1. Organisational and methodological issues

- *The aim of the course*: The course "Complex Analysis" is aimed at mastering basic concepts and tools of modern complex analysis in one variable from both of the analytic and geometric viewpoints as well as understanding the role these concepts play in mathematics and sciences.

- *Objectives of the course*: After completing the course, the students are acquainted with the following notions:
  - complex numbers and their geometric description,
  - differentiability, Cauchy-Riemann equations, holomorphic function,
  - the Cauchy integral theorem and formula, basic properties of holomorphic functions (the argument principle, the maximum principle, derivatives estimates, power series expansions, the removable singularity theorem, the Liouville theorem),
  - residue calculus,
  - convergence of holomorphic functions,
  - conformal mappings, the Riemann mapping theorem,
  - multivalued analytic functions,
  - harmonic functions and the Poisson formula, potentials in fluid dynamics and electrostatics.

  and also expected to be able to
  - understand the mathematical articles and talks in various branches of sciences like analysis, geometry and mathematical physics, where complex analysis is used as tools.
  - apply complex analysis to many problems in sciences, in particular, mathematical physics.
  - perform elementary computations in complex variables such as finding derivatives, integrals or residue calculus.
  - use methods in complex analysis to solve problems stated in terms of real analysis, which includes the integration of real functions by means of residue calculus.
  - work with power series and Laurent series expansions,
  - use these expansions to compute integrals, solve differential equations, etc.
  - analyse the behaviour of conformal mappings, in particular of fractional linear transformations and find explicit expressions for the Riemann mappings of some simple shapes.
  - apply conformal mappings to problems in fluid dynamics and electrostatics.

- *Original methodological approaches used in the course*: Complex analysis appears ubiquitous in science. For example, properties of special functions used in many applied areas are based on complex analysis. More conceptually, (two-dimensional) harmonic functions are important objects in physics. On the other hand, complex analysis is one of the most elegant branch of mathematics.

  We shall present complex analysis as a wonderful mixture of analysis and geometry and, at the same time, as a powerful tool in mathematical physics.

  In general, the course put more emphasis on concrete examples rather than on details of rigorous proofs of general theorems. The students shall thus see how "good (general) mathematics" arises out of various "good examples". Vice versa, from many examples of applications of theorems to concrete problems, the students shall understand how general theorems are powerful.

In order to make the exposition as clear as possible, sometimes details will be omitted and the students are requested to fill them in exercises. If necessary, some redundant conditions, which make theorems less general but drastically simplify the proofs, will be assumed.

- *The place of the course in the system of innovative qualifications that are formed in the course of study*: The course is offered to the first year Master of Science students in Mathematics. This is one of international M. Sc. programmes conducted by the department of Mathematics in English. The course is joint with the "Math in Moscow" programme, a student internship programme of the NRU HSE and the Independent University of Moscow, which attracts mostly North-American students (whose tuition is often covered by NSF grants and NCERC grants). We expect to have students with very different background in mathematics, to whom we shall show how various branches not only in mathematics but also in broader sciences are related with each other.

2. The contents of the course

What makes this course unique. (description of scientific and methodological features, comparison with similar courses offered by the NRU HSE and other universities in Russia and worldwide.)

The course "Complex Analysis" is one of the traditional courses offered by the "Math in Moscow" programme (a student internship programme of the Independent University of Moscow and the NRU HSE for students from all over the world). In 2012, our department converted the Master of Science programme in Mathematics into an international programme, conducted in English, and this course is offered as a course of students' choice in the cycle "Analysis", which is one of the three major cycles (along with "Algebra" and "Topology"). On the one hand, the organisation of this course follows the main highlights developed by the Math in Moscow programme. On the other hand, the scientific contents and the method of presentation correspond to the author's teaching philosophy and are different from those employed by different instructors of this course in recent years. The author of the present programme has created an original syllabus, and will implement it for the first time in the autumnal semester in 2014.

Introductory complex analysis in one variable is being taught as a standard undergraduate university course in most departments of Mathematics, Physics, and Engineering, although emphasis is put on different points, according to their philosophy. More advanced graduate-level courses are much less common in Russian universities. On the other hand, most mathematical departments in the USA offer graduate-level courses of complex analysis in one variable on a regular basis. The contents of these courses in the USA is often a part of the Ph.D. qualifying exams. Complex analysis courses are often bundled with real analysis courses, which defines some specific approach to their syllabuses.

The curriculum of the Master of Science programme in Mathematics and of the Math in Moscow programme shows the Complex Analysis course as one of the basic Master's-level courses. Therefore, some minimal requirements to the contents have been developed. They take into account that the students attending this course may have different backgrounds, which requires that course should be as self-contained as possible to compensate this difference. However, to make it possible to cover a more advanced material beyond the undergraduate level, we require a very active participation to the audiences; for example, the students should recover the proofs of some standard facts using textbooks, possibly consulting the instructor.

A more specific feature of this course programme is that it is based on an interdisciplinary approach. Complex "analysis" is viewed not only as a self-contained section of analysis but also and more importantly as the language that is useful in the study of many other subjects in mathematics and physics. We often start with examples, motivate a certain concept of a tool, then introduce it in the simplest possible way, after which we either proceed with a general discussion or just hint at how a general theory may develop. Methodological originality of this course is that we put emphasis on interplay of general theories and concrete applications.

Plan of the course. Allocation of classroom hours among the topics and types of assignments.

No	Topic	weeks	total hours	lectures	discussion/ workshops
1	complex numbers and their geometric description	1	10	2	2
2	differentiability, holomorphicity and integrals	2	20	4	4
3	basic properties of holomorphic functions	4	40	8	8
4	residue calculus	2	20	4	4
5	conformal mappings, the Riemann mapping theorem	3	30	6	6
6	harmonic functions and their applications to fluid dynamics and electrostatics	3	30	6	6
7	multivalued analytic functions	1	10	2	2
Total		16	160	32	32

3. The structure of the syllabus

Details of the syllabus, especially those of the first few sections are subject to change according to students' knowledge, since they might have quite different backgrounds. During the discussion/workshop sessions explicit examples on the subjects explained in the lectures shall be discussed.

The main reference for *all the topics* is

Lars V. Ahlfors, Complex Analysis, Third edition, McGraw Hill (1979).

Additional references are listed at the end of this section, but they are nothing more than auxiliary references. (One exception is Ref. [Ablowitz-Fokas] for applications.)

The list of quizzes, tests, homework assignments is in Section 5.

Complex numbers and their geometric description: Reviews on basic facts about complex numbers; definition and basic operations (arithmetic operations, complex conjugate, absolute value), complex plane, geometric descriptions of basic operations, polar coordinates.

Differentiability, holomorphicity and integrals: Limits of sequences of complex numbers and series. Power series and radii of convergence. Differentiation and holomorphicity; here we define holomorphic (analytic) functions as differentiable functions *with continuous derivatives* (i.e., C^1-class). This additional condition drastically reduce the burden of delicate proofs later. Nevertheless we emphasise in the course that this definition is *equivalent* to the definition without the C^1-condition. Cauchy-Riemann equations. Wirtinger derivatives. Derivatives of power series. Basic formulae for derivatives (arithmetic operations, chain rule, inverse function).

Definition of contour integrals. Basic properties of integrals; well-definedness with respect to parametrisation, concatenation of contours, estimate of the absolute value of an integral by the integral of the absolute value.

Integrals and derivatives of limit functions and series.

Basic properties of holomorphic functions: Review of Green's theorem and its complex version. Cauchy's integral theorem and its use; change of contours. Here we mention that this theorem can be proved without the C^1 condition, which guarantees the validity of Green's formula. Cauchy's integral formula. Morera's theorem. Weierstrass' theorem on convergence of holomorphic functions. Taylor expansion. Identity theorem. Liouville's theorem.

Isolated singularity and Laurent expansion. Removable singularities/poles/essential singularities. Meromorphic functions.

"Infinity" as a point on the Riemann sphere. Stereographic projection.

Residue calculus: Definition of residue, including the residue at infinity. We mention about the 1-form on the complex plane. The residue theorem. Residue calculus and its application to real integrals.

Numbers of zeros and poles. Argument principle and its applications; open mapping, inverse function theorem, maximum principle.

Conformal mappings, the Riemann mapping theorem: Holomorphic functions as mappings of domains. Conformal mapping. Fractional linear mappings. Circle-to-circle correspondence. Riemann's mapping theorem.

Harmonic functions and their applications to fluid dynamics and electrostatics: Definition of n-dimensional harmonic functions. Examples for $n = 2$; real and imaginary parts of holomorphic functions, velocity potential in hydrodynamics, electrostatic potential (cf. Ref. [Ablowitz-Fokas]). Physical explanation of Riemann's mapping theorem. Poisson integral.

Multivalued analytic functions: Examples of analytic continuation; logarithm, power functions. Direct continuation. Analytic continuation along curves. Monodromy theorem and monodromy. Reflection principle.

Additional references.

- J. Bak, D.J. Newman, Complex Analysis, 2nd. Ed., Springer (1997).
- T.W. Gamelin, Complex Analysis, Springer (2001).
- M.J. Ablowitz, A.S. Fokas, Complex Variables: Introduction and Applications, 2nd Ed. Cambridge University Press (2003).

Possible themes of course works.

- Differential equations in the complex domain (method of Frobenius, monodromy matrices, Riemann-Hilbert problems).
- Löwner equations (the Bieberbach conjecture, radial/chordal Löwner equations).
- Elliptic functions (Liouville's theorems, Weierstrass $\wp$-function, Riemann's theta function, elliptic integrals).
- Riemann surfaces (Riemann surfaces of algebraic functions, complex manifold, compact Riemann surfaces).

4. Grading policy

We will use the following means of evaluation: homework assignments, quizzes, the written midterm test, the written final exam.

Quiz: 1–2 questions for 5 minutes,
Homework: 3–6 problems for one week,
Midterm and Final: 8 problems for 3 hours.

The intermediate grade is computed at the end of the first module by the formula

$$0.5 \times (\text{the average homework grade in the first module}) + 0.5 \times (\text{the midterm grade}).$$

The final grade is computed at the end of the term by the formula

$$0.3 \times (\text{the cumulated grade}) + 0.3 \times (\text{the intermediate grade}) + 0.4 \times (\text{the final exam grade}),$$

where the cumulated grade is the average homework grade in the second module. Quiz results and classroom participation are taken into account as bonus points when computing the cumulated grade.

5. Problems for quizzes, tests, homework assignments

The following list shows samples of problems, from which homework assignments and questions for quizzes may be taken. It can also be used by students for preparation of the midterm and final exams.

5.1. Complex numbers and their geometric description.

(1) Find the polar form of the following complex numbers: (i) $\dfrac{1-i\sqrt{3}}{2}$, (ii) $(2+2i)^3$, (iii) $(1+i)^n + (1-i)^n$.
(2) Let z have the polar form $re^{i\theta}$. Express the real and imaginary parts of $1/(1-z)$ in terms of r and θ.
(3) Find all the n-th roots of 1.
(4) Prove that it is necessary and sufficient for three complex numbers α, β and γ to form a equilateral triangle in the complex plane that they satisfy

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha.$$

(5) Show that circles and lines in the complex plane are expressed by the equation

$$az\bar{z} + \bar{\beta}z + \beta\bar{z} + c = 0$$

with appropriate choice of real parameters a, c and a complex parameter β. Conversely, what set does the above equation define?
(6) Find the sums $1 + \cos\varphi + \cos 2\varphi + \cdots + \cos n\varphi$ and $\sin\varphi + \sin 2\varphi + \cdots + \sin n\varphi$. (Use the geometric series of complex numbers.)

5.2. Differentiability, holomorphicity and integrals.

(1) When do the following series converge?

$$\sum_{n=0}^{\infty} r^n \cos n\varphi, \qquad \sum_{n=1}^{\infty} r^n \sin n\varphi.$$

Find their sums.

(2) Find the radii of convergence of the following series:

(i) $\sum_{n=0}^{\infty}(az)^n$ ($a \in \mathbb{C}$, $a \neq 0$);

(ii) $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$;

(iii) $\cos z := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$;

(iv) $\sin z := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$;

(v) $\log(1+z) := \sum_{n=1}^{\infty}(-1)^{n-1} \frac{z^n}{n}$;

(vi) $F(\alpha, \beta, \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n$, where $(x)_n := x(x+1)\cdots(x+n-1)$ and $\gamma \neq -1, -2, \ldots$. (This series is called the *hypergeometric series.*)

(3) Show that the following the radii of convergence of the following three series are 1:

$$f_0(z) = \sum_{n=1}^{\infty} z^n, \qquad f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad f_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

(4) * Where on the circle $|z| = 1$ do the above series f_0, f_1 and f_2 converge?

(5) Show the formula $\overline{\left(\frac{\partial f}{\partial z}(z)\right)} = \frac{\partial}{\partial \bar{z}}\left(\overline{f(z)}\right)$.

(6) Show that $\overline{f(\bar{z})}$ is holomorphic if $f(z)$ is.

(7) Prove the formulae $(e^z)' = e^z$, $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$ and $(\log(1+z))' = \frac{1}{1+z}$, where all the functions are defined by power series as above.

(8) Find all the solution of the differential equation

$$(1+z)\frac{dy}{dz} = \alpha y$$

which has the form of the convergent series $y(z) = \sum_{n=0}^{\infty} c_n z^n$.

(9) Assume that $\alpha, \beta, \gamma \in \mathbb{C}$ and that $\gamma \notin \mathbb{Z}$. Prove that the *hypergeometric differential equation*

$$z(1-z)\frac{d^2y}{dz^2} + \big(\gamma - (\alpha+\beta+1)z\big)\frac{dy}{dz} - \alpha\beta y = 0$$

has a unique solution of the form $y = \sum_{n=0}^{\infty} c_n z^n$, $c_0 = 1$, which is given by $y(z) = F(\alpha, \beta, \gamma; z)$.

(10) Which of the following functions is holomorphic?: (i) $x + iy^2$, (ii) $(x^2 - y^2 - y) + i(2xy + x)$.

(11) Which of the following functions are real parts of holomorphic functions?: (i) $u(x,y) = x + y$ (on $\mathbb{C}$), (ii) $u(x,y) = x - y$ (on $\mathbb{C}$), (iii) $u(x,y) = xy$ (on $\mathbb{C}$), (iv) $u(x,y) = y/x$ (on

the upper half plane $x > 0$), (v) $u(x,y) = \log((x-2)^2 + y^2)$ (on the unit disk $|z| < 1$), (vi) $u(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ (on $\mathbb{C} \setminus \{0\}$).

(12) Show that a holomorphic function $f(z)$ on a domain D is constant if (i) $f(z) \in \mathbb{R}$ or (ii) $|f(z)|$ is constant.

(13) Compute $\displaystyle\int_\gamma x\,dz$, where γ is a directed line segment from 0 to $a + ib$.

(14) Show that $\displaystyle\int_\gamma f'(z)\,dz = f(\beta) - f(\alpha)$, where γ is a curve from α to β and $f(z)$ is holomorphic in the neighbourhood of γ. (Hint: what is $\dfrac{d}{dt}(f(z(t)))$?)

(15) Compute $\displaystyle\int_{|z|=r} x\,dz$ in two ways; (i) by use of a parameter, (ii) by rewriting $x = (z+\bar{z})/2 = \dfrac{1}{2}\left(z + \dfrac{r^2}{z}\right)$ on the circle and using the previous question.

(16) Compute the contour integral $\displaystyle\oint_C \bar{z}\,dz$ where the contour C is (i) the boundary of the rectangle with vertices 0, a, $a+ib$, ib $(a, b > 0)$; (ii) the circle $|z| = R$. Both contours are oriented counterclockwise.

(17) Assume that $f(z)$ is holomorphic in the neighbourhood of a curve γ. Show that $\displaystyle\int_\gamma \overline{f(z)}\, f'(z)\,dz \in i\mathbb{R}$.

5.3. **Basic properties of holomorphic functions.**

(1) (i) Let z depend on θ as $z(\theta) = a\cos\theta + ib\sin\theta$, $(a, b > 0)$. Express $\operatorname{Im}\left(\dfrac{1}{z}\dfrac{dz}{d\theta}\right)$ in terms of θ.

(ii) Compute the integral

$$I = \int_0^\pi \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}.$$

(2) (i) Let $\varphi(z,t)$ be a continuous function of $(z,t) \in D \times [a,b]$, where $D \subset \mathbb{C}$ is a domain and $a, b \in \mathbb{R}$. Assume that $\varphi(z,t)$ is holomorphic in z for any $t \in [a,b]$. Prove that $F(z) := \displaystyle\int_a^b \varphi(z,t)\,dt$ is holomorphic in z and

$$F'(z) = \int_a^b \frac{\partial\varphi}{\partial t}(z,t)\,dt.$$

(ii) Put $\varphi(z,t) = f(t)/(t-z)$ for a continuous function $f(t)$ on $[a,b]$ and $D = \mathbb{C}\setminus[a,b]$. Deforming the integration contours, show that limits $\lim\limits_{\varepsilon\downarrow 0} F(x \pm i\varepsilon)$ exists for $a < x < b$ and that

$$\lim_{\varepsilon\downarrow 0}\bigl(F(x+i\varepsilon) - F(x-i\varepsilon)\bigr) = 2\pi i f(x).$$

(3) (i) Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have positive radii of convergence, ρ_1 and ρ_2. Find the coefficients of the Taylor series of $f(z)\, g(z)$. (ii) Find the Taylor coefficients of $\dfrac{z}{e^z - 1}$ up to the order z^4.

(4) Show that the series

$$f(z) := \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

converges absolutely and uniformly on any compact set in $\mathbb{C} \setminus \mathbb{Z}$ and gives a meromorphic function. What are poles?

(5) Show that an entire function $f(z)$ with two periods ω_1 and ω_2, $f(z + \omega_i) = f(z)$ $(i = 1, 2)$, is constant if ω_1 and ω_2 span a non-degenerate parallelogram.

(6) Find all singularities and their principal parts of the following functions: (i) $z^m/(z^n - 1)$, (ii) $\cos z / z^3 \sin z$ (iii) $\pi \cot \pi z$, (iv) $z/(e^z - 1)$, (v) $1/\sin^2 z$.

5.4. **Residue calculus.**

(1) Find all the poles on $\mathbb{P}$ and their residues of (i) $\dfrac{e^{iz}}{\sin z}$, (ii) polynomial $P(z) = a_N z^N + a_{N-1} z^{N-1} + \ldots + a_0$ $(a_N \neq 0)$, (ii) $\dfrac{z^4}{z^2 - 3z + 2}$.

(2) Compute

$$\int_{|z|=1} e^z z^{-n}\, dz, \qquad \int_{C_{a,b}} z^m (1-z)^n\, dz, \qquad \int_{C_{a,b}} \frac{2z}{z^2 - 1} dz,$$

where $m, n \in \mathbb{Z}$, $C_{a,b} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a, b > 0$, $a \neq 1$.

(3) Compute

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx, \qquad \int_0^{\infty} \frac{dx}{1 + x^6}, \qquad \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{a^2 + x^2} dx, \qquad \int_{-\infty}^{\infty} \frac{x \sin \alpha x}{a^2 + x^2} dx.$$

Here $\alpha \in \mathbb{R}$, $a > 0$.

(4) Compute

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}, \qquad \int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2}, \qquad \int_0^{2\pi} \frac{\cos m\theta}{1 - 2a\cos\theta + a^2} d\theta.$$

Here $0 < b < a$, $m \in \mathbb{Z}$.

(5) Show Rouché's theorem: Let D be a domain whose boundary ∂D consists of piecewise smooth closed curves, $f(z)$ and $g(z)$ be holomorphic functions in the neighbourhood of $\bar{D}$ and $|f(z)| > |g(z)|$ on ∂D. Then $f(z)$ and $g(z)$ have the same number of zeros in D.

(6) How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? (Use Rouché's theorem.)

(7) Let ω_1 and ω_2 be two complex numbers linearly independent over $\mathbb{R}$, which means that they span a non-degenerate parallelogram. We call a meromorphic function $f(z)$ on $\mathbb{C}$ an *elliptic function* with periods ω_1 and ω_2, when it satisfies $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for any $z \in \mathbb{C}$. The parallelogram P_a $(a \in \mathbb{C})$ spanned by a, $a+\omega_1$, $a+\omega_2$ and $a+\omega_1+\omega_2$ is called a period parallelogram. Show that

(i) the set of elliptic functions with periods (ω_1, ω_2) forms a *differential field*, i.e., it is a field in algebraic sense and closed under differentiation.

(ii) the number of zeros and the number of poles of a non-constant elliptic function in a period parallelogram P_a are the same, if there are neither zeros nor poles on its boundary ∂P_a. Here numbers of zeros and poles are counted with multiplicities. (A multiple zero with multiplicity k is regarded as k zeros and so on.)

(iii) Let $a_1, \dots, a_N$ be zeros of a non-constant elliptic function and $b_1, \dots, b_N$ be its poles. Then there exists integers n_1 and n_2 such that $a_1 + \cdots + a_N - b_1 - \cdots - b_N = n_1\omega_1 + n_2\omega_2$. (Use the generalised argument principle to $z\, f'(z)/f(z)$.)

(8) Prove *Schwarz' lemma*: if a holomorphic function $f(z)$ on a disk $U = \{|z| < R\}$ satisfies $f(0) = 0$, $|f(z)| \leqq M$ $(z \in U)$, then $|f(z)| \leqq \dfrac{M|z|}{R}$. Moreover, if the equality holds at any point $z_0 \neq 0$ or $|f'(0)| = \dfrac{M}{R}$, then $f(z) = \dfrac{Me^{i\alpha}}{R} z$ for some $\alpha \in \mathbb{R}$.

(9) Prove *Hadamard's three-circle theorem*: Assume that $f(z)$ is holomorphic on the domain $D = \{z \mid r_1 < |z| < r_2\}$ and continuous on its closure. Put $M(r) := \max_{|z|=r} |f(z)|$ and $\alpha := \dfrac{\log(r_2/r)}{\log(r_2/r_1)}$. Show

$$M(r) \leqq M(r_1)^\alpha M(r_2)^{1-\alpha}.$$

When does the equality holds? (Hint: apply the maximum principle to a linear combination of $\log|f(z)|$ and $\log|z|$.)

5.5. **Conformal mappings, the Riemann mapping theorem.**

(1) Find conformal mappings which map the unit disk $\{|z| < 1\}$ bijectively to the following domains: (i) the upper half plane $\mathbb{H} := \{w \mid \operatorname{Im} w > 0\}$, (ii) $D_a := \mathbb{H} \setminus \{ix \mid x \in [0, a]\}$, (iii) $D_{\text{slit}} := \mathbb{C} \setminus (-\infty, -1/4]$.

(2) What domain is mapped to the unit disk $\{|w| < 1\}$ by the following holomorphic function?

$$w = \frac{e^{\pi i z/2} - 1}{e^{\pi i z/2} + 1}.$$

(3) Let a and b are positive real numbers and $0 < a < b$. What is the image of a circle $|z| = b$ by the conformal mapping (the *Joukowski* or *Joukowsky transformation*)

$$f_a(z) := z + \frac{a^2}{z}.$$

What is the image of $\{|z| > a\}$ by f?

(4) Show that the upper half disk $D = \{z \mid \operatorname{Im} z > 0,\ |z| < 1\}$ is mapped to the unit disk $\{|w| < 1\}$ by the conformal mapping $w = \dfrac{(z-\alpha)(\alpha z - 1)}{(\bar{\alpha} z - 1)(z - \bar{\alpha})}$, where $\alpha \in D$.

(5) Show that there is no bijective conformal mapping from a simply connected domain $D \subsetneq \mathbb{C}$ to $\mathbb{C}$.

(6) Prove that any fractional linear transformation is a composite of (i) $z \mapsto z + h$, $z \mapsto \alpha z$ and $z \mapsto 1/z$ $(h, \alpha \in \mathbb{C},\ \alpha \neq 0)$, (ii) even number of inversions with respect to cicles (or lines).

(7) Show that a fractional linear transformation maps circles and lines to circles or lines.

(8) Show that a fractional linear transformation T preserves

(i) the *cross ration* (the *unharmonic ratio*) of four numbers z_1, z_2, z_3 and z_4:

$$(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)},$$

$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$,

(ii) the *Schwarzian derivative* of a holomorphic function $f(z)$:

$$\{f(z), z\} := \frac{d^2}{dz^2} \log f' - \frac{1}{2}\big(\frac{d}{dz} \log f'\big)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

$\{T(f(z)), z\} = \{f(z), z\}$.

(9) (i) Show that a fractional linear transformation of the form

$$w = e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - 1} \qquad (\theta \in \mathbb{R}, |\alpha| < 1)$$

maps the unit disk $\{|z| < 1\}$ onto the unit disk $\{|w| < 1\}$.

(ii) Prove that any one-to-one conformal mapping from the unit disk to the unit disk is given by a fractional linear transformation of the type (i). (Use Schwarz's lemma.)

(10) (i) Show that a fractional linear transformation of the form

$$w = \frac{az + b}{cz + d} \qquad (a, b, c, d \in \mathbb{R})$$

maps the upper half plane $\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$ onto the upper half plane.

(ii) Prove that any one-to-one conformal mapping from $\mathbb{H}$ to $\mathbb{H}$ is given by a fractional linear transformation of the type (i). (Use the previous problem.)

5.6. Harmonic functions and their applications to fluid dynamics and electrostatics.

(1) (i) Assume that $f(z)$ is a holomorphic function on a domain D, which does not vanish: $f(z) \neq 0$ on D. Show that $\log |f(z)|$ is a harmonic function. (ii) More generally, show that if $u(w)$ is a harmonic function and $w = f(z)$ is a holomorphic function, then $u(f(z))$ is harmonic.

(2) Let u and v be harmonic functions on a domain D. Show that if $u = v$ on an open subdomain $D_0 \subset D$, then $u = v$ on the whole domain D.

(3) Let $\{u_m(z)\}_{m=1,2,\dots}$ be a sequence of harmonic functions on a domain D which converges to a function $u(z)$ uniformly on any compact subset of D. Show that $u(z)$ is harmonic on D. (Hint: use the Poisson integral formula.)

(4) Draw the streamlines of the flows with the following complex potential $w = w(z)$. $(a, \alpha, m \in \mathbb{R})$: (i) $w = az^\alpha$, (ii) $w = m \log z$, (iii) $w = im \log z$, (iv) $w = m/z$, (v) $w = az + m \log z$.

(5) Show that streamlines and equipotential lines are mapped to streamlines and equipotential lines by a conformal mapping.

(6) Prove that the complex potential $w(z) = U\left(z + \dfrac{R^2}{z}\right)$ defines a flow exterior to a cylinder of radius R, i.e., show that the circle $|z| = R$ is one of streamlines. What flow does the complex potential $w_\alpha(z) := w(ze^{-i\alpha})$ $(\alpha \in \mathbb{R})$ define?

(7) Combining the complex potential $w_\alpha(z)$ with the Joukowski transformation f_R, study the flow around a segment $[-2R, 2R]$. (In general, one can study the flow aroud a elliptical cylinder.)

5.7. **Multivalued analytic functions.**

(1) Express $\arcsin z$ ($z \in \mathbb{C}$) in terms of log and $\sqrt{\ }$. (Solve $z = \sin w = (e^{iw} - e^{-iw})/2i$ as an equation of w.) What are all possible values of $\arcsin x$ ($x \in \mathbb{R}$, $|x| > 1$)?

(2) (i) Let $\operatorname{Log} z$ be the principal value of $\log z$ ($z \in \mathbb{C} \setminus (-\infty, 0]$). Show

$$\operatorname{Log}\left(\frac{z}{z-1}\right) = \operatorname{Log} z - \operatorname{Log}(z-1)$$

for $z \in \mathbb{C} \setminus (-\infty, 1]$ (ii) Define $f(z)$ for $z \in \mathbb{C} \setminus (-\infty, 1]$ by $f(z) := \operatorname{Log} z - \operatorname{Log}(z-1)$. Find the limits

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i}\big(f(x + i\varepsilon) - f(x - i\varepsilon)\big)$$

for $x \in \mathbb{R} \setminus \{0, 1\}$.

(3) Find the analytic continuation $(P_t(z), D_t)$ of function element $(P_0(z), D_0) = (\operatorname{Log} z, D(1,1))$ along the circle $z = e^{\pi i t}$ ($0 \leqq t \leqq 2$). Here $D(z, r)$ is the open disk with the centre z and the radius r.

(4) Assume that a holomorphic function $f(z)$ on $\{z \mid \operatorname{Re} z > 0\}$ satisfies $f(z+1) = z\, f(z)$. Show that f can be analytically continued to a meromorphic function on $\mathbb{C}$.

(5) Compute (i) $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx$ ($0 < \alpha < 1$; Use the contour (the boundary of $\{z \mid r < |z| < R\}) \cup [r, R]$.), (ii) $\displaystyle\int_0^\infty \frac{\log x}{x^2 + a^2} dx$. (Use the contour integral around the boundary of $\{z \mid r < |z| < R, \operatorname{Im} z > 0\}$.)

(6) According to Riemann's mapping theorem, there exists a conformal mapping φ from the upper half plane H_+ to the triangle $D := \triangle b_0 b_1 b_\infty$. (cf. Fig. 1) Carathéodory's theorem assures that φ extends continuously to the closure: $\varphi : \bar{H}_+ = H_+ \cup \mathbb{R} \cup \{\infty\} \to \bar{D}$. Let us find φ in the following way.

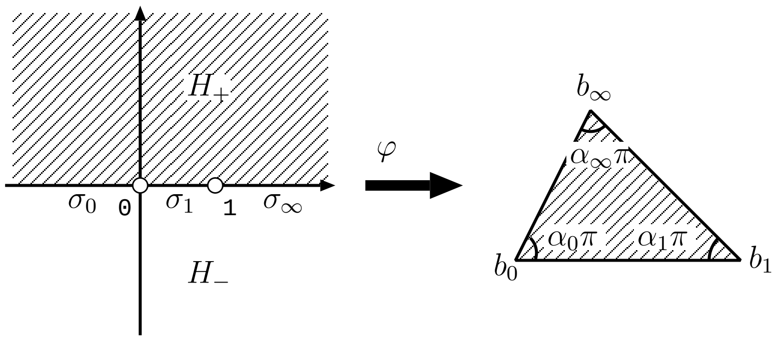

FIGURE 1

(i) Show that we may assume that $\varphi(0) = b_0$, $\varphi(1) = b_1$ and $\varphi(\infty) = \infty$ by combining φ with an appropriate fractional linear transformation.

Let us denote $\sigma_0 := (-\infty, 0)$, $\sigma_1 := (0, 1)$ and $\sigma_\infty := (1, \infty)$ and the lower half plane by H_-.

(ii) Apply the reflection principle to φ and extend it to $\varphi : H_+ \cup \sigma_1 \cup H_- \to \mathbb{C}$. What is the image $\varphi_1(H_-)$?

(iii) Apply the reflection principle to the restriction $\varphi_1|_{H_-}$ of φ_1 to H_- along σ_0. We denote the resulting function by $\varphi_{10} : H_- \cup \sigma_0 \cup H_+ \to \mathbb{C}$. Show that $\varphi_{10}(z) \neq \varpi(z)$ in general for $z \in H_+$, but that $(\log \varphi_{10}'(z))' = (\log \varphi'(z))'$.

Similarly we can extend φ_1 to $H_- \cup \sigma_\infty \cup H_+$ by the reflection principle, which we denote by $\varphi_{1\infty}$. We can define a single-valued holomorphic function $p(z)$ on $\mathbb{C} \smallsetminus \{0, 1\}$ by

$$p(z) := \begin{cases} (\log \varphi_1'(z))', & (z \in H_+ \cup \sigma_1 \cup H_-), \\ (\log \varphi_{10}'(z))', & (z \in H_+ \cup \sigma_0 \cup H_-), \\ (\log \varphi_{1\infty}'(z))', & (z \in H_+ \cup \sigma_\infty \cup H_-), \end{cases}$$

(iv) Show that $\varphi(z)$ has a form

$$\varphi(z) = b_0 + z^{\alpha_0} \tilde{\varphi}_0(z),\ \tilde{\varphi}_0(z) \text{ is holomorphic aroud } z = 0 \text{ and } \tilde{\varphi}_0(0) \neq 0,$$

in the neighbourhood of $z = 0$,

$$\varphi(z) = b_1 + (z-1)^{\alpha_1} \tilde{\varphi}_1(z),\ \tilde{\varphi}_1(z) \text{ is holomorphic around } z = 1 \text{ and } \tilde{\varphi}_1(1) \neq 0,$$

in the neighbourhood of $z = 1$,

$$\varphi(1/\zeta) = b_\infty + \zeta^{\alpha_\infty} \tilde{\varphi}_\infty(\zeta),\ \tilde{\varphi}_\infty(\zeta) \text{ is holomorphic aroud } \zeta = 0 \text{ and } \tilde{\varphi}_\infty(0) \neq 0,$$

in the neighbourhood of $\zeta = 1/z = 0$ (i.e., in the neighbourhood of $z = \infty$).

(v) $p(z)$ is

$$p(z) = \frac{\alpha_0 - 1}{z} + (\text{holomorphic function around } z = 0),$$

in the neighbourhood of $z = 0$,

$$p(z) = \frac{\alpha_1 - 1}{z - 1} + (\text{holomorphic function around } z = 1),$$

in the neighbourhood of $z = 1$,

$$p(z) = \frac{-\alpha_\infty - 1}{z} + z^{-2} \times (\text{holomorphic function around } z = \infty),$$

in the neighbourhood of $z = \infty$, which concludes that

$$p(z) = \frac{\alpha_0 - 1}{z} + \frac{\alpha_1 - 1}{z - 1}.$$

(vi) Show that there exist constants A, B such that

$$\varphi(z) = A \int_0^z t^{\alpha_0 - 1} (t-1)^{\alpha_1 - 1}\, dt + B.$$

(The integration contour lies in H_+.)

(vii) The function $\varphi(z)$ satisfies the hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2} + \bigl(c - (a+b+1)z\bigr)\frac{dF}{dz} - abF = 0.$$

for certain a, b and c. Find one of such triple (a, b, c). (In fact, $a = 0$ or $b = 0$. Here we choose $b = 0$.)

(viii) It is known that any solution of the above equation is a linear combination of the hypergeometric function $F(a, b, c; z)$ and $z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z)$. Expess the above φ in such a form.