An Introduction to the Choquet integral

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Outline

Set functions and games

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  1. If $\xi(\varnothing) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\overline{\xi}} = \xi$;
  2. If ξ is additive, then $\overline{\xi} = \xi$ (ξ is *self-conjugate*).

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  - $A \subseteq X$: event
  - $\mu(A)$: uncertainty that the event A contains the outcome of an experiment, with $\mu(A) = 0$ indicating total uncertainty, and $\mu(A) = 1$ indicating that there is no uncertainty.

Examples

Example

Let X be a set of firms. Certain firms may form a coalition in order to control the market for a given product. Then $\mu(A)$ may be taken as the annual benefit of the coalition A. $\diamondsuit$

Example

Let X be a set of voters in charge of electing a candidate for some important position (president, director, etc.) or voting a bill by a yes/no decision. Before the election, groups of voters may agree to vote for the same candidate (or for yes or no). In many cases (presidential elections, parliament, etc.), these coalitions correspond to the political parties or to alliances among them. If the result of the election is in accordance with the wish of coalition A, the coalition is said to be winning, and we set $\mu(A) = 1$, otherwise it is loosing and $\mu(A) = 0$. $\diamondsuit$

Example

Let X be a set of workers in a factory, producing some goods. The aim is to produce these goods as much as possible in a given time (say, in one day). Then $\mu(A)$ is the number of goods produced by the group A in one day. Since the production needs in general the collaboration of several workers with different skills, it is likely that $\mu(A) = 0$ if A is a singleton or a too small group. $\diamondsuit$

Examples

Example

David throws a dice, and wonders what number will show. Here $X = \{1, 2, 3, 4, 5, 6\}$, and $\mu(\{1, 3, 5\})$ quantifies the uncertainty of obtaining an odd number. $\diamondsuit$

Example

A murder has been committed. After some investigation, it is found that the guilty is either Alice, Bob or Charles. Then $X = \{\text{Alice, Bob, Charles}\}$, and $\mu(\{\text{Bob, Charles}\})$ quantifies the degree to which it is "certain" (the precise meaning of this word being conditional on the type of capacity used) that the guilty is Bob or Charles. $\diamondsuit$

Example

Glenn is an amateur of antique chinese porcelain. He enters a shop and sees a magnificent vase, wondering how old (and how expensive) this vase could be. Then X is the set of numbers from, say -3000 to 2014, i.e., the possible date expressed in years A.C. when the vase was created. For example, $\mu([1368, 1644])$ indicates tho what degree it is certain that it is a vase of the Ming period. $\diamondsuit$

Example

Leonard is planning to go to the countryside tomorrow for a picnic. He wonders if the wheather will be favorable or not. Here X is the set of possible states of the weather, like "sunny", "rainy", "cloudy", and so on. For example, $\mu(\{\text{sunny, cloudy}\})$ indicates to what degree of certainty it will not rain, and so if the picnic is conceivable or not. $\diamondsuit$

Properties

1. v is *superadditive* if for any $A, B \in 2^X$, $A \cap B = \varnothing$,
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$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcap_{i \in I} A_i\Big).$$
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3. *$v \geqslant 0$ and supermodular implies that v is monotone.*

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- The number of antichains in 2^X with $|X| = n$ is the Dedekind number $M(n)$

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

Unanimity games

- Let $A \subseteq X$, $A \neq \emptyset$. The *unanimity game centered on A* is the game u_A defined by

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

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- Unanimity games are 0-1-valued capacities.

Possibility and necessity measures

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- The conjugate of a possibility measure (resp., a necessity measure) is a necessity measure (resp., a possibility measure).

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- A particular case: the *λ-measure* ($\lambda > -1$) is a normalized capacity satisfying for every $A, B \in 2^X$, $A \cap B = \varnothing$

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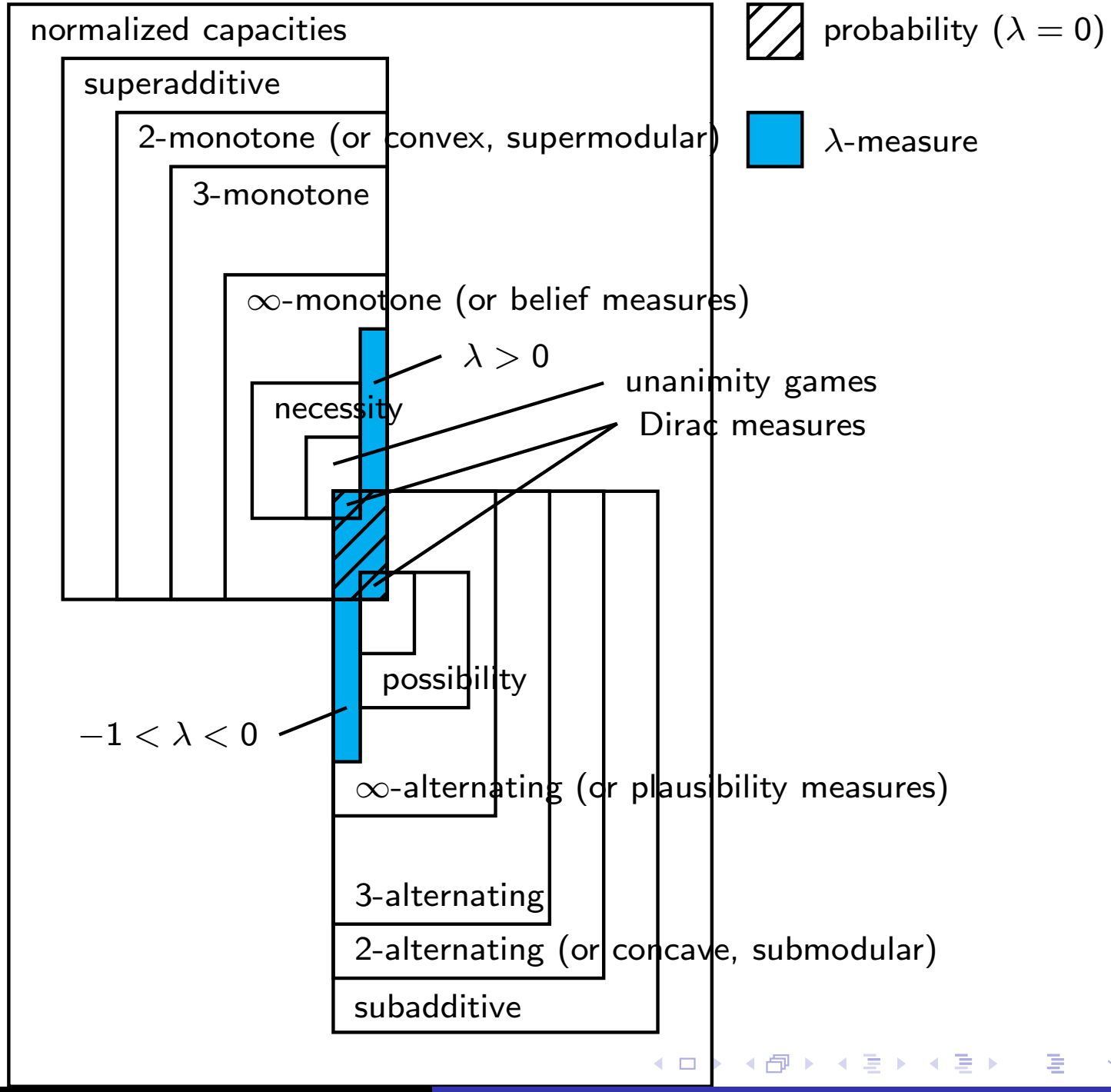
- A λ-measure is a belief measure if and only if $\lambda \geqslant 0$, and is a plausibility measure otherwise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probability ($\lambda = 0$)
λ-measure
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3-monotone
∞-monotone (or belief measures)
$\lambda > 0$
unanimity games
Dirac measures
necessity
possibility
$-1 < \lambda < 0$
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2-alternating (or concave, submodular)
subadditive

The Möbius transform

Definition

Let ξ be a set function on X. The *Möbius transform* or *Möbius inverse* of ξ is a set function m^ξ on X defined by

$$m^\xi(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \xi(B) \tag{1}$$

for every $A \subseteq X$.

Given m^ξ, it is possible to recover ξ by the formula

$$\xi(A) = \sum_{B \subseteq A} m^\xi(B) \qquad (A \subseteq X). \tag{2}$$

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5. v is a nonnegative totally monotone game if and only if $m^v \geqslant 0$.

k-additive games

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A game v on X is said to be *k-additive* for some integer $k \in \{1, \ldots, |X|\}$ if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > k$, and there exists some $A \subseteq X$ with $|A| = k$ such that $m^v(A) \neq 0$.

- A game v is *at most k-additive* for some $1 \leqslant k \leqslant |X|$ if it is k'-additive for some $k' \in \{1, \ldots, k\}$

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- The set of k-additive games on X (resp., capacities, etc.,) is denoted by $\mathcal{G}^k(X)$ (resp., $\mathcal{MG}^k(X)$, etc.)
- We denote by $\mathcal{G}^{\leqslant k}(X), \mathcal{MG}^{\leqslant k}(X)$ the set of at most k-additive games and capacities.

$$\begin{aligned}\mathcal{G}(X) &= \mathcal{G}^1(X) \cup \mathcal{G}^2(X) \cup \cdots \cup \mathcal{G}^{|X|}(X) \\ &= \mathcal{G}^1(X) \cup \mathcal{G}^{\leqslant 2}(X) \cup \cdots \cup \mathcal{G}^{\leqslant |X|}(X)\end{aligned}$$

The vector space of games

For any nonempty $A \subseteq X$ the *identity game δ_A centered at A* is the 0-1-game defined by

$$\delta_A(B) = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The set of identity games $\{\delta_A\}_{A \in 2^X \setminus \{\varnothing\}}$ and the set of unanimity games $\{u_A\}_{A \in 2^X \setminus \{\varnothing\}}$ are bases of $\mathcal{G}(X)$ of dimension $2^{|X|} - 1$.

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- In the basis of identity games, the coordinates of a game v are simply $\{v(A)\}_{A \in 2^X \setminus \{\varnothing\}}$.
- We have for any game $v \in \mathcal{G}(X)$
$$v(B) = \sum_{A \in 2^X \setminus \{\varnothing\}} \lambda_A u_A(B) = \sum_{A \subseteq B, A \neq \varnothing} \lambda_A \qquad (B \subseteq X).$$
It follows that the coefficients of a game v in the basis of unanimity games are its Möbius transform: $\lambda_A = m^v(A)$ for all $A \subseteq X, A \neq \varnothing$

Outline

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$$\begin{aligned} f &= \sum_{i=1}^{n} a_i 1_{\{x \in X \,:\, f(x) = a_i\}} \\ &= \sum_{i=1}^{n} (a_i - a_{i-1}) 1_{\{x \in X \,:\, f(x) \geqslant a_i\}} \end{aligned}$$

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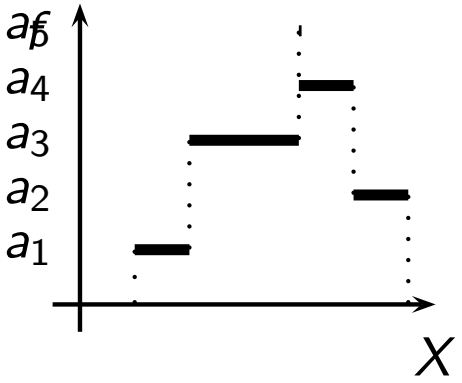

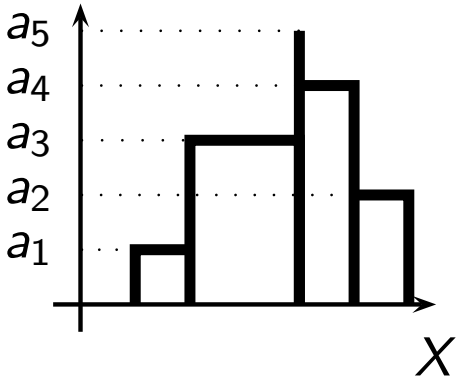

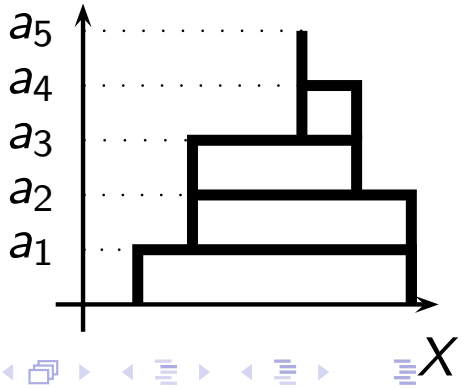

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The Choquet integral

Definition

Let $f \in B^+(\mathcal{F})$ and μ be a capacity on $(X, \mathcal{F})$. The *Choquet integral* of f w.r.t. μ is defined by

$$\int f \, \mathrm{d}\mu = \int_0^\infty G_{\mu,f}(t) \, \mathrm{d}t,$$

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A fundamental fact is the following.

Lemma

Let $A \in \mathcal{F}$ (i.e., 1_A is measurable). Then for every capacity μ

$$\int 1_A \, \mathrm{d}\mu = \mu(A).$$

The Sugeno integral

Definition

Let $f \in B^+(\mathcal{F})$ be a function and μ be a capacity on $(X, \mathcal{F})$. The *Sugeno integral* of f w.r.t. μ is defined by

$$\fint f \, \mathrm{d}\mu = \bigvee_{t \geqslant 0} (G_{\mu,f}(t) \wedge t) = \bigwedge_{t \geqslant 0} (G_{\mu,f}(t) \vee t). \tag{3}$$

Remark 1: $>$ can replace $\geqslant$ in the definition of $G_{\mu,f}$.

Remark 2: $\fint 1_A \, \mathrm{d}\mu = \mu(A)$ for any $A \in \mathcal{F}$ holds if μ is normalized.

The case of real-valued functions

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$$\check{\int} \alpha f \, \mathrm{d}\mu = \alpha \check{\int} f \, \mathrm{d}\mu \qquad (\alpha \in \mathbb{R})$$

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- Expression w.r.t. the decumulative function:
$$\int f \,\mathrm{d}\mu = \int_0^\infty \mu(f \geqslant t) \,\mathrm{d}t + \int_{-\infty}^0 \big(\mu(f \geqslant t) - \mu(X)\big) \,\mathrm{d}t.$$

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- From the decumulative function, one finds
$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu(A_i),$$
letting $a_0 = 0$, and
$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} a_i \big(\mu(A_i) - \mu(A_{i+1})\big),$$
with the convention $A_{n+1} = \varnothing$.

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With same notations, one finds

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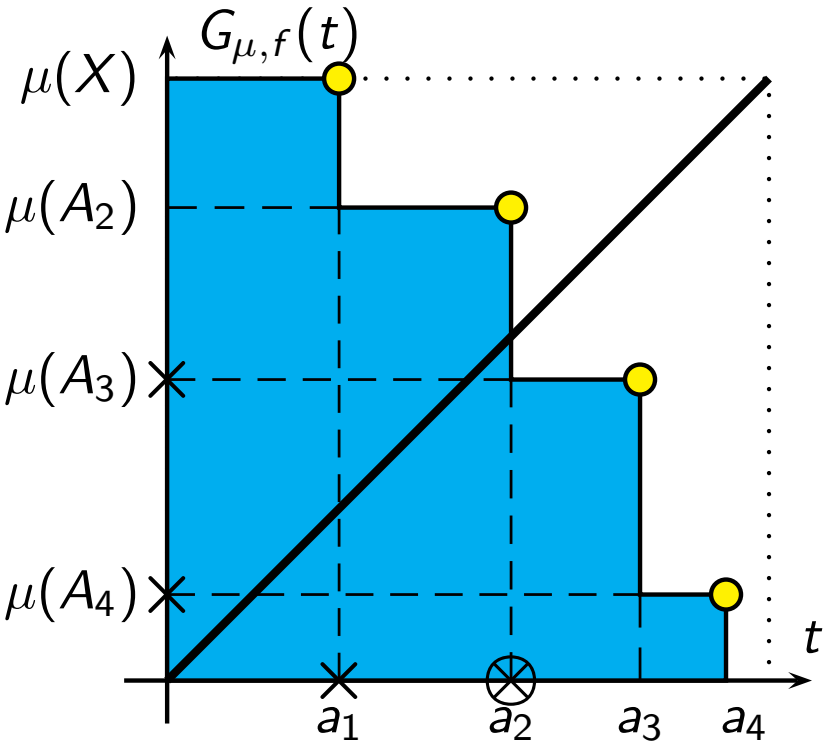

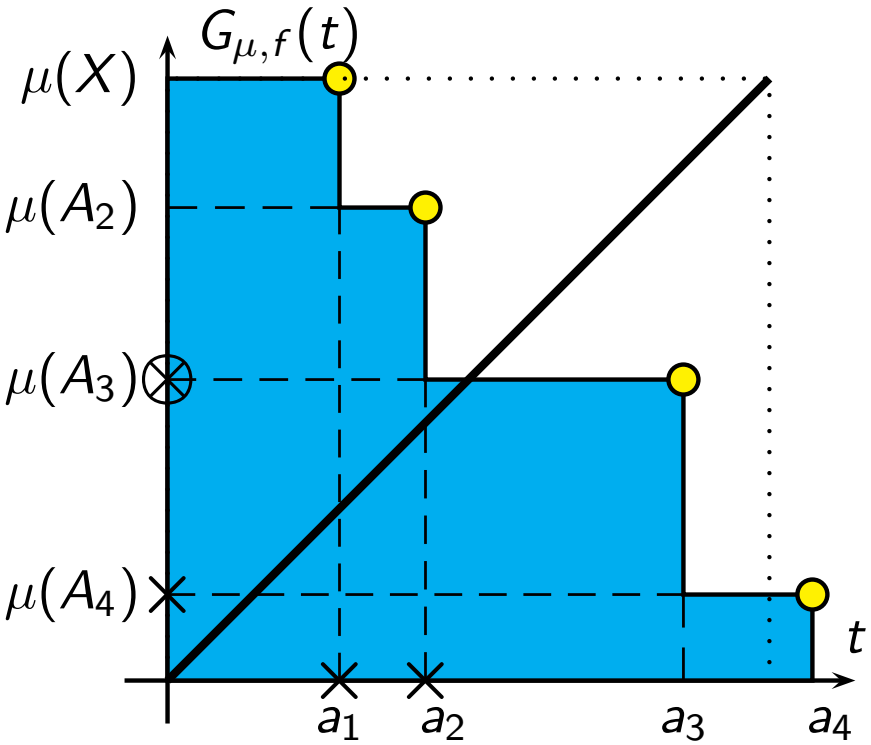

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- From the case of simple functions, we obtain directly

$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) \mu(A_\sigma^\uparrow(i))$$

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  - Finally only x_n remains and he works $f(x_n) - f(x_{n-1})$ hours.

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  - Then x_2 leaves and the group $X \setminus \{x_1, x_2\} = \{x_3, \ldots, x_n\}$ works in addition $f(x_3) - f(x_2)$, etc.,
  - Finally only x_n remains and he works $f(x_n) - f(x_{n-1})$ hours.
- Then the total production is $f(x_1)\mu(X) + (f(x_2) - f(x_1))\mu(\{x_2, \ldots, x_n\}) + (f(x_3) - f(x_2))\mu(\{x_3, \ldots, x_n\}) + \cdots + (f(x_n) - f(x_{n-1}))\mu(\{x_n\}) = \int f \, \mathrm{d}\mu$.

Outline

Basic properties of the Choquet integral

- Positive homogeneity:

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- Linearity w.r.t. the game:

$$\int f \,\mathrm{d}(v+\alpha v') = \int f \,\mathrm{d}v + \alpha \int f \,\mathrm{d}v' \qquad (v, v' \in \mathcal{BV}(\mathcal{F}), \alpha \in \mathbb{R})$$

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Let f be a function in $B^+(\mathcal{F})$, and μ a capacity on $(X, \mathcal{F})$.

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Comonotonic functions

- Two functions $f, g : X \to \mathbb{R}$ are *comonotonic* if there is no $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$

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Theorem

Let f, g be comonotonic functions on X (finite). Then for any game v, the Choquet integral is comonotonically additive, and the Sugeno integral is comonotonically maxitive and minitive for any capacity μ:

$$\int (f+g)\,\mathrm{d}v = \int f\,\mathrm{d}v + \int g\,\mathrm{d}v$$

$$⨍ (f \vee g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \;\vee\; ⨍ g\,\mathrm{d}\mu$$

$$⨍ (f \wedge g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \;\wedge\; ⨍ g\,\mathrm{d}\mu.$$

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for all $f, g : X \to \mathbb{R}$;
4. The Choquet integral is concave, that is,
$$\int (\lambda f + (1-\lambda)g)\,\mathrm{d}v \geqslant \int \lambda f\,\mathrm{d}v + (1-\lambda)\int g\,\mathrm{d}v$$
for all $\lambda \in [0,1]$, $f, g : X \to \mathbb{R}$.

Supermodular capacities

5. The Choquet integral yields the lower expected value on the core of v:

$$\int f \, \mathrm{d}v = \min_{\phi \in \mathrm{core}(v)} \int f \, \mathrm{d}\phi, \tag{4}$$

where $\mathrm{core}(v)$ is the set of additive games ϕ on X such that $\phi(X) = v(X)$ and $\phi(S) \geqslant v(S)$ for all $S \in 2^X$.

Maxitivity and minitivity of the Sugeno integral

Theorem

The following holds:

1. $\fint (f \vee g)\,\mathrm{d}\mu = \fint f\,\mathrm{d}\mu \vee \fint g\,\mathrm{d}\mu$ *for all* f, g *if and only if* μ *is maxitive;*
2. $\fint (f \wedge g)\,\mathrm{d}\mu = \fint f\,\mathrm{d}\mu \wedge \fint g\,\mathrm{d}\mu$ *for all* f, g *if and only if* μ *is minitive.*

The Choquet integral in terms of the Möbius transform

Let X be finite, v be a game, and $f : X \to \mathbb{R}$. Then

$$\int f \, \mathrm{d}v = \sum_{A \subseteq X} m^v(A) \bigwedge_{i \in A} f_i.$$

where m^v is the Möbius transform of v.

Outline

Characterization of the Choquet integral

Theorem

(Schmeidler 1986) Let $I : B(\mathcal{F}) \to \mathbb{R}$ be a functional. Define the set function $v(A) = I(1_A)$ on $\mathcal{F}$. The following propositions are equivalent:

1. *I is monotone and comonotonically additive;*
2. *v is a capacity, and for all $f \in B(\mathcal{F})$, $I(f) = \int f \, \mathrm{d}v$.*

Characterization of the Sugeno integral

Theorem

Let $|X| = n$, $\mathcal{F} = 2^X$, and let $I : (\mathbb{R}_+)^X \to \mathbb{R}_+$ be a functional. Define the set function $\mu(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:

1. *I is comonotonically maxitive, satisfies $I(\alpha 1_A) = \alpha \wedge I(1_A)$ for every $\alpha \geqslant 0$ and $A \subseteq X$ (hat function property), and $I(1_X) = 1$;*
2. *μ is a normalized capacity on X and $I(f) = \fint f \, \mathrm{d}\mu$.*

Outline

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The concave integral (Lehrer 2009)

We consider $|X| = n$, $\mathcal{F} = 2^X$.

Definition

Let $f : X \to \mathbb{R}_+$ and μ be a capacity. The *concave integral* of f w.r.t. μ is given by:

$$\int^{\text{cav}} f \, \mathrm{d}\mu = \sup \left\{ \sum_{S \subseteq X} \alpha_S \mu(S) \; : \; \sum_{S \subseteq X} \alpha_S 1_S = f, \quad \alpha_S \geqslant 0, \forall S \subseteq X \right\}. \tag{5}$$

Nota: "sup" can be replaced by "max".

The example of workers in a factory revisited

- $X = \{1, 2, 3\}$, let μ on X be defined by $\mu(1) = \mu(2) = \mu(3) = 0.2$, $\mu(12) = 0.9$, $\mu(13) = 0.8$, $\mu(23) = 0.5$ and $\mu(123) = 1$

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- How should the workers organize themselves in teams so as to maximize the total production while not exceeding their alloted time?
- The answer is given by the concave integral: team $\{1, 2\}$ is working 0.4 unit of time and team $\{1, 3\}$ is working 0.6 unit of time, which yields

$$0.9 \cdot 0.4 + 0.8 \cdot 0.6 = 0.84$$

The example of workers in a factory revisited

- By contrast, the Choquet integral computes the total productivity under the constraint that the teams form a specific chain, in this case the teams are $\{1, 2, 3\}$, $\{1, 3\}$ and $\{1\}$ for durations 0.4, 0.2 and 0.4 respectively, yielding
$$0.4 + 0.8 \cdot 0.2 + 0.2 \cdot 0.4 = 0.64$$

Properties

The following properties hold for the concave integral:

1. For every capacity μ, the concave integral $\int^{\mathrm{cav}} \cdot \, \mathrm{d}\mu$ is a concave and positively homogeneous functional, and satisfies $\int^{\mathrm{cav}} 1_S \, \mathrm{d}\mu \geqslant \mu(S)$ for all $S \in 2^X$;

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$$\int f \, \mathrm{d}\mu \leqslant \int^{\mathrm{cav}} f \, \mathrm{d}\mu,$$
and equality holds for every $f \in \mathbb{R}_+^X$ if and only if μ is supermodular.