Fuzzy measures and integrals in MCDM

Michel GRABISCH
Paris School of Economics, University of Paris I
106-112, Bd. de l'Hôpital, 75013 Paris, France.
Email: michel.grabisch@univ-paris1.fr

1 Introduction

Most of the methods in multicriteria decision making use the weighted arithmetic mean (weighted sum) when criteria have to be aggregated. This simple aggregation function, used from the ancient Greeks, is universally used and has the advantage of simplicity and readability, in the sense that any decision maker without special knowledge can understand the meaning of its parameters, namely the weights on the criteria. However, the drawbacks of the weighted sum have been pointed out many times: for instance its sensitivity to extreme values, and more importantly, its inability to explore the concave parts of the Pareto frontier in multiobjective optimization problems. On a more practical side, the weighted sum is unable to translate behaviors in aggregation like: favor high scores; focus on weak points; eliminate alternative if criterion 3 is not well satisfied; if criteria 1 and 2 are satisfied, do not overestimate their importance; etc. The first two examples in the previous list are still relatively simple since they can be handled by a weighted sum on the ordered list of scores in increasing order: this is called the ordered weighted average (OWA), and it contains as particular cases the median, the minimum and the maximum. However, the last two examples are more complex and cannot be treated, neither by the weighted sum nor by an OWA. They are typical examples of *interaction* between criteria. Interaction means that the way the score on a criterion is handled depends on the value of the scores on the other criteria. A typical example given in [4] is the following: suppose that some students are evaluated on three topics, which are mathematics, physics and literature, with an emphasis on scientific topics. There is some interaction between mathematics and physics in the sense that, since usually students good at maths are also good at physics and vice versa, a good mark in both math and physics should not be overestimated with respect to literature. On the contrary, since scientific topics are more important, bad marks in both mathematics and physics become very important. The existence of interaction phenomena, as well as the inability of the weighted sum to represent them, have been recognized from a long time ago, and the absence of an adequate tool to take them into account has provoked the appearance of the following general rule for the practitioner: *choose/define criteria so that they are independent (no interaction).* Obviously, this rule cannot always be applied, or leads to excessive simplification.

The appearance of the Choquet integral in MCDM has permitted to solve this problem, and has brought a precise definition of what interaction is. The Choquet integral not

only generalizes both the weighted sum and the OWA, but also can model the examples of interaction described above. So far, many theoretical studies as well as real applications have been done (see the overviews [4, 8, 9] and [10, Ch. 5] for further references and details).

2 Fuzzy measures and integrals

We consider a set of criteria $N = \{1, \ldots, n\}$, and denote the set of its subsets by 2^N. An alternative is an element of $X = X_1 \times \cdots \times X_n$, where X_i is the set of values for attribute i. We write $x = (x_1, \ldots, x_n)$ for $x \in X$. We make the assumption that the utility functions $u_i : X_i \to \mathbb{R}$ have been determined, up to a positive affine transformation.

A *fuzzy measure* [17] or *capacity* [2] on N is a mapping $\mu : 2^N \to [0, 1]$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$

(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity).

In applications in MCDM, $\mu(A)$ is often interpreted as the *weight of importance* of the group or coalition $A \subseteq N$. This is however an imprecise statement, and it should be interpreted in a more rigorous way as follows. Suppose that for each criterion $i \in N$ there exist two levels with an absolute meaning, e.g., a *neutral level*, denoted by $\mathbf{0}_i$, and a *satisfactory level*, denoted by $\mathbf{1}_i$ (like in the MACBETH methodology [1]). These levels are elements of X_i, and represent respectively a level which is neither good nor bad, and a level which would make the decision maker totally satisfied if he could obtain it. Alternatively, the neutral level can be replaced by the *unsatisfactory level*, which is a level which is not acceptable by the decision maker. We set for convenience $u_i(\mathbf{1}_i) = 1$ and $u_i(\mathbf{0}_i) = 0$ for all $i \in N$. We consider now for any subset $A \subseteq N$ the *binary alternative* $(\mathbf{1}_A, \mathbf{0}_{-A})$, whose coordinates are $\mathbf{1}_i$ for all $i \in A$, and $\mathbf{0}_i$ otherwise. This alternative is satisfactory for all criteria $i \in A$ and neutral (or unsatisfactory) otherwise. Then $\mu(A)$ is interpreted as the overall evaluation of $(\mathbf{1}_A, \mathbf{0}_{-A})$.

Under this interpretation, the axioms defining a fuzzy measure are natural: the overall evaluation of an alternative being neutral (or unsatisfactory) for every criterion should be 0, and its evaluation should be 1 if all criteria are satisfactory. Moreover, turning one criterion on the neutral level to a satisfactory level cannot decrease the overall evaluation (monotonicity condition).

In summary, a fuzzy measure determines the evaluation of every binary alternative. It remains to determine the evaluation of any other alternative. This is done through the use of the *Choquet integral* [2], defined as follows. Consider an alternative $x = (x_1, \ldots, x_n)$ and the vector $u(x) = (u_1(x_1), \ldots, u_n(x_n))$ of its values for the utility functions. We call $u_i(x_i)$ the *score* of x w.r.t. criterion i, and we denote it by a_i if there is no ambiguity. Rearrange the criteria so that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then the Choquet integral of $u(x) = (a_1, \ldots, a_n)$, provided all components a_i are nonnegative, is computed as follows:

$$\mathcal{C}_\mu(a_1, \ldots, a_n) = a_1 \underbrace{\mu(N)}_{=1} + \sum_{i=2}^{n} (a_i - a_{i-1}) \mu(\{i, \ldots, n\}), \quad (a_1, \ldots, a_n \geq 0) \tag{1}$$

or equivalently

$$\mathcal{C}_\mu(a_1,\ldots,a_n) = \sum_{i=1}^{n-1} a_i(\mu(\{i,\ldots,n\}) - \mu(\{i+1,\ldots,n\})) + a_n\mu(\{n\}). \tag{2}$$

Example 1. Consider $N = \{1,2,3\}$ and the fuzzy measure μ defined in the following table

A	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$\mu(A)$	0	0.2	0.2	0.4	0.5	0.6	0.7	1

Consider now the vector $u(x) = (3,1,6)$. Criteria should be first rearranged in the following order: 2, 1, 3 since $u_2(x_2) < u_1(x_1) < u_3(x_3)$. Then, using (1), the Choquet integral is computed as follows:

$$\mathcal{C}_\mu(u(x)) = 1 \times 1 + (3-1) \times \mu(\{1,3\}) + (6-3) \times \mu(\{3\}) \tag{3}$$

$$= 1 + 2 \times 0.6 + 3 \times 0.4 = 3.4 \tag{4}$$

Thus, the overall evaluation of x is 3.4. Observe that this value lies in $[1,6]$, the interval of minimal and maximal utility values on the criteria.

Formula (2) clearly shows that the Choquet integral is a kind of weighted sum, the weights being $\mu(\{i,\ldots,n\}) - \mu(\{i+1,\ldots,n\})$. However, be careful that these weights depend on x since criteria must be first reordered to get an increasing sequence of scores. Hence, precisely, since there are $n!$ possible orderings, the Choquet integral is an assembling of $n!$ weighted sums, each for a specific order of criteria (and this explains why the Choquet integral can model interaction). It can be proved that the Choquet integral is nevertheless continuous. Other important properties are:

(i) $\mathcal{C}_\mu(u(\mathbf{1}_A, \mathbf{0}_{-A})) = \mu(A)$, for all $A \subseteq N$;

(ii) $\mathcal{C}_\mu(\alpha u(x)) = \alpha \mathcal{C}_\mu(u(x))$ for all $\alpha > 0$;

(iii) $u_i(x_i) \leq u_i(x'_i)$ for $i = 1,\ldots,n$ implies $\mathcal{C}_\mu(u(x)) \leq \mathcal{C}_\mu(u(x'))$;

(iv) $\min_{i=1}^n u_i(x_i) \leq \mathcal{C}_\mu(u(x)) \leq \max_{i=1}^n u_i(x_i)$.

The first property says that the Choquet integral extends the overall evaluation provided by the fuzzy measure. The second one says that the Choquet integral is compatible with a ratio scale, and the third one that an increase of the scores cannot lead to a decrease of the overall evaluation. The last one says that the overall evaluation is comprised between the minimum and the maximum of all scores. The two last properties imply *idempotency*, that is, $\mathcal{C}_\mu(a, a \ldots, a) = a$ for any $a \in \mathbb{R}_+$.

So far the Choquet integral is defined for nonnegative arguments, preventing it to be used with interval scales. In order to extend it to any real-valued argument, there are two possibilities. The first one is called the *symmetric integral*, denoted by $\check{\mathcal{C}}_\mu$. Its name is due to the fact that $\check{\mathcal{C}}_\mu(-u(x)) = -\check{\mathcal{C}}_\mu(u(x))$, and it is compatible with ratio scales. However, it is not compatible with interval scales since it is not invariant to positive affine

transformations: $\check{\mathcal{C}}_\mu(\alpha u(x)+\beta) \neq \alpha \check{\mathcal{C}}_\mu(u(x))+\beta$ in general, for $\alpha > 0$ and $\beta \in \mathbb{R}$. Its expression is, using the notation $u(x) = (a_1, \ldots, a_n)$:

$$\check{\mathcal{C}}_\mu(a_1, \ldots, a_n) = \sum_{i=1}^{p-1} (a_i - a_{i+1})\mu(\{1, \ldots, i\}) + a_p \mu(\{1, \ldots, p\}) \\ + a_{p+1}\mu(\{p+1, \ldots, n\}) + \sum_{i=p+2}^{n} (a_i - a_{i-1})\mu(\{i, \ldots, n\}),$$

having rearranged the criteria so that $a_1 \leq \cdots \leq a_p < 0 \leq a_{p+1} \leq \cdots \leq a_n$.

The second possibility is called the asymmetric integral, and its definition merely amounts to keep the same formula (1) for $(a_1, \ldots, a_n) \in \mathbb{R}^n$. The asymmetric integral is invariant to positive affine transformations and is therefore compatible with interval scales.

3 Relation with other aggregation functions

The Choquet integral contains as particular case two important families of aggregation functions: the weighted sum (weighted arithmetic mean, abbreviated by WAM) and the ordered weighted average (OWA) [19]. We recall their definitions:

$$\mathrm{WAM}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i$$

$$\mathrm{OWA}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{(i)},$$

where $(a_{(1)}, \ldots, a_{(n)})$ is the vector of scores rearranged in nondecreasing order, i.e., $a_{(1)} \leq \cdots \leq a_{(n)}$, and $w_1, \ldots, w_n$ are weights in $[0, 1]$. We recall that OWA contains as particular case the arithmetic mean (with $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$), the minimum ($w_1 = 1$, $w_2 = \cdots = w_n = 0$), the maximum ($w_1 = \cdots = w_{n-1} = 0$, $w_n = 1$), the median and every order statistic. We stress the fact that the weights in OWA are not weights on criteria, but weights on the rank of the scores. Hence, high values for weights with low (respectively, high) indices indicate that the OWA focuses on the smallest (respectively, greatest) scores obtained, i.e., it has a conjunctive (respectively, disjunctive) behavior.

The weighted arithmetic mean is recovered with an additive fuzzy measure, while the ordered weighted average is recovered with a symmetric fuzzy measure [14]. We define these terms below.

An *additive fuzzy measure* μ satisfies $\mu(A) = \sum_{i \in A} \mu(\{i\})$ for every $A \subseteq N$. It is then easy to see that the Choquet integral reduces to the weighted arithmetic mean with $w_i = \mu(\{i\})$, for $i = 1, \ldots, n$. A *symmetric fuzzy measure* μ satisfies $\mu(A) = \mu(B)$ whenever $|A| = |B|$, i.e., μ depends only on the cardinality of the subsets. Letting $m_i = \mu(A)$ with $|A| = i$, we obtain that the Choquet integral reduces to an ordered weighted average with $w_i = m_{n-i+1} - m_{n-i}$, $i = 2, \ldots, n$, and $w_1 = 1 - \sum_{i=2}^{n} w_i$.

This shows the versatility of the Choquet integral, able to both take into account importance of criteria as well as importance of ranks. In particular, it can be more or less disjunctive or conjunctive.

4 Shapley value and interaction indices

The versatility of the Choquet integral is hindered by its exponential complexity. Indeed, the size of the model is in $O(2^n)$, which makes its use impossible for high values of n, and render its interpretation nearly impossible for the decision maker. Therefore, one needs some tool to interpret fuzzy measures, and also submodels of polynomial size. This section focusses on the first point, while the second one will be addressed in the next section.

The most popular tool in MCDM being the weighted sum, decision makers are used to deal with weights of importance for criteria. The Choquet integral being much more complex than a weighted sum, is there some similar notion to importance weights? An adequate answer to this question comes from cooperative game theory, through the notion of Shapley value [16, 15]. Since μ quantifies the overall score of alternatives being satisfactory on some set of criteria, a criterion i should be said to be important if whenever i enters a group of criteria A, the difference $\mu(A \cup i) - \mu(A)$ is high. The *Shapley importance index* $\phi_i(\mu)$ for criterion i is a weighted average of this difference taken over all possible $A \subseteq N \setminus i$:

$$\phi_i(\mu) = \sum_{A \subseteq N \setminus i} \frac{|A|!(n - |A| - 1)!}{n!} (\mu(A \cup i) - \mu(A)),$$

and the vector $(\phi_1(\mu), \ldots, \phi_n(\mu))$ is called the *Shapley value* of μ. The weights in the average ensure that $\sum_{i=1}^n \phi_i(\mu) = 1$ and invariance to permutations on N. Also, $\phi_i(\mu) \geq 0$, therefore the coefficients $\phi_1(\mu), \ldots, \phi_n(\mu)$ act like weights of importance in a weighted sum.

The Shapley value offers a simple summary of the fuzzy measure μ, but is by no means a substitute to it. That is, infinitely many fuzzy measures have the same Shapley value. We need therefore some other quantity to enrich the summary. This is brought by the *interaction index*, a notion which is intuitively appealing in MCDM. Let us first consider two criteria i, j. By a *positive interaction* or *synergy* between i and j, we mean that the satisfaction of both criteria is much more valuable than the satisfaction of them separately. We may say that they are *complementary*. This is translated quantatively as follows, taking any set $A \subseteq N \setminus \{i, j\}$:

$$\mu(A \cup \{i, j\}) - \mu(A) \geq (\mu(A \cup i) - \mu(A)) + (\mu(A \cup j) - \mu(A)),$$

which can be rewritten as

$$\mu(A \cup \{i, j\}) - \mu(A \cup i) - \mu(A \cup j) + \mu(A) \geq 0.$$

Similarly, a *negative interaction* or synergy between i and j means that the satisfaction of both is not that better than the satisfaction of one of them, leading to the reverse inequality, and in this case we may say that they are *redundant* or *substitutable*. The case of equality means that the added value by both criteria is exactly the sum of the individual added values, hence there is no interaction between i and j. The *interaction index* $I_{ij}(\mu)$ [13] is obtained by taking a weighted average of the above expression for all possible groups $A \subseteq N \setminus \{i, j\}$:

$$I_{ij}(\mu) = \sum_{A \subseteq N \setminus \{i,j\}} \frac{|A|!(n - |A| - 2)!}{(n - 1)!} (\mu(A \cup \{i, j\}) - \mu(A \cup i) - \mu(A \cup j) + \mu(A)).$$

Note that the weights are similar to those of the Shapley value.

The Shapley value together with the interaction indices for all pairs $i, j \in N$ gives a good summary of the fuzzy measure μ, although still not complete. It is possible to define interaction indices for more than two criteria. The general definition for a group of criteria $B \subseteq N$ is given as follows [6]:

$$I_B(\mu) = \sum_{A \subseteq N \setminus B} \Big(\frac{|A|!(n - |A| - |B|)!}{(n - |B| + 1)!} \sum_{K \subseteq B} (-1)^{|B \setminus K|} \mu(A \cup K) \Big).$$

Note that the Shapley value is a particular case of it when $B = \{i\}$, $i \in N$. It can be shown that the set of 2^n interaction indices $I_B(\mu)$ for $B \subseteq N$ is an exact representation of μ, in the sense that there is a one-to-one correspondence between them.

Example 2. Taking again the fuzzy measure given in Example 1, we find for the interaction indices and Shapley value:

$$\begin{aligned}
\phi_1(\mu) &= \frac{1}{3}(0.2 - 0) + \frac{1}{6}(0.5 - 0.2) + \frac{1}{6}(0.6 - 0.4) + \frac{1}{3}(1 - 0.7) = 0.25 \\
\phi_2(\mu) &= \frac{1}{3}(0.2) + \frac{1}{6}(0.3) + \frac{1}{6}(0.3) + \frac{1}{3}(0.4) = 0.3 \\
\phi_3(\mu) &= \frac{1}{3}(0.4) + \frac{1}{6}(0.4) + \frac{1}{6}(0.5) + \frac{1}{3}(0.5) = 0.45 \\
I_{12}(\mu) &= \frac{1}{2}(0.5 - 0.2 - 0.2 + 0) + \frac{1}{2}(1 - 0.6 - 0.7 + 0.4) = 0.1 \\
I_{13}(\mu) &= \frac{1}{2}(0.6 - 0.2 - 0.4 + 0) + \frac{1}{2}(1 - 0.5 - 0.7 + 0.2) = 0 \\
I_{23}(\mu) &= \frac{1}{2}(0.7 - 0.2 - 0.4 + 0) + \frac{1}{2}(1 - 0.5 - 0.6 + 0.2) = 0.1 \\
I_{123}(\mu) &= 1 - 0.5 - 0.6 - 0.7 + 0.2 + 0.2 + 0.4 - 0 = 0.
\end{aligned}$$

One sees that criterion 3 is the most important, criteria 1 and 3 are independent, and there is a slight complementarity between criteria 1 and 2, and between 2 and 3.

5 k-additive fuzzy measures

The main advantage of the representation of a fuzzy measure μ with interaction indices is that it gives a clear meaning to μ: the Shapley value expresses the importance of criteria, the interaction index for pairs of criteria expresses when they are complementary or redundant, etc. However, the interpretation of $I_B(\mu)$ is less and less clear as the size of B grows: It seems hardly possible to have a clear view of what expresses an interaction index for 4 criteria. This suggests that one should truncate the representation up to some level. We call *k-additive fuzzy measure* [6] a fuzzy measure for which the interaction indices are zero for any group of criteria of size greater than k. Hence, a 1-additive measure reduces to a vector of importance weights, and is therefore equivalent to an additive fuzzy measure. In this case, we know that the Choquet integral is merely a weighted sum. More interesting is a 2-additive fuzzy measure, since complementarity and redundancy within

pairs of criteria can be expressed. It can be shown that the Choquet integral takes the following particular form [5]:

$$\mathcal{C}_\mu(a_1,\dots,a_n) = \sum_{i,j:I_{ij}>0}(a_i\wedge a_j)I_{ij} + \sum_{i,j:I_{ij}<0}(a_i\vee a_j)|I_{ij}| + \sum_{i\in N} a_i(\phi_i - \frac{1}{2}\sum_{j\neq i}|I_{ij}|) \quad (5)$$

where $\wedge,\vee$ stand for minimum and maximum respectively, and we have omitted μ in $I_{ij}(\mu)$ and $\phi_i(\mu)$. This expression offers two important advantages:

(i) It gives a clear meaning to the interaction index in terms of aggregation behavior. A positive interaction between i,j implies a conjunctive aggregation (*both* criteria must be satisfied), while a negative interaction implies a disjunctive behavior (it is enough to satisfy one of the criteria). In addition, one can see that the Shapley value indeed acts like a weight of importance in a weighted sum, however its effect on criterion i is diminished by the sum of interaction indices pertaining to i. It follows that if criterion i has no interaction (is independent) with the other criteria, it appears only in the linear part of the Choquet integral with weight ϕ_i.

(ii) The Choquet integral is a sum of terms being either a minimum, a maximum or a monomial of degree 1. It can be shown that all coefficients of these terms are nonnegative, and moreover they sum up to 1: in short, (5) is a convex combination of terms representing elementary behaviors in decision making: disjunction and conjunction of two criteria, and dictators. All these local behaviors can be put in a pie chart, which offers a very clear view for the decision maker.

Finally, observe that the 2-additive model is in $O(n^2)$ since the number of free coefficients defining a 2-additive fuzzy measure is $n-1+\binom{n}{2} = \frac{n^2+n-2}{2}$.

Example 3. We take again the fuzzy measure μ of Example 1, with $u(x) = (3,1,6)$. Indeed, we see by Example 2 that μ is 2-additive. Therefore (5) can be applied, and it gives (see Example 2 for the values of interaction):

$$\begin{aligned}\mathcal{C}_\mu(u(x)) &= 0.1(3\wedge 1) + 0 + 0.1(1\wedge 6) + 3\Big(0.25 - \frac{0.1}{2}\Big) + 1\Big(0.3 - \frac{0.2}{2}\Big) + 6\Big(0.45 - \frac{0.1}{2}\Big)\\ &= 3.4\end{aligned}$$

We observe that as expected the result is the same as our computation in Example 1.

6 Ordinal models

So far it was assumed that utility functions were real-valued functions. In many cases however, scores are given on a finite ordinal scale, like {*bad,medium,good*}. Since the fuzzy measure is interpreted as an overall score on binary alternatives, it follows that the fuzzy measure too is defined on the same ordinal scale. However, the arithmetic operations used in the Choquet integral cannot be used on an ordinal scale, and another kind of integral has to be used, involving in its definition only minimum and maximum, which are the only operations being meaningful on an ordinal scale.

Let us denote the finite ordinal scale by L, with 0 and 1 the lower and upper bounds of L, respectively. Imposing that the overall score be 0 (respectively 1) if all scores on criteria are 0 (respectively 1), it can be shown [12] that the only possible aggregation function using only minimum and maximum is the *Sugeno integral* [17], defined by

$$\mathcal{S}_\mu(a_1, \ldots, a_n) = \bigvee_{i=1}^{n} \Big(a_i \wedge \mu(\{i, \ldots, n\}) \Big)$$

where $(a_1, \ldots, a_n)$ is the vector of scores, and criteria have been reordered so that $a_1 \leq \cdots \leq a_n$, as for the Choquet integral. To be meaningful, the fuzzy measure and all utility functions must take values in L, with $\mu(N) = 1$.

The Sugeno integral keeps many of the properties of the Choquet integral:

(i) $\mathcal{S}_\mu(u(\mathbf{1}_A, \mathbf{0}_{-A})) = \mu(A)$, for all $A \subseteq N$;

(ii) $\mathcal{S}_\mu(\alpha \wedge u(x)) = \alpha \wedge \mathcal{S}_\mu(u(x))$, $\mathcal{S}_\mu(\alpha \vee u(x)) = \alpha \vee \mathcal{S}_\mu(u(x))$;

(iii) $u_i(x_i) \leq u_i(x'_i)$ for $i = 1, \ldots, n$ implies $\mathcal{S}_\mu(u(x)) \leq \mathcal{S}_\mu(u(x'))$;

(iv) $\min_{i=1}^n u_i(x_i) \leq \mathcal{S}_\mu(u(x)) \leq \max_{i=1}^n u_i(x_i))$.

However, an important drawback of the Sugeno integral is that it can happen that two alternatives x, x' have the same overall evaluation although $u_i(x_i) > u_i(x'_i)$ for all $i \in N$ (see Example 4).

Example 4. Let us consider the scale $L = \{0, 1, \ldots, 10\}$ and use the fuzzy measure of Example 1 multiplied by 10, together with $u(x) = (3, 1, 6)$. We obtain:

$$\mathcal{S}_\mu(u(x)) = (1 \wedge 10) \vee (3 \wedge 6) \vee (6 \wedge 4) = 4.$$

Note that this is relatively close to 3.4 found in Example 1 with the Choquet integral. Now consider $u(x') = (2, 0, 5)$. We find

$$\mathcal{S}_\mu(u(x')) = (0 \wedge 10) \vee (2 \wedge 6) \vee (5 \wedge 4) = 4,$$

i.e., $\mathcal{S}_\mu(u(x)) = \mathcal{S}_\mu(u(x'))$ although x has better scores than x' on each criterion.

7 Identification of the model

Once the utility functions have been determined, the identification of the model amounts to the identification of the fuzzy measure, which most of the time will be the solution of an optimization problem under constraints. We give here two main types of optimization problem (we refer the reader to [7] and [10, Ch. 11] for a survey and more detail):

(i) **Minimizing the total squared error:** Denoting by $\mathcal{O}$ the set of available alternatives, the data set consists of the vectors $u(x)$ of scores on criteria together with the overall score $y(x)$, for all $x \in \mathcal{O}$. The aim is to minimize the distance to the desired overall scores:

$$\min_\mu \sum_{x \in \mathcal{O}} \big(\mathcal{C}_\mu(u(x)) - y(x) \big)^2.$$

The constraints of the problem are the monotonicity constraints of the fuzzy measure:

$$\mu(A \cup i) - \mu(A) \geq 0, \quad \forall i \in N, \forall A \subseteq N \setminus i$$

and possibly some information on the Shapley value and interaction indices, like a partial order or the sign of some interaction index.

(ii) **Maximum separation of alternatives:** the data consists of a preference relation $\succeq$ on a set $\mathcal{O}$ of alternatives. The objective is to maximize the distance between the overall scores obtained for these alternatives, provided they are compatible with the preference relation (constraints):

$$\begin{aligned} \max \ & \epsilon \\ \text{subject to } & \mathcal{C}_\mu(u(x)) - \mathcal{C}_\mu(u(x')) \geq \delta + \epsilon, \quad \forall x, x' \in \mathcal{O} \text{ such that } x \succeq x' \\ & \mu(A \cup i) - \mu(A) \geq 0, \quad \forall i \in N, \forall A \subseteq N \setminus i, \end{aligned}$$

where δ is a fixed threshold. As above, additional constraints can be given, pertaining to the Shapley value and interaction indices. It should be noted that contrarily to the first case, this optimization problem may have no solution.

The above optimization problems concern solely the Choquet integral, and are either quadratic (in the first case) or linear programs (in the second case). For the first case, standard quadratic programming methods can be avoided, and a faster (although suboptimal) heuristic algorithm (HLMS) can be used instead (see [3, 11]). For the Sugeno integral, only heuristic methods like Genetic Algorithms can be used (see, e.g., [18]).

References

[1] C. A. Bana e Costa and J.-C. Vansnick. The MACBETH approach: basic ideas, software and an application. In N. Meskens and M. Roubens, editors, *Advances in Decision Analysis*, pages 131–157. Kluwer Academic Publishers, 1999.

[2] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[3] M. Grabisch. A new algorithm for identifying fuzzy measures and its application to pattern recognition. In *Int. Joint Conf. of the 4th IEEE Int. Conf. on Fuzzy Systems and the 2nd Int. Fuzzy Engineering Symposium*, pages 145–150, Yokohama, Japan, March 1995.

[4] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European J. of Operational Research*, 89:445–456, 1996.

[5] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[6] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[7] M. Grabisch, I. Kojadinovic, and P. Meyer. A review of capacity identification methods for Choquet integral based multi-attribute utility theory — applications of the Kappalab R package. *Eur. J. of Operational Research*, 186:766–785, 2008.

[8] M. Grabisch and Ch. Labreuche. Fuzzy measures and integrals in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis*, pages 563–608. Kluwer Academic Publishers, 2005.

[9] M. Grabisch and Ch. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Annals of Operations Research*, 175:247–286, 2010. doi: 10.1007/s10479-009-0655-8.

[10] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Number 127 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.

[11] M. Grabisch and E. Raufaste. An empirical study of statistical properties of Choquet and Sugeno integrals. *IEEE Tr. on Fuzzy Systems*, 16(4):839–850, 2008.

[12] J.-L. Marichal. Weighted lattice polynomials. *Discrete Mathematics*, 309:814–820, 2009.

[13] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[14] T. Murofushi and M. Sugeno. Some quantities represented by the Choquet integral. *Fuzzy Sets & Systems*, 56:229–235, 1993.

[15] A. E. Roth, editor. *The Shapley Value. Essays in Honor of Lloyd S. Shapley*. Cambridge University Press, 1988.

[16] L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[17] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

[18] Z. Wang, K. S. Leung, and J. Wang. A genetic algorithm for determining nonadditive set functions in information fusion. *Fuzzy Sets and Systems*, 102:462–469, 1999.

[19] R. R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Systems, Man & Cybern.*, 18:183–190, 1988.