**Государственное образовательное учреждение**
**высшего профессионального образования**
**Научно-Исследовательский Университет – Высшая Школа Экономики**

**ФАКУЛЬТЕТ МАТЕМАТИКИ**

РАБОЧАЯ ПРОГРАММА ДИСЦИПЛИНЫ

# Basic Representation Theory

Москва
2014

# I. Syllabus of the course.

**1. Mission of the course**

Our mission is to learn the representation theory of a wide class of objects (algebras, groups, Lie algebras) during one semester at a level suitable for solving problems and understanding relations with other topics in mathematics and mathematical physics.

**2. Aims of the course**

This is an introduction to representation theory. Upon completion of this course student will know enough to understand classical applications (in Galois Theory and Quantum Mechanics) and will be ready to take more advanced courses leading to modern problems and applications of representation theory.

In more detail, the aims are to learn

- common notions and problems of representation theory;
- various instruments for dealing with finite-dimensional representations of finite groups: intertwining operators, characters, Maschke's and Burnside's theorems;
- representations of ring and algebra in a particular case of group ring and algebra.
- representations of symmetric group and related algebraic and combinatorial constructions: Young diagrams and tableau, Young symmetrizers;
- basics of Lie algebras and their representations;
- representations of $sl_2(\mathbb{C})$.

The main goal of this course is to study foundations of representation theory in order to continue education in it. In particular, they can take more advanced courses on representation theory and related subjects as well as be able to read modern books and research papers about representation theory.

**3. Novelty of methods**

The main peculiarity of this course is how we keep a balance between stating statements in a full generality and considering explicit examples. Along the way, we demonstrate not only how to deduce special cases from a general statement, but also how a general theory appears as a uniform way to describe observations from solutions of concrete problems. Despite this approach is slower, it provides a deeper understanding of the subject and make the course more useful to students of various level of initial knowledge and specialization. Moreover, it makes possible to consider (at least briefly) a vast number of objects and results in a limited time, that seems to fulfill aims of an introductory course.

**4. Role of the course**

Representation theory describe and classify actions of an abstract algebraic object (such as group, ring, associative algebra, Lie algebra) on vector spaces. Such objects was considered in the beginning of XX century in Number Theory, namely, in problems related to Galois Theory.

In 30s it appeared that Quantum Mechanics is based on Representation Theory of simplest Lie groups and algebras, in particular, representations of Lie groups $SO_3$ and $SU_2$ describes the structure of atoms and predicts the electron's orbitals. Later Representation theory was applied to various branches of mathematics, first of all to Algebraic and Differential Geometry as well as to Topology. Now the most important questions of Representation Theory are related to Quantum Field Theory and Non-Commutative Geometry, that is a generalization of Algebraic Geometry to the case of non-commutative rings.

The proposed course require only minimal amount of prerequisites, namely, standard courses of algebra and calculus. It grants the students basic knowledge and skills in Representation Theory, that can be useful for researchers and professors in mathematics and mathematical physics, as well as just expand a scope for a specialist in a different area.

## II. Essence of the course

**1. History of the course**

This course is based on courses given by S.Loktev, V.Ivanov, O.Sheinman and E.Smirnov in the Independent University of Moscow. This program is written to make the course suitable both for "Math in Moscow" program in the Independent University of Moscow and for the magister program "Mathematics" (in English) in NRU HSE.

The course consists of three topics that are usually discussed in different courses: representations of finite group, representations of Lie groups and algebras, quantum mechanics with Bohr's model of an atom. The first topic is relatively easy and traditional, it is partially contained in a general course of algebra for mathematics in Moscow State University, Independent University of Moscow and Mathematical department of NRU HSE. The second topic is usually given as a separate advanced course that lasts at least one semester. Despite the connections of the third topic to the first two, it is usually given separately and mostly for prospective specialists in physics.

There are two reasons for the intensity of this course. Students of the magister program need a concise introduction to their specialty in order to obtain a result and write a qualifying paper at the end of the second year. Students of the "Math in Moscow" program usually can not stay in Moscow for more than a semester. Nevertheless, all the students obtain enough knowledge to continue with more advanced course or other ways to study areas of mathematics and mathematical physics that requires representation theory.

**2. Thematic plan**

| subjects | amount of hours | | |
|---|---|---|---|
| | lectures | seminars | personal |
| common notions and problems of representation theory; | 2 | 2 | 2 |
| Complete reducibility of finite group representations. | 2 | 2 | 2 |
| Characters of finite group representations. | 4 | 4 | 4 |
| Number theory around characters | 2 | 2 | 2 |
| Tensor product of representations | 2 | 2 | 2 |
| Induced representation, Frobenius duality | 4 | 4 | 4 |
| Representations of the symmetric group. | 4 | 4 | 4 |
| Lie groups and Lie algebras; representations. | 4 | 4 | 4 |
| Representation of the Lie algebra $\mathrm{sl}_2$. | 2 | 2 | 2 |
| Applications to quantum mechanics. | 2 | 2 | 2 |
| total | 28 | 28 | 28 |

## III. Details of the program

Subject of representation theory. Definition of a representation of an algebra, examples. Subrepresentation and quotient representation. Direct sum of representations, irreducible and indecomposable representation. Homomorphism of representations (intertwining operator), isomorphism of representations. Schur's lemma.

Complete reducibility of finite group representations. Maschke's theorem on complete reducibility of finite group representations. Different proofs using invariant projector and invariant positively defined form. Examples of reducible indecomposable representations for infinite groups as well as over fields with positive characteristic. Uniqueness of decomposition into irreducible representations, multiplicities of irreducible subrepresentations. Intertwining operators from regular representation, decomposition of regular representation. Burnside's formula and its analog for an arbitrary field.

Characters of finite group representations. Definition of character of a complex representation as a function on a group. Character of the dual representation, the direct sum of representations and the tensor product of representations. Scalar product of functions on a group, orthogonality of matrix elements of irreducible representations, orthonormality of characters of irreducible representations. Character based criteria of irreducibility of a representation and when two representations are isomorphic. Formula for multiplicity of an irreducible subrepresentation. Completeness of characters in the space of central functions on a group, equality of number of of complex irreducible representations up to an isomorphism and number of conjugation classes.

Number theory around characters. Algebraic numbers and algebraic integers. Characters as algebraic integers. Frobenius divisibility (dimension of irrep divides order of the group). Burnside's theorem. Vanishing theorem of characters.

**Tensor product of representations.** The structure of a semiring on the set of characters. Particular computations of the decomposition of tensor product for small groups.

**Induced representation.** Definition of induced representation. The Mackey formula for characters of induced representation. Duality theorem between homomorphisms from induced representation and homomorphism to restricted representation. Representations of semidirect product of groups

**Representations of the symmetric group.** Young diagram, relation to conjugation classes of symmetric group. Construction of representations based on Young symmetrizators in the group algebra. Decomposition of restriction from $S_n$ onto $S_{n-1}$ (without proof). Examples.

**Lie groups and Lie algebras; representations.** Topological groups. Proof of Maschke's theorem and character theory for compact groups. Structure of smooth manifold. Definition and examples of Lie group, matrix Lie groups. An abstract definition of Lie algebra, examples. Tangent space of a Lie group in the unit as a Lie algebra. Definition of a representation. Construction of a tangent Lie algebra representation from a Lie group representation. Relation of subrepresentations of a Lie group with subrepresentations of a tangent Lie algebra. Examples: adjoint, coadjoint and tautological representations. Infinite-dimensional examples: representations in functions on homogeneuos space.

**Representations of a Lie algebra $\mathfrak{sl}_2$.** Highest weight representations of $\mathfrak{sl}_2$. Uniqueness of an irreducible representation with a given highest weight, its dimension. Decomposition of the tensor product of irreducible representations. Infinite-dimensional representations, Verma modules. Examples of reducible indecomposable infinite-dimensional representations.

**Applications to quantum mechanics.** Quantum mechanics settings. Bohr's model of an atom. Periodic table form the point of view of representation theory of the group $SO_3$.

## IV. Textbooks and other references.

**In English**

1. Fulton, William; Harris, Joe *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics, 129, New York: Springer-Verlag, 1991

2. J.E. Humphreys *Introduction to Lie Algebras and Representation Theory*, Springer 1973

3. A.A.Kirillov, J.Bernstein, V.Arnold, *Representation Theory*, Abdus Salam School of Mathematical Sciences, 2009

4. Sheinman, Oleg K. *Basic Representation Theory*, Moscow: MCCME, 2007.

5. Serre, Jean-Pierre, *Linear Representations of Finite Groups*, Springer-Verlag, 1977.

6. Vinberg, Èrnest B. *A Course in Algebra*, Graduate Studies in Mathematics, 56, AMS, 2003.

**In Russian**

1. Э. Б. Винберг, *Курс алгебры*, М.: Факториал, 1999.

2. Э. Б. Винберг, *Линейные представления групп*, М.: Наука, 1985.

3. И. М. Парамонова, О. К. Шейнман, *Задачи семинара “Алгебры Ли и их приложения”*, М.: МЦНМО, 2004.

4. О. К. Шейнман, *Основы теории представлений*, М.: МЦНМО, 2004.

## V. Grading procedure

The procedure described here works both for “Math in Moscow” program and for magister program at the department of mathematics.

Each week a homework is formed from several problems not discussed during the seminar. Next week student submit a written solutions and we grade it to obtain a mark. A total of these marks form 20 percents of the resulting mark. A sample of homeworks is attached to this program.

According to the regulations of NRU HSE, in the middle of the course there is a written classwork which is called *midterm exam* in “Math in Moscow”. A mark for this work forms 30 percents of the resulting mark.

Just after the course there is a written exam, which is called *final exam* in “Math in Moscow”. A mark for this work forms the remaining 50 percents of the resulting mark.

The midterm exam and thje final exam lasts 3 hours. Students are allowed to use their class notes and common materials during these exams.