \) is a martingale then \( X \) is the standard Brownian motion.

Exercise 13. Let \( X \) be a Lévy-Itô process of the form (14), whose coefficients \( \mu, \sigma \) and \( \gamma \) are deterministic and bounded. Applying the change of variable formula (17) to the function \( f(x) = e^{ix} \), show that the characteristic function of \( X_T \) is given by a generalized version of the Lévy-Khintchine formula.

Exercise 14. Let \( X \) be a Lévy-Itô process of the form (14) such that

\[ \mu_t + \frac{\sigma_t^2}{2} + \int_{\mathbb{R}} (e^{\gamma(x)} - 1 - \gamma_t(x)1_{|x|\leq 1})\nu(dx) = 0 \]

a.s. for all \( t \). Using the change of variable formula (17), show that \( e^{X_t} \) can be written in the form (15) with coefficients to be defined.

Assuming that \( \sigma_t \) and \( \int_{\mathbb{R}} (e^{\gamma(x)} - 1)^2\nu(dx) \) are a.s bounded by a constant \( C \), use the isometry relation (16) and Gronwall’s lemma to show that \( (e^{X_t}) \) is a square integrable martingale.

Gronwall’s lemma: Let \( \phi \) be a positive locally bounded function on \( \mathbb{R}^+ \) such that

\[ \phi(t) \leq a + b \int_0^t \phi(s)ds \]

for all \( t \) and two constants \( a \) and \( b \geq 0 \). Then \( \phi(t) \leq ae^{bt} \).

5 Stochastic exponential of a jump process

Proposition 7 (Stochastic exponential). Let \((X_t)_{t\geq 0}\) be a Lévy-Itô process with volatility coefficient \( \sigma \). There exists a unique cadlag process \((Z_t)_{t\geq 0}\) such that

\[ dZ_t = Z_{t-}dX_t, \quad Z_0 = 1. \]  (18)
$Z$ is given by:

$$Z_t = e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}. \quad (19)$$

**Proof.** Let

$$V_t = \prod_{0 \leq s \leq t; \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.$$ 

The first step is to show that this process exists and is of finite variation. We decompose $V$ into a product of two terms:

$$V_t' = \prod_{0 \leq s \leq t; |\Delta X_s| \leq 1/2} (1 + \Delta X_s) e^{-\Delta X_s} \quad \text{and} \quad V_t'' = \prod_{0 \leq s \leq t; |\Delta X_s| > 1/2} (1 + \Delta X_s) e^{-\Delta X_s}.$$ 

$V''$ for every $t$ is a product of finite number of factors, so it is clearly of finite variation and there are no existence problems. $V'$ is positive and we can consider its logarithm.

$$\ln V_t' = \sum_{0 \leq s \leq t; |\Delta X_s| \leq 1/2} (\ln(1 + \Delta X_s) - \Delta X_s).$$ 

Note that each term of this sum satisfies

$$0 > \ln(1 + \Delta X_s) - \Delta X_s > -\Delta X_s^2.$$ 

Therefore, the series is decreasing and bounded from below by $-\sum_{0 \leq s \leq t} \Delta X_s^2$, which is finite for every Lévy-Itô process. Hence, $(\ln V_t')$ exists and is a decreasing process. This entails that $(V_t')$ exists and has trajectories of finite variation.

The second step is to apply the Itô formula to the function $Z_t \equiv f(t, X_t, V_t) \equiv e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t$. This yields (in differential form)

$$dZ_t = -\frac{\sigma_t^2}{2} e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dt + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dX_t + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dV_t$$

$$+ \frac{\sigma_t^2}{2} e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dt + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dX_t - e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t$$

$$- e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t \Delta X_t - e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} \Delta V_t.$$
Now observe that since $V_t$ is a pure jump process,

$$dV_t \equiv \Delta V_t = V_{t-}(e^{\Delta X_t}(1 + \Delta X_t) - 1).$$

Substituting this into the above equality and making all the cancellations yields the Equation (18).

To understand why the solution is unique, observe that if $(Z_t^{(1)})$ and $(Z_t^{(2)})$ satisfy the Equation (18), then their difference $\tilde{Z}_t = Z_t^{(1)} - Z_t^{(2)}$ satisfies the same equation with initial condition $\tilde{Z}_0 = 0$. From the form of this equation, it is clear that if the solution is equal to zero at some point, it will remain zero. □

$Z$ is called the stochastic exponential or the Doléans-Dade exponential of $X$ and is denoted by $Z = \mathcal{E}(X)$.

**Relation between ordinary and stochastic exponential** It is clear from the above results that the ordinary exponential and the stochastic exponential of a Lévy process are two different notions: they do not correspond to the same stochastic process. In fact, contrary to the ordinary exponential $\exp(X_t)$, which is obviously a positive process, the stochastic exponential $Z = \mathcal{E}(X)$ is not necessarily nonnegative. It is easy to see that the stochastic exponential is always nonnegative if all jumps of $X$ are greater than $-1$, or, equivalently, $\nu((-\infty, -1]) = 0$.

It is therefore natural to ask, which of the two processes is more suitable for modeling price dynamics. The following result, due to Goll and Kallsen [24], shows that the two approaches are equivalent: if $Z > 0$ is the stochastic exponential of a Lévy process it is also the ordinary exponential of another Lévy process and vice versa. Therefore, the two operations, although they produce different objects when applied to the same Lévy process, end up by giving us the same class of positive processes.

**Proposition 8** (Relation between ordinary and stochastic exponentials).

1. Let $(X_t)_{t \geq 0}$ be a real valued Lévy process with Lévy triplet $(\sigma^2, \nu, \gamma)$ and $Z = \mathcal{E}(X)$ its stochastic exponential. If $Z > 0$ a.s. then there exists another Lévy process $(L_t)_{t \geq 0}$ with triplet $(\sigma^2_L, \nu_L, \gamma_L)$ such that $Z_t = e^{L_t}$.
where
\[
L_t = \ln Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \{ \ln(1 + \Delta X_s) - \Delta X_s \}. \tag{20}
\]

\[
\sigma_L = \sigma,
\]

\[
\nu_L(A) = \nu(\{x : \ln(1 + x) \in A\}) = \int 1_A(\ln(1 + x))\nu(dx), \tag{21}
\]

\[
\gamma_L = \gamma - \frac{\sigma^2}{2} + \int \nu(dx) \{ (e^x - 1)1_{[-1,1]}(x) - x1_{[-1,1]}(x) \}. \tag{23}
\]

2. Let \((L_t)_{t \geq 0}\) be a real valued Lévy process with Lévy triplet \((\sigma^2, \nu_L, \gamma_L)\) and \(S_t = \exp L_t\) its exponential. Then there exists a Lévy process \((X_t)_{t \geq 0}\) such that \(S_t\) is the stochastic exponential of \(X\): \(S = \mathcal{E}(X)\) where
\[
X_t = L_t + \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \{ e^{\Delta L_s} - 1 - \Delta L_s \}. \tag{22}
\]

The Lévy triplet \((\sigma^2, \nu, \gamma)\) of \(X\) is given by:
\[
\sigma = \sigma_L,
\]

\[
\nu(A) = \nu_L(\{x : e^x - 1 \in A\}) = \int 1_A(e^x - 1)\nu_L(dx), \tag{23}
\]

\[
\gamma = \gamma_L + \frac{\sigma^2}{2} + \int \nu_L(dx) \{ (e^x - 1)1_{[-1,1]}(x) - x1_{[-1,1]}(x) \}. \tag{23}
\]

Proof. 1. The condition \(Z > 0\) a.s. is equivalent to \(\Delta X_s > -1\) for all \(s\) a.s., so taking the logarithm is justified here. In the proof of Proposition 7 we have seen that the sum \(\sum_{0 \leq s \leq t} \{ \ln(1 + \Delta X_s) - \Delta X_s \}\) converges and is a finite variation process. Then it is clear that \(L\) is a Lévy process and that \(\sigma_L = \sigma\). Moreover, \(\Delta L_s = \ln(1 + \Delta X_s)\) for all \(s\). This entails that

\[
J_L([0, t] \times A) = \int_{[0, t] \times \mathbb{R}} 1_A(\ln(1 + x))J_X(ds \, dx)
\]

and also \(\nu_L(A) = \int 1_A(\ln(1 + x))\nu(dx)\). It remains to compute \(\gamma_L\). Substi-
tuting the Lévy-Itô decomposition for \((L_t)\) and \((X_t)\) into (20), we obtain
\[
\gamma L_t - \gamma t + \frac{\sigma^2 t}{2} + \int_{s \in [0,t], |x| \leq 1} x \tilde{J}_L(ds \, dx) + \int_{s \in [0,t], |x| > 1} x J_L(ds \, dx)
\]
\[
- \int_{s \in [0,t], |x| \leq 1} x \tilde{J}_X(ds \, dx) - \int_{s \in [0,t], |x| > 1} x J_X(ds \, dx)
\]
\[
- \sum_{0 \leq s \leq t} \{ \ln(1 + \Delta X_s) - \Delta X_s \} = 0.
\]

Observing that
\[
\int_{s \in [0,t], |x| \leq 1} x (J_L(ds \, dx) - J_X(ds \, dx))
\]
\[
= \sum_{0 \leq s \leq t} (\Delta X_s 1_{[-1,1]}(\Delta X_s) - \ln(1 + \Delta X_s)) 1_{[-1,1]}(\ln(1 + \Delta X_s))
\]
converges, we can split the above expression into jump part and drift part, both of which must be equal to zero. For the drift part we obtain:
\[
\gamma L - \gamma + \frac{\sigma^2}{2} - \int_{-1}^{1} \{ x \nu_L(dx) - x \nu(dx) \} = 0,
\]
which yields the correct formula for \(\gamma_L\) after a change of variable.

2. The jumps of \(S_t\) are given by \(\Delta S_t = S_{t-}(\exp(\Delta L_t) - 1)\). If \(X\) is a Lévy process such that \(S = \mathcal{E}(X)\) then since \(dS_t = S_{t-}dX_t\) then \(\Delta S_t = S_{t-}\Delta X_t\) so \(\Delta X_t = \exp(\Delta L_t) - 1\) so \(\nu\) is given by (23). In particular \(\Delta X_t > -1\) a.s. and it is easily verified that \(\ln \mathcal{E}(X)\) is a Lévy process with characteristics matching those of \(L\) only if \(X\) has characteristics given by (23). Conversely if \(X\) is a Lévy process with characteristics given by (23), using (19) we can verify as above that \(\mathcal{E}(X) = \exp L_t\).

In view of this result and given that the formulas involving the ordinary exponential are usually more tractable, the latter is more commonly used for asset price modeling. However, in some situations, the stochastic exponential may be a better choice, as shown by the following example.

**CPPI strategy in the presence of jumps** The CPPI (constant proportion portfolio insurance) is a portfolio insurance strategy which allows (in
theory) to keep the portfolio value above a fixed level, while still preserving some upside potential in case of a favorable market evolution.

To fix the ideas, suppose that the portfolio manager has promised to the investor a guaranteed capital of $N$ at maturity $T$ ($N$ can be greater or smaller than the initial investment). To achieve this, the portfolio value $V_t$ must remain at each date $t$ above the floor $B_t$, which is equal to the price of the zero-coupon bond with notional $N$ and maturity $T$. The difference $C_t = V_t - B_t$ is called cushion and the CPPI strategy uses the following algorithm:

- At each date $t$, if $V_t > B_t$, invest $mC_t$ in the risky asset, where $m > 1$ is called the multiplier, and the rest into zero-coupon bonds with maturity $T$.
- If $V_t \leq B_t$, invest all wealth in zero-coupon bonds with maturity $T$.

If the price of the risky asset is a continuous process, the portfolio value remains above the floor, and the dynamics of the cushion is given by

$$\frac{dC_t}{C_t} = m \frac{dS_t}{S_t} + (1 - m)r dt,$$

where $r$ is the interest rate. In the Black-Scholes model,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

this equation can be solved explicitly, and we obtain the final portfolio value

$$V_T = N + (V_0 - Ne^{-rT}) \exp \left( rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2} \right),$$

whose expectation is

$$E[V_T] = N + (V_0 - Ne^{-rT}) \exp (rT + m(\mu - r)T).$$

If $\mu > r$, there seems to be a paradox: there is no risk and the expected return can be made arbitrarily large by choosing $m$ big enough. This paradox is easily solved if the price trajectories are allowed to jump, since in this case, if the multiplier increases, the loss probability increases as well.

Let

$$\frac{dS_t}{S_{t-}} = r dt + dZ_t,$$
where $Z$ is a Lévy process and let $\tau = \inf\{t : V_t \leq B_t\}$ be the first date when the portfolio passes below the floor (it is possible that $\tau = \infty$). Before $\tau$, the cushion satisfies
\[
\frac{dC_t}{C_t} = mdZ_t + rdt,
\]
and the discounted cushion $C^*_t := \frac{C_t}{e^{rt}}$ is therefore given by
\[
C^*_t = \mathcal{E}(mZ)_t, \quad t < \tau.
\]

After $\tau$, the entire portfolio is invested into the risk-free asset, which means that the discounted cushion remains constant. Therefore,
\[
C^*_t = \mathcal{E}(mZ)_{t\land \tau}.
\]

The loss occurs if at some date $t \leq T$, $C^*_t \leq 0$, which can happen if and only if $Z$ has a jump in the interval $[0, T]$ whose size is less than $-1/m$. We then get (see exercise 7)
\[
P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx)\right).
\]

See [14] for details on portfolio insurance in the presence of jumps.

Exercise 15. Show that if $X$ and $Y$ are two Lévy processes and $Y$ has no diffusion component then
\[
\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + \sum \Delta X \Delta Y)
\]

Exercise 16 (Martingale property of the stochastic exponential). Let $X$ be a Lévy process and a martingale. Show that $\mathcal{E}(X)$ is a martingale as well.

**Hint:** Represent $X$ as the sum of a compound Poisson process and a Lévy process $X'$ such that $|\Delta X'| < 1$. Use Proposition 8 and the previous exercise.

Exercise 17. Use the previous exercise to show that for every Lévy process $X$ with $E[|X_t|] < \infty$,
\[
E[\mathcal{E}(X)_{t\land \tau}] = e^{E[|X_t|]}, \quad t > 0.
\]
6 Exponential Lévy models

The Black-Scholes model
\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]
can be equivalently rewritten in the exponential form \( S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \). This gives us two possibilities to construct an exponential Lévy model starting from a (one-dimensional) Lévy process \( X_t \): using the stochastic differential equation:
\[ \frac{dS_{t_{sde}}}{S_{t_{sde}}} = r dt + dX_t, \tag{24} \]
or using the ordinary exponential
\[ S_{t_{exp}} = S_{0_{exp}} e^{r t + X_t}, \tag{25} \]
where we explicitly included the interest rate \( r \) (assumed constant) in the formulas, to simplify notation later on. The subscripts \( sde \) for stochastic differential equation and \( exp \) for exponential, used here to emphasize the fact that \( S_{sde} \) and \( S_{exp} \) are different processes, will be omitted later on when there is no ambiguity. Sometimes it will be convenient to discount the price processes with the numéraire \( B(t,T) = e^{-r(T-t)} \) for some fixed maturity \( T \). In this case \( \hat{S}_t := \frac{S_t}{B(t,T)} = e^{-r(T-t)} S_t \) and the equations become
\[ \frac{d\hat{S}_t}{\hat{S}_t} = dX_t \tag{26} \]
or \[ \hat{S}_t = \hat{S}_0 e^{X_t}, \tag{27} \]

Examples of exponential Lévy models Parametric exponential Lévy models fall into two categories. In the first category, called \textit{jump-diffusion} models, the “normal” evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events — crashes and large drawdowns. Such an evolution can be represented by a Lévy process with a nonzero Gaussian component and a jump part with finitely many jumps:
\[ X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{28} \]
where \((Y_i)\) are i.i.d. and \(N\) is a Poisson process.

In the Merton model (see Example 1), which is the first model of this type, suggested in the literature, jumps in the log-price \(X\) are assumed to have a Gaussian distribution: \(Y_i \sim \mathcal{N}(\mu, \delta^2)\).

In the Kou model [30], jump sizes are distributed according to an asymmetric Laplace law with a density of the form

\[
\nu_0(dx) = [p\lambda_+ e^{-\lambda_+ x}1_{x>0} + (1-p)\lambda_- e^{-\lambda_- |x|}1_{x<0}]dx
\]

(29)

with \(\lambda_+ > 0, \lambda_- > 0\) governing the decay of the tails for the distribution of positive and negative jump sizes and \(p \in [0, 1]\) representing the probability of an upward jump. The probability distribution of returns in this model has semi-heavy (exponential) tails.

The second category consists of models with an infinite number of jumps in every interval, called infinite activity or infinite intensity models. In these models, one does not need to introduce a Brownian component since the dynamics of jumps is already rich enough to generate nontrivial small time behavior [8].

The variance gamma process [9, 35] (see Example 2) is obtained by time-changing a Brownian motion with a gamma subordinator and has the characteristic exponent of the form:

\[
\psi(u) = -\frac{1}{\kappa} \log(1 + \frac{u^2 \sigma^2 \kappa}{2} - i\theta ku).
\]

(30)

The density of the Lévy measure of the variance gamma process is given by

\[
\nu(x) = \frac{c_-}{|x|} e^{-\lambda_- |x|} 1_{x<0} + \frac{c_+}{x} e^{-\lambda_+ x} 1_{x>0},
\]

(31)

where \(c = 1/\kappa\), \(\lambda_+ = \sqrt{\theta^2 + 2\sigma^2/\kappa} - \theta \sigma^2\) and \(\lambda_- = \sqrt{\theta^2 + 2\sigma^2/\kappa} + \theta \sigma^2\).

To define the tempered stable process, introduced by Koponen [29] and also known under the name of CGMY model [8], one specifies directly the Lévy density:

\[
\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} 1_{x<0} + \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0}
\]

(32)

with \(\alpha_+ < 2\) and \(\alpha_- < 2\).
7 The Esscher transform and absence of arbitrage in exponential Lévy models

7.1 Measure changes for Lévy processes

To find out whether a given exponential Lévy model is suitable for financial modeling, one needs to ensure that it does not contain arbitrage opportunities, a property which, by the fundamental theorem of asset pricing, is guaranteed by the existence of an equivalent martingale measure. The no arbitrage equivalences for exponential Lévy models were studied in [26, 11, 49] in the one-dimensional unconstrained case and more recently in [28] in the multidimensional case with convex constraints on trading strategies. In this section, shall only review the one-dimensional result.

In the Black-Scholes model, the unique equivalent martingale measure could be obtained by changing the drift of the Brownian motion. In models with jumps, if the Gaussian component is absent, this is no longer possible, but a much greater variety of equivalent measures can be obtained by altering the distribution of jumps. The following proposition describes the possible measure changes under which a Lévy process remains a Lévy process.

Proposition 9 (see Sato [42], Theorems 33.1 and 33.2). Let \((X, \mathbb{P})\) be a Lévy process on \(\mathbb{R}^d\) with characteristic triplet \((A, \nu, \gamma)\); choose \(\eta \in \mathbb{R}^d\) and \(\varphi : \mathbb{R}^d \to \mathbb{R}\) with

\[
\int_{\mathbb{R}^d} \left( e^{\varphi(x)/2} - 1 \right)^2 \nu(dx) < \infty. \tag{33}
\]

and define

\[
U_t := \eta.X^c + \int_0^t \int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) \tilde{J}_X(ds \, dx),
\]

where \(X^c\) denotes the continuous martingale (Brownian motion) part of \(X\), and \(\tilde{J}_X\) is the compensated jump measure of \(X\).

Then \(\mathcal{E}(U)_t\) is a positive martingale such that the probability measure \(\mathbb{P}'\) defined by

\[
\frac{d\mathbb{P}'|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \mathcal{E}(U)_t, \tag{34}
\]

is equivalent to \(\mathbb{P}\) and under \(\mathbb{P}'\), \(X\) is a Lévy process with characteristic triplet.
\((A, \nu', \gamma')\) where \(\nu' = e^\theta \nu\) and

\[
\gamma' = \gamma + \int_{|x| \leq 1} x(\nu' - \nu)(dx) + A \eta.
\]

(35)

A useful example, which will be the basis of our construction of an equivalent martingale measure is provided by the Esscher transform. Let \(X\) be a Lévy process on \(\mathbb{R}^d\) with characteristic triplet \((A, \nu, \gamma)\), and let \(\theta \in \mathbb{R}^d\) be such that \(\int_{|x| > 1} e^{\theta \cdot x} \nu(dx) < \infty\). Applying a measure transformation of Proposition 9 with \(\eta = \theta\) and \(\phi(x) = \theta \cdot x\), we obtain an equivalent probability under which \(X\) is a Lévy process with Lévy measure \(\tilde{\nu}(dx) = e^{\theta \cdot x} \nu(dx)\) and third component of the characteristic triplet \(\tilde{\gamma} = \gamma + A \theta + \int_{|x| \leq 1} x(e^{\theta \cdot x} - 1)\nu(dx)\).

Using Proposition 9, the Radon-Nikodym derivative corresponding to this measure change is found to be

\[
\frac{dP'}{dP} \bigg|_{F_t} = e^{\theta \cdot X_t} \mathbb{E}\left[ e^{\theta \cdot X_1} \right] = \exp(\theta \cdot X_t - \kappa(\theta) t),
\]

(36)

where \(\kappa(\theta) := \ln \mathbb{E}\left[ \exp(\theta \cdot X_1) \right] = \psi(-i\theta)\).

Although the two definitions of an exponential Lévy model, via the ordinary exponential (27) or via the stochastic exponential (26), are equivalent, the set of Lévy processes that lead to arbitrage-free models of the form (27) does not necessarily coincide with the set that yields arbitrage-free models of the form (26). In particular, we shall see that the no-arbitrage conditions for multidimensional stochastic and ordinary exponentials are considerably different. It will be more convenient to find these conditions for models of type (26) first and then deduce the conditions for ordinary exponentials using the transformation \(X_t := \ln \mathbb{E}(Y)_t\). In the multidimensional case, this transformation must be applied to each component.

In an exponential Lévy model of type (26), the absence of arbitrage is tantamount to the existence of a probability \(Q\) equivalent to \(P\) such that \(\mathcal{E}(X)\) is a \(Q\)-martingale. We will see that when this is the case, it is always possible to find a martingale probability \(Q \sim P\) under which \(X\) remains a Lévy process, which means that \(X\) itself must be a \(Q\)-martingale (cf. Proposition 8.23 in[12]).

**Theorem 2** (Absence of arbitrage in models based on stochastic exponentials, one-dimensional case).

*Let \((X, P)\) be a real-valued Lévy process on \([0, T]\) with characteristic triplet \((\sigma^2, \nu, \gamma)\). The following statements are equivalent:*
1. There exists a probability \( Q \) equivalent to \( \mathbb{P} \) such that \((X, Q)\) is a Lévy process and a martingale.

2. Either \( X \equiv 0 \) or \((X, \mathbb{P})\) is not a.s. monotone.

3. One of the following conditions is satisfied:

   (i) \( \sigma > 0 \).

   (ii) \( \sigma = 0 \) and \( \int_{|x| \leq 1} |x| \nu(dx) = \infty \).

   (iii) \( \sigma = 0 \), \( \int_{|x| \leq 1} |x| \nu(dx) < \infty \) and one of the following is true:

   - \( \nu((-\infty, 0)) > 0 \) and \( \nu((0, \infty)) > 0 \).
   - \( \nu((-\infty, 0)) > 0 \) and \( b > 0 \).
   - \( \nu((0, \infty)) > 0 \) and \( b < 0 \).

   - The trivial case of a constant process: \( \nu = 0 \) and \( b = 0 \),

   where \( b = \gamma - \int_{|x| \leq 1} x \nu(dx) \) is the drift of \( X \).

Condition 2. implies that if an exponential Lévy model admits an arbitrage, it can be realized by a buy-and-hold strategy (if \( X \) is increasing) or a corresponding short sale (if \( X \) is decreasing).

Before proceeding with the proof of theorem 2, we will show that for one-dimensional exponential Lévy models of the form (27), the no-arbitrage conditions are actually the same as for stochastic exponentials.

**Corollary 3** (Absence of arbitrage in models based on ordinary exponential, one-dimensional case). Let \((X, \mathbb{P})\) be a real-valued Lévy process on \([0, T]\) with characteristic triplet \((\sigma^2, \nu, \gamma)\). The following statements are equivalent:

1. There exists a probability \( Q \) equivalent to \( \mathbb{P} \) such that \((X, Q)\) is a Lévy process and \( e^X \) is a martingale.

2. Either \( X \equiv 0 \) or \((X, \mathbb{P})\) is not a.s. monotone.

3. The characteristic process of \( X \) satisfies condition 3 of Theorem 2.

**Proof.** It suffices to show that \( \ln \mathcal{E}(X) \) is monotone if and only if \( X \) is monotone. From [12, Proposition 8.22] it is easy to see that \( \ln \mathcal{E}(X) \) is a finite variation process if and only if \( X \) is a finite variation process. In the finite-variation case, the stochastic exponential has a simple form:

\[
\mathcal{E}(X)_t = e^{bt} \prod_{s \leq t} (1 + \Delta X_s),
\]
and it is readily seen that the monotonicity properties of $X$ and $\log \mathcal{E}(X)$ are the same.

\textbf{Proof of theorem 2.} We exclude the trivial case $X \equiv 0$ a.s. which clearly does not constitute an arbitrage opportunity (every probability is a martingale measure).

The equivalence $2 \iff 3$ follows from [12, Proposition 3.10].

$3 \Rightarrow 1$. Define a probability $\tilde{P}$ equivalent to $P$ by

$$
\frac{d\tilde{P}|_{\mathcal{F}_T}}{dP|_{\mathcal{F}_T}} = \mathcal{E}\left(\int_0^T \int_\mathbb{R} (e^{-x^2} - 1)J_X(ds \, dx)\right)_T
$$

Under $\tilde{P}$, $X$ has characteristic triplet $(\sigma^2, \tilde{\nu}, \tilde{\gamma})$ with $\tilde{\nu} = e^{-x^2} \nu$ and $\tilde{\gamma} = \gamma + \int_{|x| \leq 1} x(e^{-x^2} - 1)\nu(dx)$. It is easy to see that $E^{\tilde{P}}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$ and all $t > 0$.

Suppose that the convex function $\lambda \mapsto E^{\tilde{P}}[e^{\lambda X_1}]$ has a finite minimizer $\lambda^*$. Then, using the dominated convergence theorem, $E^{\tilde{P}}[X_1 e^{\lambda^* X_1}] = 0$ which implies that $X$ is a $Q$-martingale with

$$
\frac{dQ|_{\mathcal{F}_t}}{d\tilde{P}|_{\mathcal{F}_t}} = \frac{e^{\lambda^* X_t}}{E[e^{\lambda^* X_1}]} \quad \text{(Essher transform)}
$$

To show the existence of a finite minimizer $\lambda^*$, it is sufficient to prove that $E^{\tilde{P}}[e^{\lambda X_1}] \to \infty$ as $\lambda \to \infty$, or, equivalently, that the function

$$
f(\lambda) = \log E^{\tilde{P}}[e^{\lambda X_1}] = \frac{\sigma^2}{2} \lambda^2 + \tilde{\gamma} \lambda + \int_{\mathbb{R}} (e^{\lambda x} - 1 - \lambda x 1_{|x| \leq 1})e^{-x^2}\nu(dx).
$$

goess to infinity as $\lambda \to \infty$. In case (i), $f''(\lambda) \geq \sigma^2$ which means that $f(\lambda) \to \infty$ as $\lambda \to \infty$. In case (ii),

$$
f'(\lambda) = \tilde{\gamma} + \int_{|x| > 1} x e^{-x^2}\nu(dx) + \int_{\mathbb{R}} x(e^{\lambda x} - 1)e^{-x^2}\nu(dx),
$$

and it is not difficult to check that $\lim_{\lambda \to -\infty} f'(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} f'(\lambda) = -\infty$ which means that $f(\lambda) \to \infty$ as $\lambda \to \infty$. In case (iii),

$$
f'(\lambda) = b + \int_{\mathbb{R}} x e^{\lambda x} e^{-x^2}\nu(dx),
$$

33
and it is easy to see, by examining one by one the different mutually exclusive
cases listed after the statement of the Theorem, that in each of these cases $f'$
is bounded from below on $\mathbb{R}$ and therefore once again, $f(\lambda) \to \infty$ as $\lambda \to \infty$.

1 $\Rightarrow$ 2. It is clear that a process cannot be a martingale under one
probability and a.s. monotone under an equivalent probability, unless it is constant. $\Box$

**Exercise 18.** Let $(X, P)$ be a Lévy process with characteristic triple $(A, \nu, \gamma)$, and let $Q$ be a probability measure defined by

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{E[e^{\theta X_t}]}$$

for $\theta \in \mathbb{R}$ such that $E[e^{\theta X_t}] < \infty$. Compute the characteristic triple of $X$
under $Q$.

**Exercise 19.**

- Let $X$ be a Lévy process and let $f : \mathbb{R} \to (0, \infty)$. What condition must
be imposed on the function $f$ for the sum

$$\sum_{t \in [0,1] : \Delta X_t \neq 0} f(\Delta X_t)$$

to converge a.s.?

- Let $(X, P)$ be a Lévy process with Lévy measure $\frac{1}{|x|^{1+\alpha}}$ and let $(X, Q)$
be a Lévy process with Lévy measure $\frac{1}{|x|^{1+\alpha'}}$ (with $\alpha > 0$ and $\alpha' > 0$).
Use the previous question to show that $P \sim Q$ implies $\alpha = \alpha'$. Check
your result using Proposition 9.

8 \hspace{2em} European options in exp-Lévy models

Given the results of section 7, in any “reasonable” exponential Lévy model
we can assume that there exists a probability measure $Q$ equivalent to $P$
such that the discounted prices of all assets are $Q$-martingales. In practice,
this measure is usually found by calibrating the exponential Lévy model to
market quoted prices of European options [4, 13], and the first step in using
the model is therefore to obtain fast pricing algorithms for European calls
and puts.
Prices of European options in exponential Lévy models can be computed directly from the characteristic function of $X$ which is explicitly known from the Lévy-Khintchine formula. This idea was first introduced to finance by Carr and Madan [9] (for European calls and puts) and later extended and generalized by many authors including [39, 34, 33, 18]. The result given below is a slight generalization of the one in [18], allowing both discontinuous pay-off functions and Lévy processes without a bounded density, such as variance gamma.

We start with a one-dimensional risk-neutral exponential Lévy model in the form (27). Under the risk-neutral probability, the process $e^{X_t}$ must therefore be a martingale, a condition which can be expressed in terms of the characteristic triplet of $X$:

$$\gamma + \frac{A}{2} + \int_{\mathbb{R}} (e^{iy} - 1 - y1_{|y|\leq 1})\nu(dy) = 0.$$ 

We consider a European option with pay-off $G(S_T) = G(\hat{S}_T)$ at time $T$ and denote by $g$ its log-payoff function: $G(e^X) \equiv g(x)$. As above, we denote by $\Phi_t$ the characteristic function of $X_t$.

**Proposition 10.** Suppose that there exists $R \neq 0$ such that

\begin{align}
&g(x)e^{-Rx} \text{ has finite variation on } \mathbb{R}, \\
g(x)e^{-Rx} \in L^1(\mathbb{R}), \\
&E[e^{RX_T-t}] < \infty \text{ and } \int_{\mathbb{R}} \frac{\Phi_{T-t}(u - iR)|}{1 + |u|} du < \infty.
\end{align}

Then the price at time $t$ of the European option with pay-off function $G$ satisfies

$$P(t, S_t) := e^{-r(T-t)}E[G(S_T)|F_t] = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\Phi_{T-t}(-u - iR)\hat{S}_t^{R-iu} du,$$

where

$$\hat{g}(u) := \int_{\mathbb{R}} e^{ix} g(x) dx.$$

**Proof.** By integration by parts for Stieltjes integrals,

$$\hat{g}(u + iR) = \int_{\mathbb{R}} g(x)e^{ix(u+iR)} dx = \frac{i}{u+iR} \int_{\mathbb{R}} e^{ix(u+iR)} dg(x).$$
This implies in particular that
\[ |\hat{g}(u + iR)| \leq \frac{C}{|u + iR|}, \quad u \in \mathbb{R}. \] (42)

Suppose that \( R > 0 \) (the case \( R < 0 \) can be treated in a similar manner) and consider the function
\[ f(x) = e^{Rx} \int_{-\infty}^{\infty} p(z) \, dz, \]
where \( p \) denotes the distribution of \( X_{T-t} \). From the assumption (39) it follows
\[ \int_{\mathbb{R}} e^{Rx} p(dx) < \infty \]
and therefore \( \lim_{x \to \infty} f(x) = 0 \). Clearly also \( \lim_{x \to -\infty} f(x) = 0 \). By integration by parts,
\[ \int_{-N}^{N} f(x) dx = \frac{1}{R} \int_{-N}^{N} e^{Rx} p(dx) + \frac{1}{R} (f(N) - f(-N)). \]

This shows that \( f \in L^1(\mathbb{R}) \) and it follows that
\[ \int_{\mathbb{R}} e^{-iux} f(x) dx = \frac{\Phi_{T-t}(-u - iR)}{R - iu}. \]

From condition (39) it follows that \( f \) can be recovered by Fourier inversion (cf. [40, Theorem 9.11]):
\[ f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} \Phi_{T-t}(-u - iR) \frac{dx}{R - iu} \] (43)

Let us now turn to the proof of (40). From (41), (43) and Fubini’s theorem,
\[ \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR) \Phi_{T-t}(-u - iR) \hat{S}_t^{R - iu} du \] (44)
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} g(x)e^{Rx} \int_{\mathbb{R}} du \frac{e^{iux} \Phi_{T-t}(-u - iR)e^{(R-iu)\log \hat{S}_t}}{R - iu} \] (45)
\[ = \int_{\mathbb{R}} g(x)e^{-R(x - \log \hat{S}_t)} f(x - \log \hat{S}_t) = \int_{\mathbb{R}} g(x) \int_{x - \log \hat{S}_t}^{\infty} p(dz) \] (46)
\[ = \int_{\mathbb{R}} g(x + \log \hat{S}_t) p(dx) = E^Q[G(\hat{S}_T)|\mathcal{F}_t] = E^Q[G(S_T)|\mathcal{F}_t]. \] (47)

\[ \square \]
Example 3. The digital option has pay-off $G(S_T) = 1_{S_T \geq K}$. In this case for all $R > 0$ conditions (37) and (38) are satisfied and

$$
\hat{g}(u + iR) = \frac{K^{iu-R}}{R - iu}.
$$

Example 4. The European call option has pay-off $G(S_T) = (S_T - K)^+$. Therefore, conditions (37) and (38) are satisfied for all $R > 1$,

$$
\hat{g}(u + iR) = \frac{K^{iu+1-R}}{(R - iu)(R - 1 - iu)},
$$

and the price of a call option can be written as an inverse Fourier transform:

$$
C(t, S_t) = e^{-r(T-t)} \int_{\mathbb{R}} \frac{K^{iu+1-R}S_t^{R-iu}\Phi_{T-t}(-u - iR)}{(R - iu)(R - 1 - iu)} du
= S_t \frac{e^{k_f(u+1-R)}\Phi_{T-t}(-u - iR)}{2\pi (R - iu)(R - 1 - iu)} du
$$

(48)

where $k_f$ is the log forward moneyness defined by $k_f = \ln(K/S_t) - r(T - t)$. This property allows to compute call option prices for many values of $k_f$ in a single computation using the Fast Fourier Transform algorithm.

9 Gap options

The gap options are a class of exotic equity derivatives offering protection against rapid downside market moves (gaps). These options have zero delta, allowing to make bets on large downside moves of the underlying without introducing additional sensitivity to small fluctuations, just as volatility derivatives allow to make bets on volatility without going short or long delta. The market for gap options is relatively new, and they are known under many different names: gap options, crash notes, gap notes, daily cliquets, gap risk swaps etc. The gap risk often arises in the context of constant proportion portfolio insurance (CPPI) strategies [14] and other leveraged products such as the leveraged credit-linked notes. The sellers of gap options (who can be seen as the buyers of the protection against gap risk) are typically major banks who want to get off their books the risk associated to CPPI or other leveraged products. The buyers of gap options and the sellers of the protection are usually hedge funds looking for extra returns.
The pay-off of a gap option is linked to the occurrence of a gap event, that is, a 1-day downside move of sufficient size in the underlying. The following single-name gap option was commercialized by a big international bank in 2007 under the name of gap risk swap:

**Example 5 (Single-name gap option).**

- The protection seller pays the notional amount $N$ to the protection buyer at inception and receives Libor + spread monthly until maturity or the first occurrence of the gap event, whichever comes first, plus the notional at maturity if no gap event occurs.

- The gap event is defined as a downside move of over 10% in the DJ Euro Stoxx 50 index within 1 day (close to close).

- If a gap event occurs between dates $t-1$ and $t$, the protection seller immediately receives the reduced notional $N(1-10\times(0.9-R))^+$, where $R = \frac{S_t}{S_{t-1}}$ is the index performance at gap, after which the product terminates.

The gap options are therefore similar to equity default swaps, with a very important difference, that in EDS, the price change from the inception date of the contract to a given date is monitored, whereas in gap options, only 1-day moves are taken into account.

The gap options are designed to capture stock jumps, and clearly cannot be priced within a diffusion model with continuous paths, since any such model will largely underestimate the gap risk. For instance, for a stock with a 25% volatility, the probability of having an 10% gap on any one day during one year is $3 \times 10^{-8}$, and the probability of a 20% gap is entirely negligible.

Suppose that the time to maturity $T$ of a gap option is subdivided onto $N$ periods of length $\Delta$ (e.g. days): $T = N\Delta$. The return of the $k$-th period will be denoted by $R_k^\Delta = \frac{S_k}{S_{(k-1)\Delta}}$. For the analytic treatment, we formalize the single-asset gap option as follows.

**Definition 10 (Gap option).** Let $\alpha$ denote the return level which triggers the gap event and $k^*$ be the time of first gap expressed in the units of $\Delta$: $k^* := \inf\{k : R_k^\Delta \leq \alpha\}$. The gap option is an option which pays to its holder the amount $f(R_{k^*}^\Delta)$ at time $\Delta k^*$, if $k^* \leq N$ and nothing otherwise.

Supposing that the interest rate is deterministic and equal to $r$, it is easy to see that the pay-off structure of example 5 can be expressed as a linear combination of pay-offs of definition 10.
We first treat the case where the log-returns are independent and stationary.

**Proposition 11.** Let the log-returns \((R^\Delta_k)_{k=1}^N\) be i.i.d. and denote the distribution of \(\log R^\Delta_1\) by \(p^\Delta(dx)\). Then the price of a gap option as of definition 10 is given by

\[
G^\Delta = e^{-r\Delta} \int_{-\infty}^\beta f(e^x)p^\Delta(dx) \frac{1 - e^{-rT} \left( \int_\beta^\infty p^\Delta(dx) \right)^N}{1 - e^{-r\Delta} \int_\beta^\infty p^\Delta(dx)},
\]

with \(\beta := \log \alpha < 0\).

**Proof.**

\[
G^\Delta = E\left[ e^{-\Delta^* r} f(R^\Delta_{k^*}) 1_{k^* \leq N} \right] = \sum_{n=1}^N P[k^* = n] E[f(R^\Delta_n)|k^* = n] e^{-\Delta^* r}
\]

\[
= \sum_{n=1}^N P[R_n^\Delta \leq \alpha] E[f(R_n^\Delta)|R_n^\Delta \leq \alpha] e^{-\Delta^* r} \prod_{l=1}^{n-1} P[R_l^\Delta > \alpha]
\]

\[
= e^{-r\Delta} \int_{-\infty}^\beta f(e^x)p^\Delta(dx) \frac{1 - e^{-rT} \left( \int_\beta^\infty p^\Delta(dx) \right)^N}{1 - e^{-r\Delta} \int_\beta^\infty p^\Delta(dx)}.
\]

\[
N\int_{-\infty}^\beta f(e^x)p^\Delta(dx).
\]

**Numerical evaluation of prices** Formula (49) allows to compute gap option prices by Fourier inversion. For this, we need to be able to evaluate the cumulative distribution function \(F^\Delta(x) := \int_{-\infty}^x p^\Delta(d\xi)\) and the integral

\[
\int_{-\infty}^\beta f(e^x)p^\Delta(dx).
\]

Let \(\phi^\Delta\) be the characteristic function of \(p^\Delta\), and suppose that \(p^\Delta\) satisfies \(\int |x|p^\Delta(dx) < \infty\) and \(\int_\mathbb{R} \frac{\phi^\Delta(u)}{1+|u|} du < \infty\). Let \(F'\) be the CDF and \(\phi'\) the characteristic function of a Gaussian random variable with zero mean and standard deviation \(\sigma' > 0\). Then by Lemma 1 in [14],

\[
F^\Delta(x) = F'(x) + \frac{1}{2\pi} \int_\mathbb{R} e^{-iu\phi^\Delta(u)} \frac{\phi^\Delta(u) - \phi_{\Delta}(u)}{iu} du.
\]
The Gaussian random variable is only needed to obtain an integrable expression in the right hand side and can be replaced by any other well-behaved random variable.

The integral (50) is nothing but the price of a European option with payoff function \(f\) and maturity \(\Delta\). For arbitrary \(f\) it can be evaluated using the Fourier transform method as described in section 8. For the numerical evaluation, the integrals must be truncated to a finite interval \([-L, L]\). Since \(\Delta\) is small, the characteristic function \(\phi_{\Delta}(u)\) decays slowly at infinity, which means that \(L\) must be sufficiently large (for example, in a jump-diffusion model with volatility \(\sigma\), \(L \sim \frac{C}{\sigma \sqrt{\Delta}}\), where \(C\) is a constant which depends on the desired precision, such as \(C = 5\) — see [50]). The computation of the integrals will therefore be rather costly. For this reason, we do not recommend to use the exact formula, and propose an approximation, which is based on an expansion of \(G_{\Delta}\) around the value \(\Delta = 0\). In other words, instead of using a numerical method whose computational complexity increases when \(\Delta\) is small, we suggest an explicit formula whose precision improves when \(\Delta \to 0\).

### Approximate pricing formula

Suppose that \(S_t = S_0 e^{X_t}\), where \(X\) is a Lévy process. This means that \(p_{\Delta}\) as defined above is the distribution of \(X_{\Delta}\).

Since \(r\Delta \sim 10^{-4}\) and the probability of having a gap on a given day \(\int_{-\infty}^{\beta} p_{\Delta}(dx)\) is also extremely small, with very high precision,

\[
G_{\Delta} \approx \int_{-\infty}^{\beta} f(e^{x})p_{\Delta}(dx) \frac{1 - e^{-rT-N\int_{-\infty}^{\beta} p_{\Delta}(dx)}}{r \Delta + \int_{-\infty}^{\beta} p_{\Delta}(dx)}. \tag{52}
\]

Our second approximation is less trivial. From [41], we know that for all Lévy processes and under very mild hypotheses on the function \(f\), we have

\[
\int_{-\infty}^{\beta} g(x)p_{\Delta}(dx) \sim \Delta \int_{-\infty}^{\beta} g(x)\nu(dx),
\]

as \(\Delta \to 0\), where \(\nu\) is the Lévy measure of \(X\). Consequently, when \(\Delta\) is nonzero but small, we can replace the integrals with respect to the density with the integrals with respect to the Lévy measure in formula (52), obtaining an approximate but explicit expression for the gap option price:

\[
G_{\Delta} \approx G_0 = \lim_{\Delta \to 0} G_{\Delta} = \int_{-\infty}^{\beta} f(e^{x})\nu(dx) \frac{1 - e^{-rT-N\int_{-\infty}^{\beta} \nu(dx)}}{r + \int_{-\infty}^{\beta} \nu(dx)}. \tag{53}
\]
This approximation is obtained by making the time interval at which returns
are monitored (a priori, one day), go to zero. In other words, \( G_0 \) is the zero-
order term of the Taylor expansion of \( G_\Delta \) around the point \( \Delta = 0 \). The error
of this approximation will therefore decay proportionally to \( \Delta \) when \( \Delta \to 0 \).

A modified gap option For a better understanding of the risks of a gap
option, it is convenient to interpret the pricing formula (53) as an exact price
of a modified gap option rather than the true price of the original option.
From now on, we define the single-asset gap option as follows.

Definition 11 (Modified gap option). Let \( \tau = \inf \{ t : \Delta X_t \leq \beta \} \) be the
time of the first jump of \( X \) smaller than \( \beta \). The gap option as a product
which pays to its holder the amount \( f \left( \frac{S_\tau}{S_{\tau-}} \right) = f(e^{\Delta X_\tau}) \) if \( \tau \leq T \) and zero
otherwise.

The price of this product is given by

\[
G = E^Q[e^{-r\tau} f(e^{\Delta X_\tau})1_{\tau \leq T}]
\]

which is easily seen to be equal to \( G_0 \):

Proposition 12. Suppose that the underlying follows an exponential Lévy
model: \( S_t = S_0 e^{X_t} \), where \( X \) is a Lévy process with Lévy measure \( \nu \).
Then the price of the gap option as of definition 11, or, equivalently, the approximate
price of the gap option as of definition 10 is given by

\[
G = \int_{-\infty}^{\beta} f(e^x) \nu(dx) \frac{1 - e^{-rT-T} \int_{-\infty}^{\beta} \nu(dx)}{r + \int_{-\infty}^{\beta} \nu(dx)}
\]

with \( \beta := \log \alpha \).

The gap option then arises as a pure jump risk product, which is only
sensitive to negative jumps larger than \( \beta \) in absolute value, but not to small
fluctuations of the underlying. In particular, it has zero delta.

10 Hedging in exponential Lévy models

Exponential Lévy models generally correspond to incomplete markets, mak-
ing exact replication impossible. Hedging must therefore be interpreted as
approximation of the terminal pay-off with an admissible portfolio. The usual practice is to minimize the expected squared deviation of the hedging portfolio from the contingent claim, an approach known as quadratic hedging. The resulting strategies are often explicitly computable and, more importantly, they are linear, because the hedging portfolios can be interpreted as orthogonal projections of contingent claims onto the closed linear subspace of hedgeable portfolios. To hedge a book of options written on the same underlying, a trader can therefore compute the hedge ratio for every option in the book and then add them up, just like this is typically done with delta hedging. This greatly reduces the computational cost of hedging and is an important advantage of quadratic hedging compared to other, e.g., utility-based approaches.

To define the criterion to be minimized in a mean square sense, two approaches are possible. In the first approach \cite{7, 37, 27}, the hedging strategy is supposed to be self-financing, and one minimizes the quadratic hedging error at maturity, that is, the expected squared difference between the terminal value of the hedging portfolio and the option’s pay-off:

\[
\inf_{V_0, \phi} E\left[ |V_T(\phi) - H|^2 \right] \quad \text{where} \quad V_T(\phi) = V_0 + \int_0^T \phi^0_t dS^0_t + \int_0^T \phi_t dS_t, \quad \text{(54)}
\]

where \( S^0 \) is the risk-free asset. If the interest rate is constant, we can choose the zero-coupon bond with maturity \( T \) as the risk-free asset: \( S^0_t = e^{-r(T-t)} \) and after discounting this problem becomes:

\[
\inf_{V_0, \phi} E[|V_T(\phi) - H|^2], \quad \text{where} \quad V_T = \hat{V}_0 + \int_0^T \phi_t d\hat{S}_t.
\]

In the second approach \cite{21, 20, 44, 47}, strategies that are not self-financing are allowed, but they are required to replicate the option’s pay-off exactly: \( V_T(\phi) = H \). In an incomplete market, this means that the option’s seller will have to continuously inject / withdraw money from the hedging portfolio. The cumulative amount of funds injected or withdrawn is called the cost process. It is given by

\[
C_t(\phi) = V_t(\phi) - G_t(\phi),
\]

where

\[
V_t(\phi) = \phi^0_t S^0_t + \phi_t S_t
\]
and $G$ is the gain process given by
\[ G_t = \int_0^t \phi_0^s dS_0^s + \int_0^t \phi_s dS_s. \]
The discounted cost process is then given by
\[ \hat{C}_t = \phi_0^t + \phi_t \hat{S}_t - \int_0^t \phi_s d\hat{S}_s. \]

The risk-minimizing strategy, as introduced by Föllmer and Sondermann [21], is a strategy which replicates the option’s pay-off, and has the cost process which varies as little as possible, that is, this strategy minimizes, at each date $t$, the residual cost given by
\[ E[(\hat{C}_T - \hat{C}_t)^2 | F_t]. \] (55)
over all admissible continuations of the strategy from date $t$ onwards. The risk-minimizing strategy always exists in the martingale case (when the discounted stock price is a martingale), but in the general case, it may fail to exist even in the most simple examples [44]. Motivated by this difficulty, Föllmer and Schweizer [20] introduced the notion of locally risk minimizing strategy, which corresponds to finding the extremum of (55) with respect to suitably defined small perturbations of the strategy, or, in other words, measuring the riskiness of the cost process locally in time. Local risk minimization is discussed in detail in section 10.2.

The expectations in (55) and (54) are taken with respect to some probability which we have to specify. To begin, let us assume that we have chosen a martingale measure $Q$ and the expectations in (54) and (55) are taken with respect to $Q$. In particular, $\hat{S}$ is a martingale under $Q$. Assume now that $H \in L^2(\Omega, \mathcal{F}, Q)$ and $\hat{S}$ is also square-integrable. If we consider portfolios of the form:
\[ S = \{ \phi \text{ caglad predictable and } E|\int_0^T \phi_t d\hat{S}_t|^2 < \infty \} \] (56)
then the set $\mathcal{A}$ of attainable pay-offs is a closed linear subspace of $L^2(\Omega, \mathcal{F}, Q)$, and the quadratic hedging problem becomes an orthogonal projection:
\[ \inf_{V_0, \phi} E|V_T(\phi) - H|^2 = \inf_{A \in \mathcal{A}} \|H - A\|^2_{L^2(Q)}. \] (57)
The solution is then given by the well-known Galtchouk-Kunita-Watanabe decomposition [31, 22], which states that any random variable \( H \in L^2(\Omega, F, Q) \) can be represented as

\[
H = E[H] + \int_0^T \phi_t^H d\hat{S}_t + N_t^H, 
\]

where \((N_t^H)\) is a square integrable martingale orthogonal to \( \hat{S} \). The optimal hedging strategy is then given by \( \phi^H \) and the initial cost of the hedging portfolio is \( V_0 = e^{-r(T-t)} E[H] \).

Introducing the martingale \( \hat{H}_t := E[H|F_t] \) generated by \( H \), we have

\[
\hat{H}_t = E[H] + \int_0^t \phi_s^H d\hat{S}_s + N_t^H,
\]

and the orthogonality implies

\[
\langle \hat{H} - \int_0^t \phi_s^H d\hat{S}_s, \hat{S} \rangle \equiv 0,
\]

which means that the optimal hedge ratio may be expressed more explicitly using the predictable covariation of the option price and the stock price:

\[
\phi_t^H = \frac{d\langle \hat{H}, \hat{S} \rangle_t}{d\langle \hat{S}, \hat{S} \rangle_t}.
\]

In the martingale setting, optimizing the global hedging error (54) we obtain a strategy which is also risk minimizing in the sense of equation (55). For any strategy \( \phi \), we have

\[
E[(\hat{C}_T - \hat{C}_t)^2|F_t] = (\hat{H}_t - \hat{V}_t)^2 + E\left[\left(H - \hat{H}_t - \int_t^T \phi_s d\hat{S}_s\right)^2|F_t\right]
\]

\[
= (\hat{H}_t - \hat{V}_t)^2 + E[(N_T - N_t)^2|F_t] + E\left[\left(\int_t^T (\phi_s - \phi_s^H) d\hat{S}_s\right)^2|F_t\right].
\]

To minimize this expression, we clearly need to take \( \phi = \phi^H \) and choose \( \phi^0 \) such that \( \hat{V}_t = \hat{H}_t \) for all \( t \). In this case, the discounted cost process is given by

\[
\hat{C}_t = \hat{V}_t - \int_0^t \phi_s^H d\hat{S}_s = E[H] + N_t^H.
\]
We shall see in section 10.2 that in the martingale setting, the strategy $\phi^H$ which minimizes the terminal hedging error also coincides with the locally risk minimizing strategy of Föllmer and Schweizer [20]. Moreover, it is often easy to compute in terms of option prices. This is no longer true if $\hat{S}$ is not a martingale. However using the risk-neutral second moment of the hedging error as a criterion for measuring risk is not very natural: $Q$ represents a pricing rule and not a statistical description of market events, so the profit and loss (P&L) of a portfolio may have a large variance while its “risk neutral” variance can be small. Nevertheless, to estimate the expected return of a stock, and therefore, to distinguish it from a martingale, one needs historical stock return observations covering an extended period of time, often exceeding the lifetime of the option. Option hedging, on the other hand, is a “local” business, where one tries to cancel out the daily movements of option prices with the daily movements of the underlying and locally, every stock behaves like a martingale.

10.1 Quadratic hedging in exponential-Lévy models under the martingale measure

Although the quadratic hedging problem is “solved” by the Galtchouk-Kunita-Watanabe decomposition, from a practical point of view the problem is of course to compute the risk minimizing hedge $\phi^H_t$. Formulas for $\phi^H_t$ with various degrees of explicitness and under various assumptions on the driving process $X$ and on the pay-off $G$ were given in [7, 19, 5, 25] and several other papers. In particular [15] provide the expressions for hedge ratios in the case when the hedging portfolio itself contains options. In the case of European pay-offs and exponential Lévy models, the problem was solved in [27] using Fourier analysis techniques. Their method, reviewed in section 10.2 covers the general case as well as the martingale case. In this section, we provide another Fourier-based result, which is specialized to the martingale setting but works under different regularity assumptions on the pay-off than in [27], which include, for instance, digital options.

**Proposition 13** (Quadratic hedge in exponential-Lévy models, martingale case). Let $X$ be a Lévy process with Lévy measure $\nu$, diffusion coefficient $\sigma$, and characteristic function $\Phi$, such that $e^X$ is a martingale and assume:

1. The log-payoff function satisfies the conditions (37) and (38).
ii. The integrability condition (39) holds for all $t < T$.

iii. The Lévy measure of $X$ satisfies

$$\int_{|x|>1} e^{2(x\vee Rx)} \nu(dx) < \infty.$$  \hspace{1cm} (60)

Then the optimal quadratic hedging for a European option with pay-off $G(S_T)$ at date $T$ in an exponential Lévy model $S_t = S_0 e^{rt + X_t}$ amounts to holding a position in the underlying

$$\phi_t = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR) \Phi_{T-t}(-u - iR) \hat{S}_t^{R-iu-1} \Upsilon(R - iu) du$$  \hspace{1cm} (61)

where $\Upsilon(y) = \frac{\kappa(y+1) - \kappa(y) - \kappa(1)}{\kappa(2) - 2\kappa(1)}$, and $\kappa(z) := \log E[e^{zX_1}]$, \hspace{1cm} (62)

or, equivalently, $\phi_t = \phi(t, S_t)$ where:

$$\phi(t, S) = \frac{\sigma^2 \partial P}{\partial S}(t, S) + \frac{1}{2} \int \nu(dz)(e^z - 1)[P(t, Se^z) - P(t, S)]$$

$$\sigma^2 + \int (e^z - 1)^2 \nu(dz)$$  \hspace{1cm} (63)

with $P(t, S) = e^{-r(T-t)} E^Q[G(S_T)|S_t = S]$ the option price at date $t$ when the underlying is at the level $S$.

Remark 1. Condition (60), which is the only assumption imposed in addition to those of Proposition 10, guarantees that both the price process $S_t$ and the option pay-off $G(S_T)$ are square integrable.

Proof. By Itô formula, the discounted stock price dynamics is given by

$$\hat{S}_T = \hat{S}_0 + \int_0^T \hat{S}_t \sigma dW_t + \int_0^T \int_\mathbb{R} \hat{S}_t(e^x - 1) \hat{J}_X (dt \times dx).$$  \hspace{1cm} (64)

To prove the proposition using the formula (59), we now need to obtain a similar integral representation for the option’s discounted price function $\hat{P}(t, S_t) = e^{r(T-t)} P(t, S_t)$.

Let $t < T$. Applying the Itô formula under the integral sign in (40), we find

$$\hat{P}(t, S_t) - \hat{P}(0, S_0) = \frac{1}{2\pi} \int_\mathbb{R} du \hat{g}(u + iR) \int_0^t \Phi_{T-s}(-u - iR)(R - iu) \hat{S}_s^{R-iu} \sigma dW_s$$

$$+ \frac{1}{2\pi} \int_\mathbb{R} du \hat{g}(u + iR) \int_0^t \Phi_{T-s}(-u - iR) \hat{S}_s^{R-iu} \int_\mathbb{R} (e^{(R-iu)z} - 1) \hat{J}_X (ds \times dz).$$  \hspace{1cm} (65)
Let us first assume that $\sigma > 0$ and study the first term in the right-hand side of (65), which can be written as

$$
\int_\mathbb{R} \mu(du) \int_0^t H^u_s dW_s
$$

where

$$
\mu(du) = |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)| du
$$

is a finite positive measure on $\mathbb{R}$ and

$$
H^u_s = \frac{\sigma \hat{g}(u + iR)\Phi_{T-s}(-u - iR)}{2\pi |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)|} (R - iu) \hat{S}_s^{R-iu}
$$

By the Fubini theorem for stochastic integrals (see [38, page 208]), we can interchange the two integrals in (66) provided that

$$
E \int_0^t \mu(du) |H^u_s|^2 ds < \infty
$$

Under the assumption (39) it is easy to check that

$$
\frac{\Phi_{T-s}(-u - iR)}{|\Phi_{T-t}(-u - iR)|} \leq C
$$

for all $s \leq t \leq T$ for some constant $C > 0$ which does not depend on $s$ and $t$. To prove (67) it is then sufficient to check

$$
E \int_0^t \int_\mathbb{R} |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)||\hat{S}_s^{2(R-iu)}|^2 (R - iu)^2 dudt < \infty
$$

which holds because

$$
|\Phi_{T-t}(-u - iR)| \leq Ce^{-(T-t)\frac{2\mu^2}{2}}
$$

Therefore, the first term on the right-hand side of (65) is equal to

$$
\int_0^t \tilde{\sigma}_s dW_s, \quad \tilde{\sigma}_s = \frac{\sigma}{2\pi} \int_\mathbb{R} du \hat{g}(u + iR)\Phi_{T-s}(-u - iR)(R - iu) \hat{S}_s^{R-iu}.
$$

This also shows that $\tilde{\sigma}_s = \sigma S_s \frac{\partial P(s_s_s)}{\partial s}$. 47
Let us now turn to the second term in the right-hand side of (65). Here we need to apply the Fubini theorem for stochastic integrals with respect to a compensated Poisson random measure [3, Theorem 5] and the applicability condition boils down to

\[ E \int_0^t \int_\mathbb{R} |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)||\hat{S}_s^{2R-iu}|^2 \int_\mathbb{R} |e^{(R-iu)z} - 1|^2 \nu(dz)dudt < \infty \]

If \( \sigma > 0 \), this is once again guaranteed by (68), and when \( \sigma = 0 \),

\[ \int_\mathbb{R} |e^{(R-iu)z} - 1|^2 \nu(dz) = \psi(-2iR) - 2\Re\psi(-u - iR). \]

Since, for some \( C < \infty \),

\[ |\Re\psi(-u - iR)\Phi_{T-t}(-u - iR)| = C e^{\frac{T-t}{2}\Re\psi(-u - iR)}, \]

the integrability condition is satisfied and we conclude that

\[ \hat{P}(t, S_t) - \hat{P}(0, S_0) = \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_\mathbb{R} \tilde{\gamma}_s(z) J_X(ds \times dz) \quad (70) \]

for all \( t < T \) with \( \tilde{\sigma} \) as above and

\[ \tilde{\gamma}_s(z) = \frac{1}{2\pi} \int_\mathbb{R} du \hat{g}(u + iR)\Phi_{T-s}(-u - iR)\hat{S}_s^{R-iu}(e^{(R-iu)z} - 1) \quad (71) \]

The optimal (risk-minimizing) hedge is obtained from formula (59):

\[ \hat{\phi}_t = \frac{\sigma \hat{S}_t \tilde{\sigma}_t + \hat{S}_t \int_\mathbb{R} \nu(dz)(e^z - 1)\tilde{\gamma}_t(z)}{\hat{S}_t^2(\sigma^2 + \int_\mathbb{R} (e^z - 1)^2 \nu(dz))}. \]

Substituting the expressions for \( \tilde{\sigma} \) and \( \tilde{\gamma} \) in terms of option prices into the above expression, we obtain (63) directly. On the other hand, the Fourier representations (69) and (71) and an application of Fubini’s theorem yield (61).

As a by-product, the martingale representation (70) also yields the expression for the residual risk of a hedging strategy:

\[
E[\epsilon(\phi)^2] = E \left[ \int_0^T dt \int_\mathbb{R} \nu(dz) \left( \hat{P}(t, S_{t-}e^z) - \hat{P}(t, S_{t-}) - \hat{S}_{t-}\phi_t(e^z - 1) \right)^2 \right] 
+ E \left[ \int_0^T \hat{S}_{t-}^2 \left( \phi_t - \frac{\partial P}{\partial S}(t, S_{t-}) \right)^2 \sigma^2 dt \right]. \quad (72)
\]
This allows us to examine whether there are any cases where the hedging error can be reduced to zero, i.e., where one can achieve a perfect hedge for every option and the market is complete. Hedging error is zero if and only if, for almost all $t$, there exists $k \in \mathbb{R}$ with:

$$(\sigma S_t \frac{\partial P}{\partial S}, (P(t, S_t e^z) - P(t, S_t))_{z \in \text{supp } \nu}) = k(\sigma S_t, (S_t(e^z - 1))_{z \in \text{supp } \nu})$$

This is only true in two (trivial) cases:

- The Lévy process $X$ is a Brownian motion with drift: $\nu = 0$ and we retrieve the Black-Scholes delta hedge
  $$\phi_t = \Delta_{BS}(t, S_t) = \frac{\partial P}{\partial S}(t, S_t).$$

- The Lévy process $X$ is a Poisson process with drift: $\sigma = 0$ and there is a single possible jump size: $\nu = \delta_{x_0}(x)$. In this case the hedging error equals
  $$E \left[ \int_0^T dt \left( \hat{P}(t, S_{t-} e^{x_0}) - \hat{P}(t, S_{t-}) - \hat{S}_{t-} \phi_t (e^{x_0} - 1) \right)^2 \right]$$
  so by choosing
  $$\phi_t = \frac{P(t, S_{t-} e^{x_0}) - P(t, S_{t-})}{S_{t-} (e^{x_0} - 1)}$$
  we obtain a replication strategy.

In other cases, the market is incomplete (an explicit counter-example may be constructed using power option with pay-off $H_T = (S_T)^\alpha$).

### 10.2 Quadratic hedging in exponential Lévy models under the historical measure

Throughout this section, to simplify notation, we suppose that the interest rate is equal to zero; the formulas for the general case can be obtained by working with discounted claims. Let $S$ be the price process of the underlying, and suppose that it can be written in the form

$$S_t = S_0 + M_t + \int_0^t \alpha_s d\langle M \rangle_s$$  \hspace{1cm} (73)
for some square integrable martingale $M$ and some predictable process $\alpha$.

If $S$ is an exponential of a Lévy process $X$ with Lévy measure $\nu$ satisfying $\int_{|x|>1} e^{2x} \nu(dx) < \infty$ and diffusion coefficient $\sigma$, which can be written as

$$S_t = S_0 + \int_0^t \gamma S_u du + \int_0^t S_u \sigma dW_u + \int_0^t \int_{\mathbb{R}} S_u - (e^z - 1) \tilde{J}_X(du \times dz),$$  

then the representation (73) holds with

$$M_t = \int_0^t S_u \sigma dW_u + \int_0^t \int_{\mathbb{R}} S_u - (e^z - 1) \tilde{J}_X(du \times dz)$$

$$\langle M \rangle_t = \int_0^t S_u^2 \left( \sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz) \right) du$$

$$\alpha_t = \frac{\gamma}{S_t \left( \sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz) \right)}$$

We then introduce the so-called mean-variance tradeoff process

$$K_t := \int_0^t \alpha_s^2 d\langle M \rangle_s.$$  

In an exponential Lévy model, the mean-variance tradeoff is deterministic:

$$K_t = \frac{\gamma^2 t}{\sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz)}.$$

**Local risk minimization** The locally risk minimizing strategy [20, 45] is a (not necessarily self-financing) trading strategy whose discounted cost process $\hat{C}$ is a martingale orthogonal to $M$. This strategy is optimal in the sense that we eliminate all the risk associated to the underlying with hedging, and the only part of risk that remains in the cost process is the risk which is orthogonal to the fluctuations of the underlying, and hence, cannot be hedged with it. If the market is complete, then all risk is explained by the underlying and the cost process of a locally minimizing strategy becomes constant, that is, the strategy becomes self-financing. As already mentioned, the locally risk minimizing strategy also has the interpretation of minimizing the residual risk (55) with respect to suitably defined small perturbations of the strategy [45]. Since the cost process is nonconstant, the locally risk
minimizing strategy is not a self-financing strategy in general however since $C$ is a martingale with mean zero this strategy is self-financing on average.

The locally risk minimizing strategy is closely related to an extension of the Kunita-Watanabe decomposition to semimartingale setting, known as the Föllmer-Schweizer decomposition [20, 44, 47, 48].

**Definition 12.** Let $H \in L^2(\mathbb{P})$ be a contingent claim. A sum $H = H_0 + \int_0^T \phi^H_u dS_u + L^H_T$ is called the Föllmer-Schweizer decomposition of $H$ if $H_0$ is $\mathcal{F}_0$-measurable, $\phi^H$ is an admissible trading strategy and $L^H$ is a square integrable martingale with $L^H_0 = 0$, orthogonal to $M$.

Given a Föllmer-Schweizer decomposition for the claim $H$, the locally risk minimizing strategy for $H$ can be constructed by taking $\phi_t = \phi^H_t$ for all $t$, and choosing $\phi^0$ such that the cost process is $C_t = H_0 + L^H_t$ for all $t$, which amounts to $\phi^0_t = H_0 + L^H_t - \phi^H_t S_t - \int_0^t \phi^H_u dS_u$.

**Relationship with the minimal martingale measure** Define a process $Z$ via $Z := \mathbb{E}(\cdot \mid \mathcal{F}_t)$ and assume that $Z$ is a strictly positive square integrable martingale. Then we can define a new measure $\mathbb{Q}^M$ by $d\mathbb{Q}^M \mid \mathcal{F}_t = Z_t$.

By Girsanov-Meyer theorem ([38], theorem 36 in chapter 3), we have that (i) $\mathbb{Q}^M$ is a martingale measure, that is, $S$ becomes a martingale under $\mathbb{Q}$ and (ii) any square integrable martingale which is orthogonal to $M$ under $\mathbb{P}$ remains a martingale under $\mathbb{Q}$ (although it may no longer be orthogonal to $M$). This measure is known as the minimal martingale measure [1, 47].

The minimal martingale measure allows to express the Föllmer-Schweizer decomposition in a more explicit form. First, compute the process $L^H$:

$$L^H_t = E^{\mathbb{Q}^M}[L^H_T \mid \mathcal{F}_t] = E^{\mathbb{Q}^M}[H \mid \mathcal{F}_t] - H_0 - \int_0^t \phi^H_u dS_u.$$

Since $L^H_0 = 0$, the initial capital for the Föllmer-Schweizer strategy is $H_0 = E^{\mathbb{Q}^M}[H]$. Let $H^M_t := E^{\mathbb{Q}^M}[H \mid \mathcal{F}_t]$. The orthogonality condition under $\mathbb{P}$ then yields an analogue of formula (59):

$$\phi^H_t = \frac{d\mathbb{Q}^M}{d\mathbb{P}}([H \mid \mathcal{F}_t], S)_{\mathcal{F}_t}.$$

In models with jumps, the minimal martingale measure does not always exist as a probability measure (but may turn out to be a signed measure).
In an exponential-Lévy model of the form (74), the density of the minimal martingale measure simplifies to \( Z = \mathcal{E}(U) \) with

\[
U_t = -\frac{\gamma}{\sigma^2 + \int_{\mathbb{R}}(e^z - 1)^2 \nu(dz)} \left\{ \sigma W_t + \int_0^t \int_{\mathbb{R}} (e^z - 1) \tilde{J}(ds \times dz) \right\}.
\]

By Proposition 9, this yields a probability change if

\[
\frac{\gamma(e^x - 1)}{\sigma^2 + \int_{\mathbb{R}}(e^z - 1)^2 \nu(dz)} < 1 \quad \forall x \in \text{supp } \nu,
\]

which imposes a strong restriction on the drift parameter \( \gamma \). If this condition is not satisfied, the Föllmer-Schweizer decomposition may still exist, but the interpretation using the minimal martingale measure is no longer valid, and the initial capital may turn out to be negative.

The existence of a Föllmer-Schweizer decomposition has been studied by many authors (see for example [1, 47]), and in particular it was shown that the decomposition always exists in the case of exponential Lévy models. For these models, explicit formulas for the coefficients of this decomposition for European options are given in [27]:

**Proposition 14** (Föllmer-Schweizer decomposition for European options in exponential Lévy models [27]).

- **Case of exponential pay-offs.** Let \( z \in \mathbb{C} \) with \( S_t^z \in L^2(\mathbb{P}) \). Then the contingent claim \( H(z) = S_T^z \) admits a Föllmer-Schweizer decomposition with

  \[
  \phi(z)_t = \Upsilon(z)e^{\eta(z)(T-t)}S_t^{z-1}
  \]

  \[
  L(z)_t = e^{\eta(z)(T-t)}S_t^z - e^{\eta(z)T}S_0^z - \int_0^t \phi(z)_udS_u,
  \]

  where the coefficients \( \Upsilon \) and \( \eta \) are given by

  \[
  \Upsilon(z) = \frac{k(z + 1) - k(z) - k(1)}{k(2) - 2k(1)}, \quad \eta(z) = k(z) - k(1)\Upsilon(z),
  \]

  and \( k(z) = \log E[e^{zX_1}] \) is the Laplace exponent of \( X \).
Case of arbitrary payoffs. Let the option payoff be \( H = f(S_T) \) with \( f \) of the form
\[
f(s) = \int s^2 \Pi(dz)
\]
for some finite complex measure \( \Pi \) on a strip \( \{ z \in \mathbb{C} : R' \leq \Re z \leq R \} \), where \( R', R \in \mathbb{R} \) are chosen such that \( E[e^{2RX_1}] < \infty \) and \( E[e^{2R'X_1}] < \infty \). Then \( H \) admits a Föllmer-Schweizer decomposition with coefficients
\[
\phi_t^H = \int \phi(z)_t \Pi(dz)
\]
\[
L_t^H = \int L(z)_t \Pi(dz)
\]

Variance-optimal hedging
An alternative approach is to choose a self-financing strategy \( \phi \) and the initial capital \( V_0 \) such as to minimize
\[
E^P \left[ (V_0 + G_T(\phi) - H)^2 \right].
\]
under the statistical measure \( \mathbb{P} \). This approach, known as mean-variance hedging or variance optimal hedging, is described in many papers including [7, 37, 48, 6, 27, 10]. The general results concerning existence of optimal strategies are given in [10]. Schweizer [46] studies the case where the mean-variance tradeoff process \( K \) is deterministic and shows that in this case, the variance-optimal hedging strategy is also linked to the Föllmer-Schweizer decomposition. Hubalek et al. [27] exploit these results to derive explicit formulas for the hedging strategy in the case of Lévy processes. The following proposition uses the notation of Proposition 14.

**Proposition 15** (Mean variance hedging in exponential Lévy models [27]).
Let the contingent claim \( H \) be as in the second part of Proposition 14. Then the variance optimal initial capital and the variance optimal hedging strategy are given by
\[
V_0 = H_0
\]
\[
\phi_t = \phi_t^H + \frac{\lambda}{\hat{S}_t} (H_t - V_0 - G_t(\phi)), \quad (75)
\]
where $\lambda = \frac{\kappa(1)}{\kappa(2) - 2\kappa(1)}$ and

$$H_t = \int S_t^z e^{\eta(z)(T-t)} \Pi(dz).$$

In the case of exponential Lévy models, and in all models with deterministic mean-variance tradeoff, the variance optimal initial wealth is therefore equal to the initial value of the locally risk minimizing strategy. This allows to interpret the above result as a “stochastic target” approach to hedging, where the locally risk minimizing portfolio $H_t$ plays the role of a “stochastic target” which we would like to follow because it allows to approach the option’s pay-off with the least fluctuations. Since the locally risk-minimizing strategy is not self-financing, if we try to follow it with a self-financing strategy, our portfolio may deviate from the locally risk minimizing portfolio upwards or downwards. The strategy (75) measures this deviation at each date and tries to compensate it by investing more or less in the stock, depending on the sign of the expected return ($\lambda$ is the expected excess return divided by the square of the volatility).

References


