Strategic Network Formation and Network Allocation Rules

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Part 2 - Strategic Network Formation

Strategic network formation

- There are many settings in which choice plays a central role in determining relationships.
- Individuals have discretion in which relationships they form and maintain and how much effort or time they devote to different relationships:
  - trading relationships, political alliances, employer-employee relationships, marriages, professional collaborations, citations, email correspondence, friendship, ...
- The models of strategic formation answer the question why the networks take a particular structure rather than how they take this form.
- Two challenges from a strategic point of view:
  - modeling the costs and benefits that arise from various networks - this enables to model how networks develop in the face of individual incentives to form or severe links, and to provide measures of overall societal welfare
  - predicting how individual incentives translate into network outcome.

Efficiency, Pareto efficiency and Pairwise stability

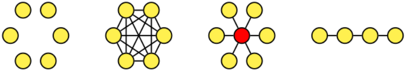
- Efficient network (E) = maximize the total utility of all actors
- Pareto efficient network (PE) = there does not exist any network that would be at least as good as the PE network for all players and strictly better for some players
- Efficiency $\Rightarrow$ Pareto efficiency
- Pairwise stable network (PS) = no player wants to sever a link and no two players both want to add a link.
- Some limitations of PS (that lead to some refinements of PS):
  - PS considers only deviations on a single link at a time
  - PS considers only deviations by at most a pair of players at a time.

Network notation (1/2)

- $N = \{1, 2, \ldots, n\}$ - set of players (agents, nodes)
- Network g – set of pairs $\{i, j\}$ (denoted ij), with $i, j \in N$, $i \neq j$.
- Link ij describes a relationship between i and j
- i and j are directly connected iff $ij \in g$
- Degree $\eta_i(g)$ of i counts the number of links i has in g, i.e.,

$$\eta_i(g) = |\{j \in N \mid ij \in g\}|$$

- Empty network $g^\emptyset$, complete network g^N, star g^*, line g^L

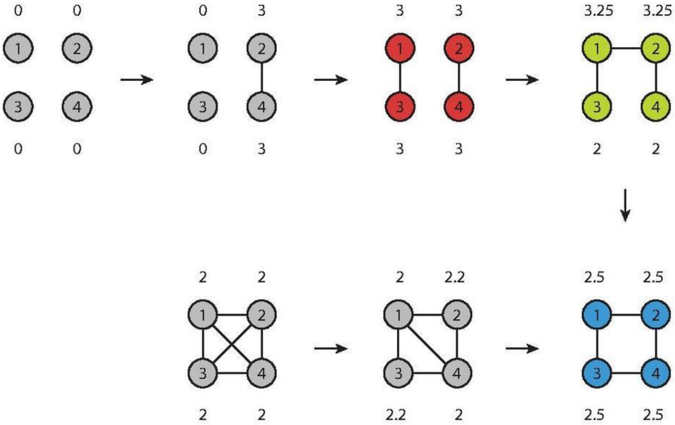


Network notation (2/2)

- Let g^N be the set of all subsets of N of size 2
- The set of all possible networks g on N is $G := \{g | g \subseteq g^N\}$
- Let $u_i : G \to \mathbb{R}$ denote the utility of player i from network g
- $g + ij$ - network obtained by adding link ij to g
- $g - ij$ - network obtained by deleting link ij from g
- A network $g \in G$ is pairwise stable (PS) if:
  1. $\forall\ ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$ and
  2. $\forall\ ij \notin g$, if $u_i(g) < u_i(g + ij)$ then $u_j(g) > u_j(g + ij)$.
- A network $g \subseteq g^N$ is (strongly) efficient (E) if

$$\sum_{i \in N} u_i(g) \geq \sum_{i \in N} u_i(g') \ \textit{for all}\ g' \subseteq g^N.$$

Which networks here are E, PE and PS?



The connections model

- Introduced by Jackson & Wolinsky (1996):

$$u_i^{JW}(g) = \sum_{j \neq i} \delta^{l_{ij}(g)} - c\eta_i(g)$$

with $0 < \delta < 1$ as benefit term, $l_{ij}(g)$ the distance between i and j, $c > 0$ the costs for a link and $\eta_i(g)$ the degree of i.

- Numerous variations of the connections model, e.g., Johnson & Gilles (2000), Carayol & Roux (2005, 2009)
- Generalized "distance-based-model" by Bloch & Jackson (2007):

$$u_i^{dist}(g) = \sum_{j \neq i} f(l_{ij}(g)) - c\eta_i(g)$$

with f nonincreasing in $l_{ij}(g)$.

Efficient networks in the connections model (J & W, 1996)

Proposition

The unique SE network in the symmetric connections model is:

1. the complete network g^N if $c < \delta - \delta^2$
2. a star g^* if $\delta - \delta^2 < c < \delta + \frac{(n-2)\delta^2}{2}$
3. no links if $\delta + \frac{(n-2)\delta^2}{2} < c$.

Pairwise stable networks in the connections model (J & W, 1996)

Proposition

In the symmetric connections model:

1. A pairwise stable graph has at most one (non-empty) component.
2. For $c < \delta - \delta^2$, the unique PS network is the complete graph g^N.
3. For $\delta - \delta^2 < c < \delta$, a star g^* encompassing all players is PS, but not necessarily the unique PS graph.
4. For $\delta < c$, any PS network which is non-empty is such that every player has at least two links (and thus is inefficient).

A dynamic version of the connections model (Watts, 2001)

How can we predict which networks are likely to emerge from a multitude of PS networks?

- refining the equilibrium concept
- examining some dynamic process.

Random ordering over links. At any point in time, any link is as likely as any other to be identified:

- If the link has not yet been added $\rightarrow$ if at least one player involved would benefit from adding it and the other would be at least as well off, then the link is added.
- If the identified link has already been added $\rightarrow$ it is deleted if either player would (myopically) benefit from its deletion.

A dynamic version of the connections model (Watts, 2001)

We can deduce which PS network will be reached in the symmetric connections model under this dynamic process:

- If $c > \delta$, the process ends at the empty network (even if there are non-empty networks strictly preferred by all players to $g^{\emptyset}$).
- If $\delta - c > \delta^2$, the efficient complete network will be reached.

Proposition

Assume that $\delta - \delta^2 < c$. For $3 < n < \infty$, there is a positive probability $0 < p(star) < 1$ that the formation process will converge to a star. However, as n increases, $p(star)$ decreases, and as n goes to infinity, $p(star)$ goes to 0.

The co-author model (Jackson & Wolinsky, 1996)

Nodes are interpreted as researchers and a link represents a collaboration between two researchers.

The utility function of each player i in network g is given by

$$u_i^{co}(g) = \sum_{j:ij \in g} w_i(\eta_i, j, \eta_j) - c(\eta_i)$$

where $w_i(\eta_i, j, \eta_j)$ is the utility of i derived from a link with j when i and j are involved in η_i and η_j projects, respectively, and $c(\eta_i)$ is the cost to i of maintaining η_i links.

$$u_i^{co}(g) = \sum_{j:ij \in g} \left(\frac{1}{\eta_i(g)} + \frac{1}{\eta_j(g)} + \frac{1}{\eta_i(g)\eta_j(g)} \right)$$

for $\eta_i(g) > 0$ and $u_i^{co}(g) = 0$ if $\eta_i(g) = 0$.

Proposition

In the co-author model, if n is even, then the efficient network structure consists of $\frac{n}{2}$ separate pairs.

If a network is PS and $n \geq 4$, then it is inefficient and can be partitioned into fully intraconnected components, each of which has a different number of members.

Moreover, if m is the number of members of one component of a PS network and $\widetilde{m}$ is the number of members of a different component that is no larger than the first, then $m > \widetilde{m}^2$.

Externalities

Is it good or bad if my co-author has many collaborators?

"You should spend the next week typing down names of all co-authors on your paper."

The role of externalities

- Externalities occur when the utility to one individual is affected by the actions of others, although those actions do not directly involve the individual in question.
- Effects of link formation on other agents (positive and/or negative). Some examples:
  - Connections model (Jackson & Wolinsky (1996)) - externalities from links are positive: other agents benefit from links they are not involved in since they reach contacts in fewer steps.
  - Co-author model (Jackson & Wolinsky (1996)) - a direct or indirect contact who is busy with some of his contacts is less available for other contacts or activities.
- Nevertheless: In many situations we see positive and negative externalities in parallel (e.g., knowledge production and exchange, open job positions)

Models accounting for negative externalities

- ''Degree-based-model'' by Morrill (2011):

$$u_i^{Mor}(g) = \sum_{j:ij \in g} \phi(\eta_j(g)) - c\eta_i(g)$$

  with ϕ decreasing in $\eta_j(g)$.
- ''Unequal connections'' by Goyal & Joshi (2006):
  - General and extensive analysis
  - Playing the field game and local spillovers game

$$\pi_i^{pfg}(g) = \Phi(\eta_i(g), L(g_{-i})), \text{ where } L(g_{-i}) = \sum_{j \neq i} \eta_j(g_{-i})$$

$$\pi_i^{lsg}(g) = \Psi_1(\eta_i(g)) + \sum_{j:ij \in g} \Psi_2(\eta_j(g)) + \sum_{j:ij \notin g} \Psi_3(\eta_j(g))$$

  - Existence and characterization of equilibrium networks

Models accounting for negative and positive externalities

- Billand, Bravard & Sarangi (2012a,b,c) - global/local spillovers
- Currarini (2007), Buechel & Hellmann (2012) - role of positive and negative externalities
- Hellmann (2012) - how externalities affect the existence and uniqueness of PS networks
- Haller (2012) - examples with negative externalities, in which the values of information are endogenously determined and depend on the network; it is harder to access the information from an agent which has more direct neighbors
- Moehlmeier, Rusinowska, Tanimura (2013) - A degree-distance-based connections model with negative and positive externalities

"Degree-distance-based" variation of the connections model:

$$\widetilde{u}_i(g) = \sum_{j \neq i} b(l_{ij}(g), \eta_j(g)) - c\eta_i(g)$$

where

- $b : \{1, ..., n-1\}^2 \to \mathbb{R}^+$ is the net benefit function
- $b(l_{ij}(g), k)$ is decreasing in degree k for all $l_{ij}(g)$
- $b(l, \eta_j(g))$ is decreasing in distance l for all $\eta_j(g)$
- and if $l_{ij}(g) = \infty$, we set $b(\infty, \eta_j) = 0$ for every $\eta_j \in \{0, 1, ..., n-1\}$.

Functional form close to the original connections model:

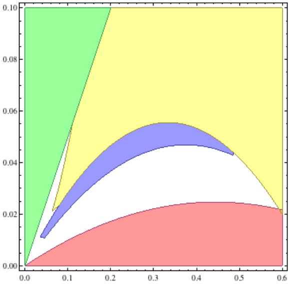
$$u_i(g) = \sum_{j \neq i} \frac{1}{1 + \eta_j(g)} \delta^{l_{ij}(g)} - c\eta_i(g)$$

Partial characterization of PS structures (as in JW (1996)):

- The case of very large decay: PS graphs are not necessarily connected and may consist of more than one component.
- Very low decay: PS graphs are minimally connected.
- Main analysis: We focus on the case of networks with small diameters (as in Jackson-Wolinsky).
- We characterize the stability regions for the $g^{\emptyset}, g^{*}, g^{N}$: non empty stability regions.
- Striking is that g^{*} and g^{N} can now be simultaneously PS.
- We exhibit other network structures with short diameters that are PS in our model but could not be in JW (1996).

A degree-distance-based connections model

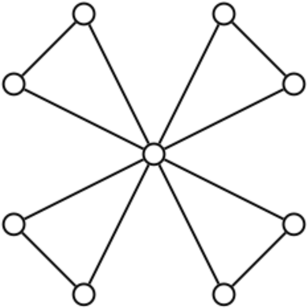
Illustration of stability regions for $n = 9$



- Green area: $g^{\emptyset}$, Red area: g^{N}, Yellow area: g^{*}, Orange area: g^{N} and g^{*}
- Blue area: Windmill/flower architecture

Example: The windmill/star with peripheral links

One node i, the center, is linked to all other nodes. Every node other than the center, $j \neq i$, has the same degree η_m.



- Many networks including $g^{\emptyset}, g^{*}$, g^{N}, g^{L} and a disconnected structure with components of size 2 can be SE in our framework for some level of link costs. However, under some conditions on the benefit function, g^{*} is the strongly efficient network for a very wide range of link costs.
- Without characterizing the efficient network, we can show that PS networks in our model are not necessarily strongly efficient.
- The strongly efficient network is not necessarily pairwise stable.
- We identify conditions under which the star is uniquely strongly efficient.
- Our framework can generate overconnectedness in the sense that the efficient network is included in the stable one. This could never occur in the original Jackson-Wolinsky model.

Small worlds in an islands-connections model

- How small world properties of networks can be explained from a strategic point of view?
- Two modifications of the original connections model:
  - if the minimum path length between 2 players $> D$, they do not get any value from each other
  - there is a geographic structure of costs: K islands, each has J players; forming a link between agents of the same island costs c, and between agents of different islands costs C; $C > c > 0$.
- Introduced by Jackson & Rogers (2005):

$$u_i^{JR}(g) = \sum_{j \neq i : l_{ij}(g) \leq D} \delta^{l_{ij}(g)} - \sum_{j : ij \in g} c_{ij}$$

where

$$c_{ij} = \begin{cases} c & \text{if } i \text{ and } j \text{ are on the same islands} \\ C & \text{otherwise} \end{cases}$$

Proposition

If $c < \delta - \delta^2$ and $C < \delta + (J-1)\delta^2$, then any network that is pairwise stable or efficient is such that:

(1) the players on any given island are completely connected to one another

(2) the diameter and average path length are no greater than $D + 1$, and

(3) if $\delta - \delta^3 < C$, then a lower bound on individual, average, and overall clustering is $\frac{(J-1)(J-2)}{J^2K^2}$.

For (3) it is assumed that $\delta - \delta^2 \neq C$

General tension between stability and efficiency

There are settings in which all PS networks are inefficient, and sometimes all PS networks are even Pareto inefficient.

Transfers-taxing and subsidizing links:

- To what extend the problem of reconciling stability and efficiency can be dealt with by transfers among players? E.g., government intervention to tax and subsidize different links (e.g., subsidizing R&D partnerships)
- Is it possible to make transfer payments among players so that at least some efficient networks are stable?
- A transfer rule is a function $t : G \to \mathbb{R}^N$ such that $\sum_i t_i(g) = 0$ for all g
- In the presence of transfers, player i's payoff $= u_i(g) + t_i(g)$
- Egalitarian transfer rule t^e: $t_i^e(g) = \frac{\sum_j u_j(g)}{n} - u_i(g)$

Some conditions on transfers - component balance

- A transfer rule t is component balanced if there are no net transfers across components of the network, i.e., $\sum_{i \in S} t_i(g) = 0$ for every g and every component S.
The value of a given component is allocated to the members of that component.
- Component balance can be applied in situations with no externalities across components (e.g., when u is component-decomposable).
- A profile of utility functions u is component-decomposable if $u_i(g) = u_i(g|_{N_i^n(g)})$ for all i and g, where $N_i^n(g)$ is the set of all players at distance of no more than n from i, so it is the set of all players in i's component.

Some conditions on transfers - equal treatment of equals

- Two players are complete equals relative to a network and a profile of utility functions if they are completely symmetric relative to all players in the network, all other players see them as interchangeable in forming a network, and the two players have the same utility function.
- A transfer rule satisfies equal treatment of equals relative to a profile of utility functions u if $t_i(g) = t_j(g)$ when i and j are complete equals relative to u and g.
- Two players who are identical according to all criteria should end up with the same transfers or allocations.

Incompatibility of pairwise stability and efficiency

Proposition
There exist component-decomposable utility functions such that every pairwise stable network relative to any component-balanced transfer rule satisfying equal treatment of equals is inefficient.

Proposition (Dutta & Mutuswami, 1997)
If the profile of utility functions is component-decomposable and all nonempty networks generate positive total utility, then there exists a component-balanced transfer rule such that some efficient network is pairwise stable. Moreover, while transfers will sometimes fail to satisfy equal treatment of equals, they can be structured to treat completely equal players equally on at least one network that is both efficient and pairwise stable.