An Introduction to the Choquet integral

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Outline

1. Capacities and set functions
2. The Choquet and Sugeno integrals
3. Properties
4. Characterizations
5. Particular cases
6. The concave integral
Set functions and games

- $X$: finite universe. \textit{Set function on $X$}: $\xi : 2^X \rightarrow \mathbb{R}$. 
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  1. If $\xi(\emptyset) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\xi} = \xi$;
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Measures and capacities

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A **capacity** $\mu : 2^X \to \mathbb{R}$ is a grounded monotone set function, i.e., $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. 
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  - $A \subseteq X$: event
  - $\mu(A)$: uncertainty that the event $A$ contains the outcome of an experiment, with $\mu(A) = 0$ indicating total uncertainty, and $\mu(A) = 1$ indicating that there is no uncertainty.
Examples

Example
Let $X$ be a set of firms. Certain firms may form a coalition in order to control the market for a given product. Then $\mu(A)$ may be taken as the annual benefit of the coalition $A$.

Example
Let $X$ be a set of voters in charge of electing a candidate for some important position (president, director, etc.) or voting a bill by a yes/no decision. Before the election, groups of voters may agree to vote for the same candidate (or for yes or no). In many cases (presidential elections, parliament, etc.), these coalitions correspond to the political parties or to alliances among them. If the result of the election is in accordance with the wish of coalition $A$, the coalition is said to be winning, and we set $\mu(A) = 1$, otherwise it is losing and $\mu(A) = 0$.

Example
Let $X$ be a set of workers in a factory, producing some goods. The aim is to produce these goods as much as possible in a given time (say, in one day). Then $\mu(A)$ is the number of goods produced by the group $A$ in one day. Since the production needs in general the collaboration of several workers with different skills, it is likely that $\mu(A) = 0$ if $A$ is a singleton or a too small group.
Examples

Example
David throws a dice, and wonders what number will show. Here $X = \{1, 2, 3, 4, 5, 6\}$, and $\mu(\{1, 3, 5\})$ quantifies the uncertainty of obtaining an odd number.

Example
A murder has been committed. After some investigation, it is found that the guilty is either Alice, Bob or Charles. Then $X = \{\text{Alice, Bob, Charles}\}$, and $\mu(\{\text{Bob, Charles}\})$ quantifies the degree to which it is “certain” (the precise meaning of this word being conditional on the type of capacity used) that the guilty is Bob or Charles.

Example
Glenn is an amateur of antique Chinese porcelain. He enters a shop and sees a magnificent vase, wondering how old (and how expensive) this vase could be. Then $X$ is the set of numbers from, say $-3000$ to $2014$, i.e., the possible date expressed in years A.C. when the vase was created. For example, $\mu([1368, 1644])$ indicates to what degree it is certain that it is a vase of the Ming period.

Example
Leonard is planning to go to the countryside tomorrow for a picnic. He wonders if the weather will be favorable or not. Here $X$ is the set of possible states of the weather, like “sunny”, “rainy”, “cloudy”, and so on. For example, $\mu(\{\text{sunny, cloudy}\})$ indicates to what degree of certainty it will not rain, and so if the picnic is conceivable or not.
Properties

1. $\nu$ is superadditive if for any $A, B \in 2^X$, $A \cap B = \emptyset$, $\nu(A \cup B) \geq \nu(A) + \nu(B)$.

(subadditive if the reverse inequality holds)
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3. \( v \) is **\( k \)-monotone** \((k \geq 2)\) if for \( A_1, \ldots, A_k \in 2^X \),
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   v\left( \bigcup_{i=1}^{k} A_i \right) \geq \sum_{\substack{I \subseteq \{1, \ldots, k\} \\ I \neq \emptyset}} (-1)^{|I|+1} v\left( \bigcap_{i \in I} A_i \right).
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4. \( v \) \( k \)-alternating \((k \geq 2)\) if for \( A_1, \ldots, A_k \in 2^X \),
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Theorem

Let \( v \) be a game on \( X \). The following holds.

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Theorem
Let $v$ be a game on $X$. The following holds.

1. $v$ superadditive $\implies \overline{v} \succeq v$.

2. $v$ is $k$-monotone (resp., $k$-alternating) for some $k \geq 2$ if and only if $\overline{v}$ is $k$-alternating (resp., $k$-monotone). In particular, $v$ is supermodular (resp., submodular) if and only if $\overline{v}$ is submodular (resp., supermodular).
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3. $v \succeq 0$ and supermodular implies that $v$ is monotone.
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0-1 capacities

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- The number of antichains in \( 2^X \) with \( |X| = n \) is the Dedekind number \( M(n) \).

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Unanimity games

Let $A \subseteq X$, $A \neq \emptyset$. The \textit{unanimity game centered on }$A$\textit{ is the game }$u_A$\textit{ defined by}

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$
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- Unanimity games are 0-1-valued capacities.
A possibility measure or maxitive measure on a $X$ is a normalized capacity $\Pi$ on $X$ satisfying

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A necessity measure or minitive measure is a normalized capacity $N$ satisfying

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The conjugate of a possibility measure (resp., a necessity measure) is a necessity measure (resp., a possibility measure).
Belief and plausibility measures

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- A particular case: the $\lambda$-measure ($\lambda > -1$) is a normalized capacity satisfying for every $A, B \in 2^X$, $A \cap B = \emptyset$

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Belief and plausibility measures

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\[ \mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A) \mu(B) \]

- A $\lambda$-measure is a belief measure if and only if $\lambda \geq 0$, and is a plausibility measure otherwise.
normalized capacities

- superadditive
  - 2-monotone (or convex, supermodular)
  - 3-monotone
  - ∞-monotone (or belief measures)

- necessity
  - λ > 0
  - unanimity games
  - Dirac measures

- possibility
  - ∞-alternating (or plausibility measures)
  - 3-alternating
  - 2-alternating (or concave, submodular)

- subadditive

probability (λ = 0)

λ-measure

λ > 0

−1 < λ < 0

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An introduction to the Choquet integral
The Möbius transform

Definition
Let $\xi$ be a set function on $X$. The Möbius transform or Möbius inverse of $\xi$ is a set function $m^\xi$ on $X$ defined by

$$m^\xi(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \xi(B)$$

(1)

for every $A \subseteq X$.

Given $m^\xi$, it is possible to recover $\xi$ by the formula

$$\xi(A) = \sum_{B \subseteq A} m^\xi(B) \quad (A \subseteq X).$$

(2)
Properties

1. \( \nu \) is additive if and only if \( m^\nu(A) = 0 \) for all \( A \subseteq X, \ |A| > 1 \). Moreover, we have \( m^\nu(\{i\}) = \nu(\{i\}) \) for all \( i \in X \).
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2. $\nu$ is monotone if and only if

$$\sum_{i \in L \subseteq K} m^\nu(L) \geq 0 \quad (K \subseteq X, \quad i \in K).$$
Properties

1. $v$ is additive if and only if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > 1$. Moreover, we have $m^v({i}) = v({i})$ for all $i \in X$.

2. $v$ is monotone if and only if
   \[ \sum_{i \in L \subseteq K} m^v(L) \geq 0 \quad (K \subseteq X, \quad i \in K). \]

3. Let $k \geq 2$ be fixed. $v$ is $k$-monotone if and only if
   \[ \sum_{L \in [A,B]} m^v(L) \geq 0 \quad (A, B \subseteq X, \quad A \subseteq B, \quad 2 \leq |A| \leq k). \]
Properties

1. \( v \) is additive if and only if \( m^v(A) = 0 \) for all \( A \subseteq X, \ |A| > 1 \). Moreover, we have \( m^v(\{i\}) = v(\{i\}) \) for all \( i \in X \).

2. \( v \) is monotone if and only if

\[
\sum_{i \in L \subseteq K} m^v(L) \geq 0 \quad (K \subseteq X, \ i \in K).
\]

3. Let \( k \geq 2 \) be fixed. \( v \) is \( k \)-monotone if and only if

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\sum_{L \in [A,B]} m^v(L) \geq 0 \quad (A, B \subseteq X, \ A \subseteq B, \ 2 \leq |A| \leq k).
\]

4. If \( v \) is \( k \)-monotone for some \( k \geq 2 \), then \( m^v(A) \geq 0 \) for all \( A \subseteq X \) such that \( 2 \leq |A| \leq k \).
Properties

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5. \( v \) is a nonnegative totally monotone game if and only if \( m^v \geq 0 \).
A game $\nu$ on $X$ is said to be $k$-additive for some integer $k \in \{1, \ldots, |X|\}$ if $m^\nu(A) = 0$ for all $A \subseteq X$, $|A| > k$, and there exists some $A \subseteq X$ with $|A| = k$ such that $m^\nu(A) \neq 0$.

A game $\nu$ is at most $k$-additive for some $1 \leq k \leq |X|$ if it is $k'$-additive for some $k' \in \{1, \ldots, k\}$.
**Definition**

A game \( v \) on \( X \) is said to be *\( k \)-additive* for some integer \( k \in \{1, \ldots, |X|\} \) if \( m^v(A) = 0 \) for all \( A \subseteq X, |A| > k \), and there exists some \( A \subseteq X \) with \( |A| = k \) such that \( m^v(A) \neq 0 \).

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- The set of \( k \)-additive games on \( X \) (resp., capacities, etc., ) is denoted by \( G^k(X) \) (resp., \( MG^k(X) \), etc. )
Definition

A game \( v \) on \( X \) is said to be \( k \)-additive for some integer \( k \in \{1, \ldots, |X|\} \) if \( m^v(A) = 0 \) for all \( A \subseteq X, |A| > k \), and there exists some \( A \subseteq X \) with \( |A| = k \) such that \( m^v(A) \neq 0 \).

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- We denote by \( \mathcal{G}^{\leq k}(X), \mathcal{M} \mathcal{G}^{\leq k}(X) \) the set of at most \( k \)-additive games and capacities.

\[
\mathcal{G}(X) = \mathcal{G}^1(X) \cup \mathcal{G}^2(X) \cup \cdots \cup \mathcal{G}^{|X|}(X) \\
= \mathcal{G}^1(X) \cup \mathcal{G}^{\leq 2}(X) \cup \cdots \cup \mathcal{G}^{\leq |X|}(X)
\]
The vector space of games

For any nonempty \( A \subseteq X \) the *identity game* \( \delta_A \) centered at \( A \) is the 0-1-game defined by

\[
\delta_A(B) = \begin{cases} 
1, & \text{if } A = B \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem**

The set of identity games \( \{\delta_A\}_{A \in 2^X \setminus \{\emptyset\}} \) and the set of unanimity games \( \{u_A\}_{A \in 2^X \setminus \{\emptyset\}} \) are bases of \( \mathcal{G}(X) \) of dimension \( 2^{|X|} - 1 \).

- In the basis of identity games, the coordinates of a game \( v \) are simply \( \{v(A)\}_{A \in 2^X \setminus \{\emptyset\}} \).
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\textbf{Theorem}

The set of identity games \( \{\delta_A\}_{A \in 2^X \setminus \{\emptyset\}} \) and the set of unanimity games \( \{u_A\}_{A \in 2^X \setminus \{\emptyset\}} \) are bases of \( G(X) \) of dimension \( 2^{|X|} - 1 \).

\begin{itemize}
  \item In the basis of identity games, the coordinates of a game \( \nu \) are simply \( \{\nu(A)\}_{A \in 2^X \setminus \{\emptyset\}} \).
  \item We have for any game \( \nu \in G(X) \)
    \[ \nu(B) = \sum_{A \in 2^X \setminus \{\emptyset\}} \lambda_A u_A(B) = \sum_{A \subseteq B, A \neq \emptyset} \lambda_A \quad (B \subseteq X). \]
\end{itemize}

It follows that the coefficients of a game \( \nu \) in the basis of unanimity games are its Möbius transform: \( \lambda_A = m^\nu(A) \) for all \( A \subseteq X, A \neq \emptyset \).
Outline

1. Capacities and set functions
2. The Choquet and Sugeno integrals
3. Properties
4. Characterizations
5. Particular cases
6. The concave integral
Simple functions

- $X$ nonempty set
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- We assume $\text{ran} f = \{a_1, \ldots, a_n\}$, supposing $0 \leq a_1 < a_2 < \cdots < a_n$. Then

$$f = \sum_{i=1}^{n} a_i 1_{\{x \in X : f(x) = a_i\}}$$

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- $X$ nonempty set, $\mathcal{F}$ algebra on $X$ (closed under finite union and complementation)
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- Let $f \in B(\mathcal{F})$, $\mu$ a capacity on $\mathcal{F}$. The decumulative (distribution) function of $f$ w.r.t. $\mu$ is

$$G_{\mu,f}(t) = \mu(\{x \in X : f(x) \geq t\}) \quad (t \in \mathbb{R})$$
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  - it has a compact support, namely $[0, \text{ess sup}_{\mu} f]$. 

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The Choquet integral

Definition
Let \( f \in B^+(\mathcal{F}) \) and \( \mu \) be a capacity on \((X, \mathcal{F})\). The Choquet integral of \( f \) w.r.t. \( \mu \) is defined by

\[
\int f \, d\mu = \int_0^\infty G_{\mu,f}(t) \, dt,
\]

where the right hand-side integral is the Riemann integral.
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A fundamental fact is the following.

**Lemma**
Let $A \in \mathcal{F}$ (i.e., $1_A$ is measurable). Then for every capacity $\mu$

$$\int 1_A \, d\mu = \mu(A).$$
The Sugeno integral

Definition
Let $f \in B^+(\mathcal{F})$ be a function and $\mu$ be a capacity on $(X, \mathcal{F})$. The 
Sugeno integral of $f$ w.r.t. $\mu$ is defined by

$$\int f \, d\mu = \bigvee_{t \geq 0} (G_{\mu,f}(t) \wedge t) = \bigwedge_{t \geq 0} (G_{\mu,f}(t) \vee t). \tag{3}$$

Remark 1: $>$ can replace $\geq$ in the definition of $G_{\mu,f}$.

Remark 2: $\int 1_A \, d\mu = \mu(A)$ for any $A \in \mathcal{F}$ holds if $\mu$ is normalized.
The case of real-valued functions

For any $f \in B(\mathcal{F})$ we write

$$f = f^+ - f^-, \text{ with } f^+ = 0 \lor f, \quad f^- = (-f)^+.$$
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The **symmetric Choquet integral** is defined by
\[ \tilde{\int} f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu. \]
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- Homogeneity (ratio scale invariance):
  \[ \tilde{\int} \alpha f \, d\mu = \alpha \tilde{\int} f \, d\mu \quad (\alpha \in \mathbb{R}) \]
The case of real-valued functions

The asymmetric Choquet integral is defined by

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$
The case of real-valued functions

- The asymmetric Choquet integral is defined by
  \[ \int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu \]

- Positive homogeneity and translation invariance (interval scale invariance)
  \[ \int (\alpha f + \beta 1_X) \, d\mu = \alpha \int f \, d\mu + \beta \mu(X) \quad (\alpha > 0, \beta \in \mathbb{R}) \]
The case of real-valued functions

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  \[ \int (\alpha f + \beta 1_X) \, d\mu = \alpha \int f \, d\mu + \beta \mu(X) \quad (\alpha > 0, \beta \in \mathbb{R}) \]

- Expression w.r.t. the decumulative function:
  \[ \int f \, d\mu = \int_0^\infty \mu(f \geq t) \, dt + \int_{-\infty}^0 (\mu(f \geq t) - \mu(X)) \, dt. \]
The Choquet integral of simple functions

- \( f \): simple, measurable nonnegative function with \( \text{ran} f = \{a_1, \ldots, a_n\} \), and \( 0 \leq a_1 < a_2 < \cdots < a_n \)
The Choquet integral of simple functions

- $f$: simple, measurable nonnegative function with $\text{ran} f = \{a_1, \ldots, a_n\}$, and $0 \leq a_1 < a_2 < \cdots < a_n$
- $A_i = \{x \in X : f(x) \geq a_i\}$, for $i = 1, \ldots, n$
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- $A_i = \{x \in X : f(x) \geq a_i\}$, for $i = 1, \ldots, n$
- From the decumulative function, one finds
  \[
  \int f \, d\mu = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu(A_i),
  \]
  letting $a_0 = 0$, and
  \[
  \int f \, d\mu = \sum_{i=1}^{n} a_i (\mu(A_i) - \mu(A_{i+1})),
  \]
  with the convention $A_{n+1} = \emptyset$. 

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With same notations, one finds

\[ \int f \, d\mu = \bigvee_{i=1}^{n} (a_i \land \mu(A_i)) \]

\[ \int f \, d\mu = \bigwedge_{i=0}^{n} (a_i \lor \mu(A_{i+1})) \]

with the convention \( A_{n+1} = \emptyset \) and \( a_0 = 0 \).
The Sugeno integral of simple functions

\[ G_{\mu,f}(t) \]

\[ \mu(X) \]

\[ \mu(A_2) \]

\[ \mu(A_3) \]

\[ \mu(A_4) \]

\[ \times \]

\[ \times \]

\[ \times \]

\[ \times \]

\[ t \]

\[ a_1 \]

\[ a_2 \]

\[ a_3 \]

\[ a_4 \]

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An introduction to the Choquet integral
The Choquet integral on finite sets

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- $X = \{x_1, \ldots, x_n\}$
- $f : X \rightarrow \mathbb{R}_+$, $f_i = f(x_i)$
- Choose a permutation $\sigma$ on $X$ s.t. $f_{\sigma(1)} \leq f_{\sigma(2)} \leq \cdots \leq f_{\sigma(n)}$. 

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The Choquet integral on finite sets

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- $A_{\sigma}^\uparrow(i) = \{x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\}$ \hspace{1cm} (i = 1, \ldots, n)
- From the case of simple functions, we obtain directly

\[
\int f \, d\mu = \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) \mu(A_{\sigma}^\uparrow(i))
\]

\[
\int f \, d\mu = \sum_{i=1}^{n} f_{\sigma(i)} (\mu(A_{\sigma}^\uparrow(i)) - \mu(A_{\sigma}^\uparrow(i + 1)))
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with the conventions $f_{\sigma(0)} = 0$ and $A_{\sigma}^\uparrow(n + 1) = \emptyset$. 
The Sugeno integral on finite sets

With the same notations we obtain

\[
\int f \, d\mu = \bigvee_{i=1}^{n} (f_\sigma(i) \land \mu(A^\uparrow_\sigma(i)))
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Example: workers in a factory (ctd)

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  - Then $x_2$ leaves and the group $X \setminus \{x_1, x_2\} = \{x_3, \ldots, x_n\}$ works in addition $f(x_3) - f(x_2)$, etc.,
Example: workers in a factory (ctd)

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Each worker starts at 8:00, works continuously but they leave at different times. We denote by $f(x_i)$ the number of worked hours for $x_i$, and label the workers so that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$.

The productivity per hour of a group $A \subseteq X$ is given by $\mu(A)$.

The total number of goods produced in a day is given by:

- The entire group $X$ has worked $f(x_1)$ hours;
- Then $x_1$ leaves and the group $X \setminus \{x_1\} = \{x_2, \ldots, x_n\}$ works in addition $f(x_2) - f(x_1)$ hours;
- Then $x_2$ leaves and the group $X \setminus \{x_1, x_2\} = \{x_3, \ldots, x_n\}$ works in addition $f(x_3) - f(x_2)$, etc.,
- Finally only $x_n$ remains and he works $f(x_n) - f(x_{n-1})$ hours.
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  - Finally only $x_n$ remains and he works $f(x_n) - f(x_{n-1})$ hours.
- Then the total production is
  $$f(x_1)\mu(X) + (f(x_2) - f(x_1))\mu(\{x_2, \ldots, x_n\}) + (f(x_3) - f(x_2))\mu(\{x_3, \ldots, x_n\}) + \cdots + (f(x_n) - f(x_{n-1}))\mu(\{x_n\}) = \int f \, d\mu.$$
Outline

1. Capacities and set functions
2. The Choquet and Sugeno integrals
3. Properties
4. Characterizations
5. Particular cases
6. The concave integral
Positive homogeneity:

$$\int \alpha f \, d\nu = \alpha \int f \, d\nu \quad (\alpha \geq 0)$$
Basic properties of the Choquet integral

- Positive homogeneity:
  \[ \int \alpha f \, d\nu = \alpha \int f \, d\nu \quad (\alpha \geq 0) \]

- Monotonicity w.r.t. the integrand: for any capacity \( \mu \),
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  \[ \nu \leq \nu' \Rightarrow \int f \, d\nu \leq \int f \, d\nu' \quad (\nu, \nu' \in BV(F)) \]

- Linearity w.r.t. the game:
  \[ \int f \, d(\nu + \alpha \nu') = \int f \, d\nu + \alpha \int f \, d\nu' \quad (\nu, \nu' \in BV(F), \alpha \in \mathbb{R}) \]
Basic properties of the Choquet integral

Boundaries: \( \inf f \) and \( \sup f \) are attained:

\[
\inf f = \int f \, d\mu_{\min}, \quad \sup f = \int f \, d\mu_{\max},
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with \( \mu_{\min}(A) = 0 \) for all \( A \subset X, A \in \mathcal{F} \), \( \mu_{\min}(X) = 1 \), and \( \mu_{\max}(A) = 1 \) for all nonempty \( A \in \mathcal{F} \);
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- **Continuity**

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Basic properties of the Sugeno integral

Let $f$ be a function in $B^+(\mathcal{F})$, and $\mu$ a capacity on $(X, \mathcal{F})$.

- Positive $\wedge$-homogeneity:

$$\int (\alpha 1_X \wedge f) \, d\mu = \alpha \wedge \int f \, d\mu \quad (\alpha \geq 0)$$
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  \]
Basic properties of the Sugeno integral

Let $f$ be a function in $B^+(\mathcal{F})$, and $\mu$ a capacity on $(X, \mathcal{F})$.

- Positive $\land$-homogeneity:

$$\int (\alpha 1_X \land f) \, d\mu = \alpha \land \int f \, d\mu \quad (\alpha \geq 0)$$

- Positive $\lor$-homogeneity if $\text{ess sup}_\mu f \leq \mu(X)$:

$$\int (\alpha 1_X \lor f) \, d\mu = \alpha \lor \int f \, d\mu \quad (\alpha \in [0, \text{ess sup}_\mu f])$$

- Hat function: for every $\alpha \geq 0$ and for every $A \in \mathcal{F}$,

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Basic properties of the Sugeno integral

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\[ \mu \leq \mu' \Rightarrow \int f \, d\mu \leq \int f \, d\mu' \quad (\mu, \mu' \text{ on } (X, \mathcal{F})) \]
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\[ \int f \, d(\mu \lor (\alpha \land \mu')) = \int f \, d\mu \lor (\alpha \land \int f \, d\mu') \quad (\mu, \mu' \text{ capacities on } (X, \mathcal{F})) \]
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- Boundaries:
  \[ \text{ess inf}_\mu f \leq \int f \, d\mu \leq (\text{ess sup}_\mu f) \land \mu(X) \]
Comonotonic functions

Two functions $f, g : X \to \mathbb{R}$ are *comonotonic* if there is no $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$
Comonotonic functions

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- Equivalently when $X$ is finite, $f, g$ are comonotonic if there exists a permutation $\sigma$ on $X$ such that $f_{\sigma(1)} \leq \cdots \leq f_{\sigma(n)}$ and $g_{\sigma(1)} \leq \cdots \leq g_{\sigma(n)}$
Comonotonic functions

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**Theorem**

Let $f, g$ be comonotonic functions on $X$ (finite). Then for any game $v$, the Choquet integral is comonotonically additive, and the Sugeno integral is comonotonically maxitive and minitive for any capacity $\mu$:

\[
\int (f + g) \, dv = \int f \, dv + \int g \, dv
\]

\[
\int (f \lor g) \, d\mu = \int f \, d\mu \lor \int g \, d\mu
\]

\[
\int (f \land g) \, d\mu = \int f \, d\mu \land \int g \, d\mu.
\]
Supermodular capacities

For any game $\nu$, the following conditions are equivalent:

1. $\nu$ is supermodular;
Supermodular capacities

For any game $\nu$, the following conditions are equivalent:

1. $\nu$ is supermodular;
2. The Choquet integral is superadditive, that is,

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For any game $v$, the following conditions are equivalent:

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3. The Choquet integral is supermodular, that is,

$$\int (f \lor g) \, dv + \int (f \land g) \, dv \geq \int f \, dv + \int g \, dv$$

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For any game $\nu$, the following conditions are equivalent:

1. $\nu$ is supermodular;
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   for all $f, g : X \to \mathbb{R}$;
3. The Choquet integral is supermodular, that is,
   \[ \int (f \vee g) \, d\nu + \int (f \wedge g) \, d\nu \geq \int f \, d\nu + \int g \, d\nu \]
   for all $f, g : X \to \mathbb{R}$;
4. The Choquet integral is concave, that is,
   \[ \int (\lambda f + (1 - \lambda)g) \, d\nu \geq \int \lambda f \, d\nu + (1 - \lambda) \int g \, d\nu \]
   for all $\lambda \in [0, 1], f, g : X \to \mathbb{R}$. 
5. The Choquet integral yields the lower expected value on the core of $v$:

$$\int f \, dv = \min_{\phi \in \text{core}(v)} \int f \, d\phi,$$

(4)

where core($v$) is the set of additive games $\phi$ on $X$ such that $\phi(X) = v(X)$ and $\phi(S) \geq v(S)$ for all $S \in 2^X$. 

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Maxitivity and minitivity of the Sugeno integral

Theorem

The following holds:

1. \( \int (f \lor g) \, d\mu = \int f \, d\mu \lor \int g \, d\mu \) for all \( f, g \) if and only if \( \mu \) is maxitive;

2. \( \int (f \land g) \, d\mu = \int f \, d\mu \land \int g \, d\mu \) for all \( f, g \) if and only if \( \mu \) is minitive.
Let $X$ be finite, $\nu$ be a game, and $f : X \rightarrow \mathbb{R}$. Then

$$\int f \, d\nu = \sum_{A \subseteq X} m^\nu(A) \bigwedge_{i \in A} f_i.$$ 

where $m^\nu$ is the Möbius transform of $\nu$. 
Outline

1. Capacities and set functions
2. The Choquet and Sugeno integrals
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Theorem

(Schmeidler 1986) Let $I : B(\mathcal{F}) \to \mathbb{R}$ be a functional. Define the set function $v(A) = I(1_A)$ on $\mathcal{F}$. The following propositions are equivalent:

1. $I$ is monotone and comonotonically additive;
2. $v$ is a capacity, and for all $f \in B(\mathcal{F})$, $I(f) = \int f \, dv$. 
Theorem

Let $|X| = n$, $\mathcal{F} = 2^X$, and let $I : (\mathbb{R}_+)^X \to \mathbb{R}_+$ be a functional. Define the set function $\mu(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:

1. $I$ is comonotonically maxitive, satisfies $I(\alpha 1_A) = \alpha \land I(1_A)$ for every $\alpha \geq 0$ and $A \subseteq X$ (hat function property), and $I(1_X) = 1$;

2. $\mu$ is a normalized capacity on $X$ and $I(f) = \int f \, d\mu$. 

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An introduction to the Choquet integral
Outline

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We consider $|X| = n$, $\mathcal{F} = 2^X$.

**Definition**

Let $f : X \to \mathbb{R}_+$ and $\mu$ be a capacity. The *concave integral* of $f$ w.r.t. $\mu$ is given by:

$$\int^{\text{cav}} f \, d\mu = \sup \left\{ \sum_{S \subseteq X} \alpha_S \mu(S) : \sum_{S \subseteq X} \alpha_S 1_S = f, \quad \alpha_S \geq 0, \forall S \subseteq X \right\}.$$  

(5)

Nota: “sup” can be replaced by “max”. 

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The example of workers in a factory revisited

- \( X = \{1, 2, 3\} \), let \( \mu \) on \( X \) be defined by
  \[ \mu(1) = \mu(2) = \mu(3) = 0.2, \quad \mu(12) = 0.9, \quad \mu(13) = 0.8, \]
  \[ \mu(23) = 0.5 \text{ and } \mu(123) = 1 \]
The example of workers in a factory revisited

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- Each worker is given an amount of time: $f_1 = 1$ for worker 1, $f_2 = 0.4$ and $f_3 = 0.6$ for workers 2 and 3
The example of workers in a factory revisited

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- How should the workers organize themselves in teams so as to maximize the total production while not exceeding their allotted time?

- The answer is given by the concave integral: team $\{1, 2\}$ is working 0.4 unit of time and team $\{1, 3\}$ is working 0.6 unit of time, which yields
  $$0.9 \cdot 0.4 + 0.8 \cdot 0.6 = 0.84$$
The example of workers in a factory revisited

By contrast, the Choquet integral computes the total productivity under the constraint that the teams form a specific chain, in this case the teams are \{1, 2, 3\}, \{1, 3\} and \{1\} for durations 0.4, 0.2 and 0.4 respectively, yielding

\[ 0.4 + 0.8 \cdot 0.2 + 0.2 \cdot 0.4 = 0.64 \]
The following properties hold for the concave integral:

1. For every capacity \( \mu \), the concave integral \( \int^{cav} \cdot d\mu \) is a concave and positively homogeneous functional, and satisfies
   \[ \int^{cav} 1_S \, d\mu \geq \mu(S) \text{ for all } S \in 2^X; \]
Properties

The following properties hold for the concave integral:

1. For every capacity $\mu$, the concave integral $\int^{\text{cav}} f \, d\mu$ is a concave and positively homogeneous functional, and satisfies $\int^{\text{cav}} 1_S \, d\mu \geq \mu(S)$ for all $S \in 2^X$;

2. For every $f \in \mathbb{R}_+^X$ and capacity $\mu$,

$$\int^{\text{cav}} f \, d\mu = \min \{ I(f) \mid I : \mathbb{R}_+^X \to \mathbb{R} \text{ concave, positively homogeneous, and such that } I(1_S) \geq \mu(S), \forall S \subseteq X \}$$
The following properties hold for the concave integral:

1. For every capacity $\mu$, the concave integral $\int^\text{cav} \cdot d\mu$ is a concave and positively homogeneous functional, and satisfies $\int^\text{cav} 1_S d\mu \geq \mu(S)$ for all $S \in 2^X$.

2. For every $f \in \mathbb{R}^X_+$ and capacity $\mu$,

$$\int^\text{cav} f d\mu = \min \{ I(f) \mid I : \mathbb{R}^X_+ \to \mathbb{R} \text{ concave, positively homogeneous, and such that } I(1_S) \geq \mu(S), \forall S \subseteq X \}$$

3. For every $f \in \mathbb{R}^X_+$ and capacity $\mu$,

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3. For every $f \in \mathbb{R}_+^X$ and capacity $\mu$,

$$\int^\text{cav} f \, d\mu = \min_{P \text{ additive }, P \geq \mu} \int f \, dP$$

4. For every $f \in \mathbb{R}_+^X$ and capacity $\mu$,

$$\int f \, d\mu \leq \int^\text{cav} f \, d\mu,$$

and equality holds for every $f \in \mathbb{R}_+^X$ if and only if $\mu$ is supermodular.