No-Betting Pareto Dominance

Itzhak Gilboa, Larry Samuelson
and David Schmeidler

HEC Paris/Tel Aviv
Yale
Interdisciplinary Center Herzlyia/Tel Aviv/Ohio State

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I. Introduction

I.1 Trade

- Suppose Alice and Bob each have Cobb Douglas utility functions for two goods:

$$\begin{aligned} u_A(x_1, x_2) &= \frac{2}{3}\ln x_1 + \frac{1}{3}\ln x_2 \\ u_B(x_1, x_2) &= \frac{1}{3}\ln x_1 + \frac{2}{3}\ln x_2 \end{aligned}$$

- They each have endowment $(1/2, 1/2)$.

- There is a unique competitive equilibrium of this economy in which Alice consumes $\left(\frac{2}{3}, \frac{1}{3}\right)$ and Bob consumes $\left(\frac{1}{3}, \frac{2}{3}\right)$.

I.2 Speculation

- Now suppose there is one good, "money." Ann and Bill have identical utility functions, $u(x) = \ln x$.

- There are two states. Ann and Bill each have half a unit of money in each state. Ann thinks state 1 has probability 2/3, Bill thinks state 1 has probability 1/3. Their expected utility functions are:

$$\begin{aligned} u_A(x_1, x_2) &= \frac{2}{3}\ln x_1 + \frac{1}{3}\ln x_2 \\ u_B(x_1, x_2) &= \frac{1}{3}\ln x_1 + \frac{2}{3}\ln x_2 \end{aligned}$$

- There is a unique competitive equilibrium of this economy in which Alice consumes $\left(\frac{2}{3}, \frac{1}{3}\right)$ and Bob consumes $\left(\frac{1}{3}, \frac{2}{3}\right)$.

I.3 No-Betting-Pareto Dominance

- We find the Pareto gains from trade compelling in the first example, where they are based on differences in tastes. We are less convinced by the gains from trade in the second case, which we regard as a pure bet.

- We offer a *No-Betting Pareto* relation that refines (i.e., is a subset of) the standard Pareto relation and that excludes "betting."

- The No-Betting Pareto relation allows belief-based trading in which people share risks.

- Others have worried about Pareto comparisons with different beliefs. The most directly related paper is by Markus Brunnermeier, Alp Simsek and Wei Xiong.

II. Foundations

- Our approach is conservative.
  - We rank no pairs of alternatives that are not already Pareto ranked.
  - We require compelling evidence in order to exclude a Pareto ranking.
- Our agents are rational, satisfying Savage's axioms. One may distrust Pareto comparisons because people are not rational, but that is not our point here.
- Utilities and probabilities are different. Utilities are related to desirability, and probabilities to belief.

- Given this distinction, we must think about how we interpret probabilities. We exclude two extreme cases that render our analysis inapplicable:
  - Probabilities are not objective, or measurable. Otherwise, one would simply measure the appropriate probabilities and use them.
  - Probabilities are not purely subjective. For example, it makes sense to debate what probabilities "should be" (unlike tastes).
- Beliefs often differ. We see more trade and more betting than the agreeing-to-disagree and no-trade results would otherwise allow.

III. The Model

III.1 The Elements

- Agents $N = \{1, ..., n\}$
- A measurable state space (S, Σ)
- Outcomes X
- Simple acts: $F = \left\{ f : S \to X \;\middle|\; \begin{array}{c} f \text{ has finite range} \\ \text{and is } \Sigma\text{-measurable} \end{array} \right\}.$
- Agent i has $\succsim_i$ over F represented by maximization of

$$\int_S u_i(f(s))dp_i.$$

III.2 Definitions

- Pareto domination: $f \succ_P g$ iff for all $i \in N$, $f \succsim_i g$, and for some $k \in N$, $f \succ_k g$
- For a pair (f, g), interpreted as a change from g to f, agent i is *involved* in (f, g) if $u_i(f(\cdot)) \neq u_i(g(\cdot))$.
- $N(f, g) \subset N$ is the set of agents who are involved in the pair (f, g).
- (f, g) is an *improvement*, denoted by $f \succ_* g$, if $N(f, g) \neq \varnothing$ and, for all $i \in N(f, g)$, $f \succ_i g$.

III.3 No-Betting-Pareto

Given $f, g \in F$, we say that f *No-Betting Pareto dominates* g, denoted $f \succ_{NBP} g$, if:

- f improves upon g;
- There exists a probability measure p_0 such that, for all $i \in N(f, g)$,

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0.$$

- Notice that we do not require that the agents agree on the distributions of f, g, only that they can rationalize trade by hypothetical beliefs.

III.4 An Example

Suppose that f and g are given by the following:

State	p_A	p_B	f		g	
			U_A	U_B	U_A	U_B
1	.4	.3	2	1	1	2
2	.3	.325	2	2	2	1
3	.3	.375	1	2	2	2

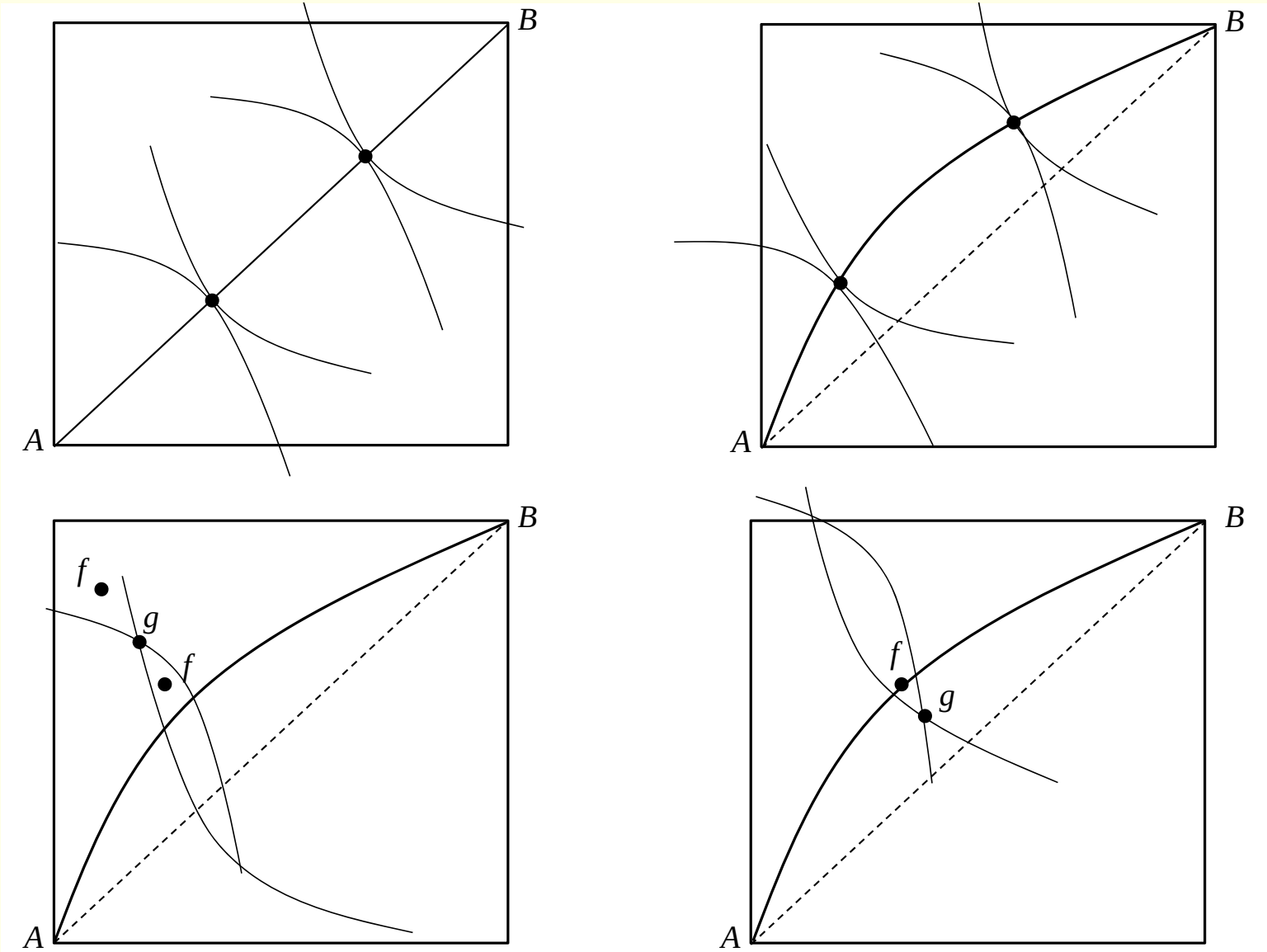
.

Amy and Bruce both prefer f to g. Indeed, Amy prefers f to g iff $p_A(1) > p_A(3)$ and Bruce prefers f to g iff $p_B(2) > p_B(1)$. There are beliefs that satisfy both inequalities, for example $(.325, .375, .3)$. Note, however, that no such belief is in the convex hull of Amy and Bruce's beliefs, as both p_A and p_B agree that state 3 is at least as likely as state 2. $\square$

III.5 A Special Case: Setting

Suppose there are two agents, two states, and an act identifies the amount of money each agent has in each state.

Then we can illustrate the No-Betting-Pareto concept in an Edgeworth box.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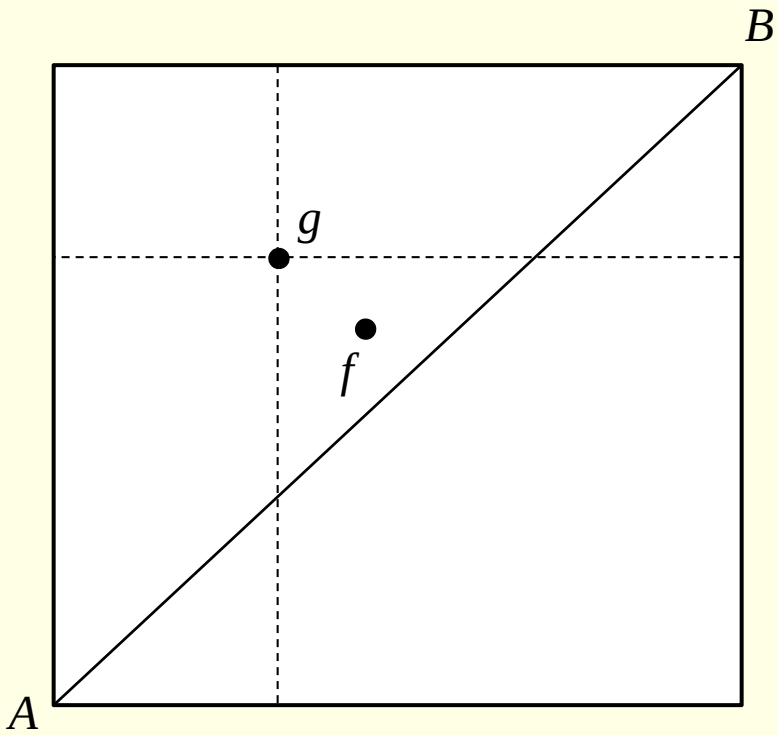
III.6 A Special Case: Characterization

In this special case, an improvement (f, g) satisfies the second condition for No-Betting-Pareto dominance if and only if it moves the agents closer to full insurance.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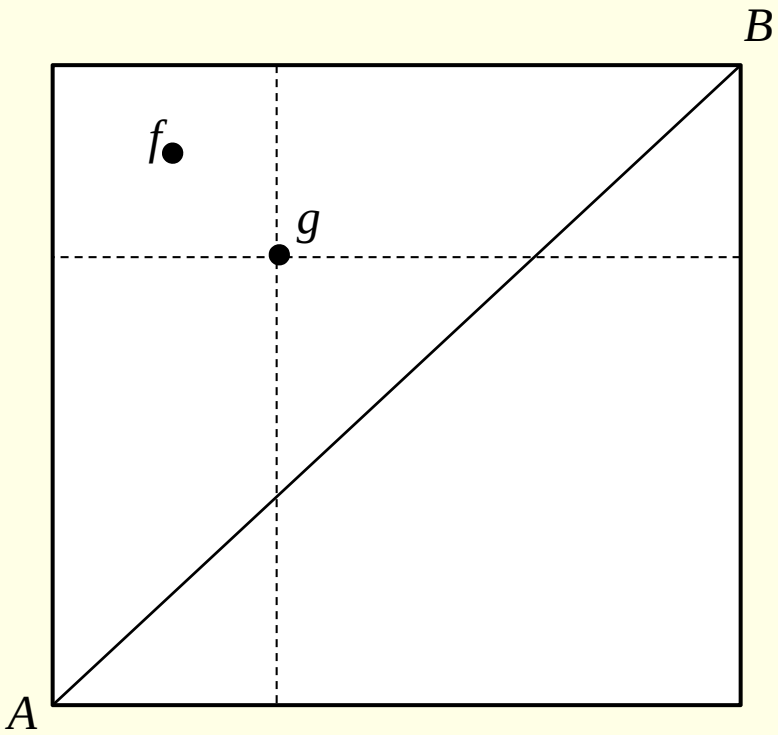
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Proposition 1 *Condition (ii) of No-Betting Pareto is satisfied in the left panel, but not the right panel.*

Why?

On the left, choose p_0 to be the probability according to which f has the same expected value as does g. Both agents would then be indifferent between f and g under p_0, were they risk neutral. Given that they are risk averse, and that g is a (p_0)-mean-preserving spread of f, both agents strictly prefer f to g under p_0 and hence condition (ii) holds.

On the right, assume that condition (ii) were to hold for the two agents with beliefs p_0. Given that these risk-averse agents strictly prefer f to g, despite the fact that f is a spread of g, such strict preferences would certainly hold were the agents risk neutral. However, for no p_0 can the difference $f - g$ simultaneously increase the p_0-expected value for both agents.

In this setting, we regard movements away from full insurance as bets.

Our criterion thus excludes (only) Pareto improvements that are bets, motivating the name No-Betting Pareto dominance.

In more general settings, it is not obvious what constitutes a move toward or away from full insurance, and not obvious what constitutes a bet.

Our task is then to identify the appropriate generalizations of the notions that are obvious in the special case.

IV. Characterizations

IV.1 Combining Agents

An alternative (and equivalent, when agents share a common belief) approach to Pareto efficiency is to define efficient allocations as those that maximize the weighted sum of agents' utilities.

We can establish a similar link for our case of different beliefs.

Proposition 2 *Consider acts f and g with $N(f,g) \neq \varnothing$. There exists a probability vector p_0 such that, for all $i \in N(f,g)$,*

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0$$

if and only if, for every distribution over the set of agents involved, denoted by $\lambda \in \Delta(N(f,g))$, there exists a state $s \in S$, such that

$$\sum_{i \in N(f,g)} \lambda(i)\, u_i(f(s)) > \sum_{i \in N(f,g)} \lambda(i)\, u_i(g(s)).$$

- If the agents want to replace g by f, for each agent there must be at least one state at which the agent is better off under f.

- The characterization indicates this is also true for every convex combination of agents.

- The λ-combination can be interpreted as a utilitarian social welfare function, or as an expected utility calculation behind the veil of ignorance.

IV.2 Betting

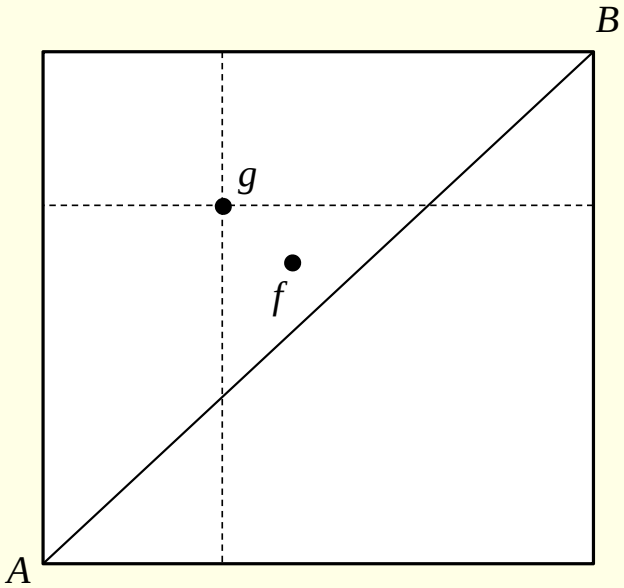
- $X = \mathbb{R}^N$, $x = (x_1, ..., x_n) \in X$.
- $u_i((x_1, ..., x_n)) = u_i(x_i)$
- u_i is differentiable, strictly monotone and (weakly) concave.
- A pair (f, g) is *feasible* if $\sum_{i \in N(f,g)} f(s)_i \leq \sum_{i \in N(f,g)} g(s)_i$ for all s.
- A feasible improvement (f, g) is a *bet* if $g(s)_i$ is independent of s for each $i \in N(f, g)$.

Proposition 3 *If* (f, g) *is a bet, then it cannot be the case that* $f \succ_{NBP} g$.

What else is excluded?

The previous result shows that No-Betting Pareto excludes the most obvious examples of bets.

These are not the only things we would want to exclude, and these are not the only excluded improvements.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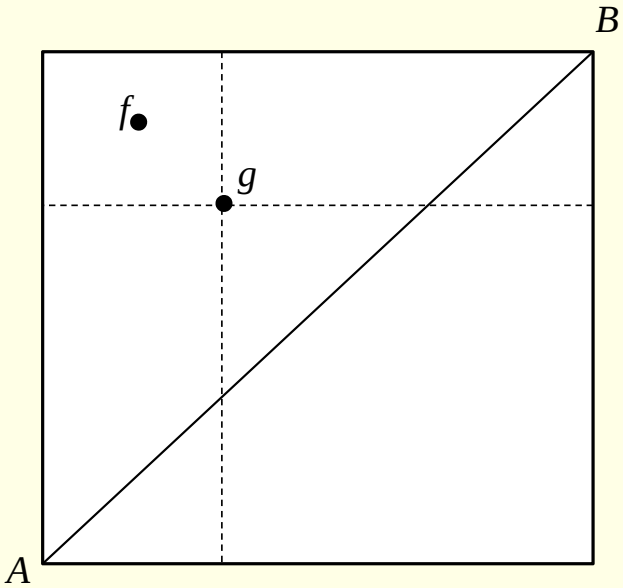
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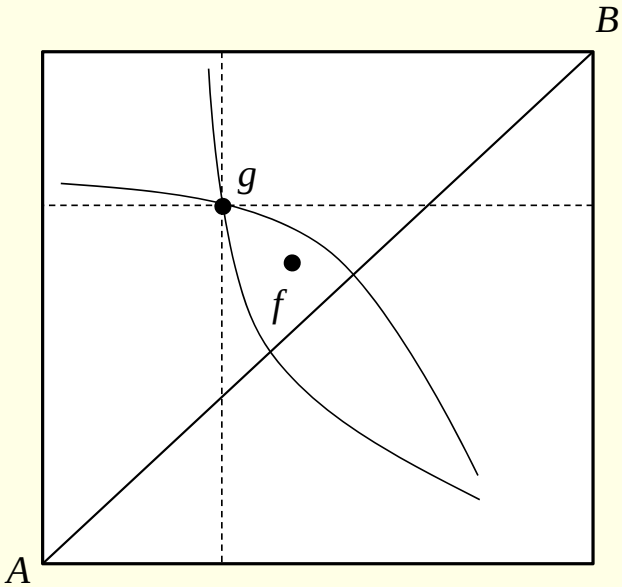
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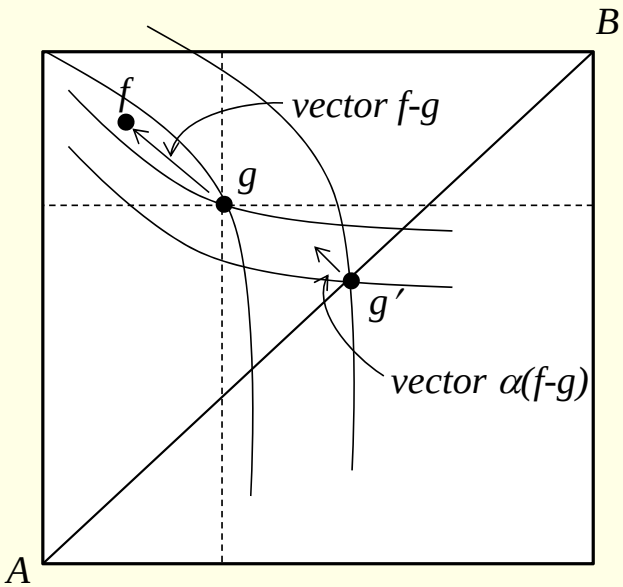
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Proposition 4 *The following are equivalent:*

(i) There does not *exist a probability vector* p_0 *such that, for all* $i \in N(f,g)$,

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0$$

(ii) There exists an alternative $d \in F$ *satisfying*

$$\sum_{i \in N} d(s)_i = 0 \qquad \forall s \in S.$$

that also has the following property: for every $g' \in F$ *such that* $g'(s)_i$ *is independent of* s *for each* $i \in N(f,g)$ *and lies in the interior of* R*, and for every profile of beliefs* $(p_i)_i$ *such that* $f \succ_* g$*, there exists* $\alpha > 0$ *such that* $(g' + \alpha d, g')$ *is a bet (for the utilities* $(u_i)_i$ *and the beliefs* $(p_i)_i$*).*

- An implication is that we can exploit the agents. If they strictly prefer a zero-sum exchange, then we can construct a negative-sum exchange that they also strictly prefer.

- It initially seems trivial to conclude that we can exploit agents with different beliefs.

- The key to the result is that we can design a single such bet that works for all beliefs and all initial full-insurance allocations.

- Future work 1: Are there other characterizations of "betting"? Is this the most useful?

IV.3 Transitivity

Proposition 5 *The relation $\succ_{NBP}$ is acyclic but it need not be transitive.*

For example, let $p_1 = (1, 0)$, $p_2 = (0, 1)$ and

Then in the following, $f \succ_{NBP} h \succ_{NBP} g$, but not $f \succ_{NBP} g$.

		State 1	State 2
g :	Agent 1	0	0
	Agent 2	0	0

		State 1	State 2
h :	Agent 1	$+2$	-1
	Agent 2	-3	$+2$

		State 1	State 2
f :	Agent 1	$+4$	-4
	Agent 2	-4	$+4$

However,

- Define $f \succ_{fNBP} g$ if (f, g) is feasible and $f \succ_{NBP} g$, then
- If (f, g) is a bet, then it cannot be the case that $f \succ^{t}_{fNBP} g$.

IV.4 Incompleteness

- Future work 2: The No-Betting Pareto relation is a refinement of the Pareto relation, and so is less complete than the Pareto relation. How can such an incomplete relation be useful?

- We can imagine settings in which agents must demonstrate that $f \succ_{NBP} g$ in order to make the trade.

- The No-Betting Pareto relation does not require than analyst to know agents' beliefs, only that they are willing to trade. We consider this an advantage.

- The No-Betting Pareto relation does require the analyst "know" the state space.

V. Future Work 3: Ambiguity

V.1 The Setting

- Suppose that each agent has a set of priors and maximizes the minimum (over this set of priors) expected utility.

- We similarly adjust our no-betting Pareto criterion, saying that $f \succ_{NBP} g$ if f is an improvement of g and there is a single set of priors with respect to which every involved agent finds f better than g.

- Suppose f is an improvement of g. Then any trade that can be justified with a single prior p_0 can be justified with a set of priors, namely $\{p_0\}$.

- Does the set-valued approach allows us to justify anything else?

V.2 Betting

- Let (f, g) be a bet. Then it cannot be that $f \succ_{NBP} g$ under ambiguity.

- Because (f, g) is a bet, we know that for any single probability, there is some agent who prefers g to f. But then given any set of probabilities, there must always be an agent who prefers g to f when the latter is evaluated according to the worst probability.

V.3 Other Trades

There may exist (f, g) for which it is impossible that $f \succ_{NBP} g$ in the absence of ambiguity, but $f \succ_{NBP} g$ under ambiguity.

For example:

	s_1	s_2
f	$.4, .4$	$.4, .4$
g	$1, 0$	$0, 1$

Then we can have $f \succ_{NBP} g$ under ambiguity, but not risk. This appears to open up a prospect for risk-sharing that we would like to allow.

Another example:

	s_1	s_2	s_3	s_4
f	$-2,1$	$1,-2$	$0,-1$	$-1,0$
g	$1,-1$	$-1,1$	$0,-1$	$-1,0.$

Again, we can have $f \succ_{NBP} g$ under ambiguity, but not risk. Here, however, in moving from g to f, the agents have essentially reversed the roles of states s_1 and s_2, while paying one unit in the process.

VI. Discussion