Rationality of Voting and Voting Systems: Lecture II

Rationality of Voting Systems

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Rules make a difference

Example

4 voters	3 voters	2 voters
A	E	D
B	D	C
C	B	B
D	C	E
E	A	A

5 options, 5 winners

Highest average ranking → Borda Count

Example

2 voters	2 voters	2 voters	1 voter
D	A	B	D
C	D	A	C
B	C	D	B
A	B	C	A

This yields the ranking DABC.
Now remove D. This gives: CBA, i.e. reversal of collective preference over A, B and C.
Fishburn: it is possible that the Borda winner wins in only one of the proper subsets of the alternative set.
Obviously, fiddling with the alternative set opens promising vistas for outcome control.

Pairwise victories → Condorcet extensions

Example

Condorcet's paradox

4 voters	4 voters	4 voters
A	B	C
C	A	B
B	C	A

Surely, there is no winner here, or what? If so, then removing this kind of "component" from any larger profile or adding it to some profile should not change the winners, right?

Surprise?

Example

A profile with a strong Condorcet winner

7 voters	4 voters
A	B
B	C
C	A

Adding the Condorcet paradox profile to this one results in a new Condorcet winner. N.B. the Borda winner remains the same in the 11- and 23-voter profiles.

Borda's paradox

Example

4 voters	3 voters	2 voters
A	B	C
B	C	B
C	A	A

Borda's points:

- plurality voting results in a bad outcome
- a superior system exists (Borda Count)

Improving Borda Count: Nanson's rule

How does it work? Compute Borda scores and eliminate all candidates with no more than average score. Repeat until the winner is found.

Properties:

- Guarantees Condorcet consistency
- Is nonmonotonic

Nanson's rule is nonmonotonic

Example

30	21	20	12	12	5
C	B	A	B	A	A
A	D	B	A	C	C
D	C	D	C	B	D
B	A	C	D	D	B

The Borda ranking: $A \succ C \succ B \succ D$ with D's score 97 being the only one that does not exceed the average of 150. Recomputing the scores for A, B and C, results in both B and C failing to reach the average of 100. Thus, A wins. Suppose now that those 12 voters who had the ranking $B \succ A \succ C \succ D$ improve A's position, i.e. rank it first, *ceteris paribus*. Now, both B and D are deleted and the winner is C.

Improving plurality rule: plurality runoff

Properties:

- Does not elect Condorcet losers
- Is nonmonotonic

Example

6 voters	*5 voters*	*4 voters*	*2 voters*
A	C	B	B
B	A	C	A
C	B	A	C

Black's system: a synthesis of two ideas

How does it work? Pick the Condorcet winner. If none exists, choose the Borda winner.

Properties:

- Satisfies Cordorcet criteria
- Is monotonic
- Is inconsistent

Example

4 voters	*3 voters*	*3 voters*	*2 voters*	*2 voters*
A	B	A	B	C
B	C	B	C	A
C	A	C	A	B

Some systems and performance criteria

	Criterion								
Voting system	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
Amendment	1	1	1	1	0	0	0	0	0
Copeland	1	1	1	1	1	0	0	0	0
Dodgson	1	0	1	0	1	0	0	0	0
Maximin	1	0	1	1	1	0	0	0	0
Kemeny	1	1	1	1	1	0	0	0	0
Plurality	0	0	1	1	1	1	0	0	1
Borda	0	1	0	1	1	1	0	0	1
Approval	0	0	0	1	0	1	1	0	1
Black	1	1	1	1	1	0	0	0	0
Pl. runoff	0	1	1	0	1	0	0	0	0
Nanson	1	1	1	0	1	0	0	0	0
Hare	0	1	1	0	1	0	0	0	0

Criteria

- a: the Condorcet winner criterion
- b: the Condorcet loser criterion
- c: the strong Condorcet criterion
- d: monotonicity
- e: Pareto
- f: consistency
- g: Chernoff property
- h: independence of irrelevant alternatives
- i: invulnerability to the no-show paradox

The no-show paradox

Theorem

Moulin, Pérez: all Condorcet extensions are vulnerable to the no-show paradox.

Example

26%	47%	2%	25%
A	B	B	C
B	C	C	A
C	A	A	B

The strong version

Example

2 seats	3 seats	2 seats	2 seats
c	b	a	a
b	a	c	b
a	c	b	c

The amendment agenda: b vs. c and the winner vs. a results in b. Suppose that the two right-most voters abstain. Then a (the abstainers' favorite) wins.

Nanson's method and preference truncation

Example

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

Here Nanson's method results in B. However, if the 2 voters with preference ranking CBDA reveal only their first-ranked option, C, the outcome is C, obviously a superior option from their point of view.

Comments

Abstention can obviously be regarded as an extreme form of preference truncation and, thus, these two paradoxes are closely related. To the same family of paradoxes belongs also the twins' paradox or the " twins not welcome" phenomenon. This paradox occurs whenever adding k copies or "clones" of voter i leads to an outcome which is worse than the original one for i.

Dodgson's rule and the twins paradox

Example

42 voters	26 voters	1 voters	11 voters
B	A	E	E
A	E	D	A
C	C	B	B
D	B	A	D
E	D	C	C

In this profile B is the (strong) Condorcet winner. Adding 20 copies of the one voter with ranking EDBAC leads to A being closest to Condorcet winner. This is worse than B from the point of view of the clones. Hence we have an instance of the twins' paradox.

Kemeny's rule and non-show paradox

Example

5 voters	4 voters	3 voters	3 voters
D	B	A	A
B	C	D	D
C	A	C	B
A	D	B	C

Here the Kemeny winner is D. Now, add 4 voters with DABC ranking. Then the resulting Kemeny ranking would have had A on top. Hence, we have an instance of the strong no-show paradox. Adding the DABC voters one by one to the 15-voter profile demonstrates the twins' paradox.

Maskin monotonicity

Definition

Maskin monotonicity. Let R^N be a profile of n voters and and a procedure that, given this profile, results in alternative x being chosen. Let now another profile S^N be constructed so that at least all those (and possibly some others) individuals who prefer x to y in R^N do so in S_N as well and this holds for all alternatives $y \neq x$. Maskin monotonicity now requires that x be chosen in S^N.

Remark

N.B. Maskin monotonicity is a very strong property. It implies monotonicity.

Plurality fails on Maskin monotonicity

Example

2 voters	1 voter	1 voter	1 voter
A	B	C	D
B	C	B	C
C	A	A	B
D	D	D	A

Obviously, A is the plurality winner. Lift now B above C and D in the two left-most voters' rankings and the new winner is B. Yet, A's position *vis-à-vis* the other alternatives has not been changed. (In fact one could improve A's position by lifting it above B in the right-most voter's ranking).

Does non-monotonicity imply no-show?

Example

	monotonic systems	non-monotonic systems
vulnerable	Copeland	alternative vote
invulnerable	Borda count	?

Campbell and Kelly's result

Theorem

Non-monotonicity does not imply the no-show paradox.

Proof by way of a pretty implausible system. To wit, consider $x \in X$ and $J \subset N$, the active voters. Define the choice rule g so that

$$g(J, P) = x$$

if x is bottom-ranked by all $i \in J$. Otherwise,

$$g(J, P) = y$$

where y is top-ranked by smallest number of voters not ranking x at the bottom.

Campbell and Kelly, cont'd

This strange rule is non-monotonic since an improvement of the ranking of a winner – if it is bottom-ranked – makes it very often non-winning. Yet, this system is not vulnerable to no-show paradox since no group can improve the outcome from what it is by not voting.

Remark

N.B. This system is neither anonymous nor neutral.

System choice in simple settings

1. A satisfies the criterion, while B doesn't, i.e. there are profiles where B violates the criterion, but such profiles do not exist for B.
2. in every profile where A violates the criterion, also B does, but not vice versa.
3. in *practically all profiles* where A violates the criterion, also B does, but not vice versa ("A dominates B almost everywhere").
4. in a plausible probability model B violates the criterion with higher probability than A.
5. in those political cultures that we are interested in, B violates the criterion with higher frequency than A.

The role of culture

- impartial culture: each ranking is drawn from uniform probability distribution over all rankings
- impartial anonymous culture: all profiles (i.e. distributions of voters over preference rankings) equally likely
- unipolar cultures
- bipolar cultures

Lessons from probability and simulation studies

- cultures make a difference (Condorcet cycles, Condorcet efficiencies, discrepancies of choices)
- none of the cultures mimics “reality”
- IC is useful in studying the proximity of intuitions underlying various procedures

Literature

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