Elusive Pro-competitive Effects and Harm from Trade *

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Abstract

Krugman's trade model with asymmetric countries, one sector, one production factor and unspecified utility functions is examined. (i) Under non-CES preferences, welfare losses from small decrease in trade costs are guaranteed near autarky because of market distortion, and losses are higher for smaller countries. (ii) The flatter demand curve is, the smaller are gains from trade in spite of pro-competitive effects (our numerical estimates use AHARA utility). (iii) A bigger country has higher wage, variety and price advantage (wage HME) and thereby welfare advantage over the smaller country. However, when trade is beneficial, the smaller country gains more.

JEL Codes: F12, L13

Keywords: Market distortions, Trade gains, Variable markups, Demand elasticity.

Introduction

Gains from trade are an evergreen issue both in theory and policy debates, see Melitz and Redding (2012), Helpman (2011). Recently Arkolakis et al. (2012a) invoked interest to two questions. First, they show how to measure gains and how small the estimated gains can be under CES methodology (constant elasticity of substitution). Second, replying to critique that CES ignores pro-competitive effects, Arkolakis et al. (2012b) have shown even *smaller* estimated gains under variable elasticity of substitution (VES) than CES! This looks surprising, because under CES the basic prices and firm sizes remain constant, that is why consumers benefit from trade mainly through increasing variety. By contrast, typical VES demand curves are more flat than CES (sub-convex), flatness generates pro-competitive and price-decreasing effects of trade. Here consumer gets "double benefit" from trade liberalization Krugman (1979): additional variety and lower prices. How it comes that *double benefit is less than single one*? Intrigued, instead of comparing empirically *estimated* gains, we investigate the issue theoretically. In the same class of situations, should a flatter demand (other things equal) generate higher or lower *predicted* gains than CES?

To isolate this basic logical question from inter-sectoral effects and firms-selection effects, we use the simplest theoretical tool: Krugman's (1979) trade model. It includes only one sector (no numerarie), one factor, iceberg trade costs, two or several asymmetric countries, any sub-convex additive utilities, homogenous firms with linear production costs.[1] We start analysis with simulations comparing CES and sub-convex VES. Similarly to what is said on *estimated* gains, we find that all VES (flat) demands bring less *predicted* gains

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[1]According to Mrázová and Neary (2013), an elementary utility function is *sub-convex* if related demand is increasingly-elastic (more flat than CES), in the opposite case it is super-convex. CES separates these classes being the limiting case for both.

than comparable CES demands, the flatter the smaller! So, "double benefit" fails, and we do not have a clear explanation why (see some suggestions in related section).

Even more surprisingly, an example (Fig.2 below) shows negative trade gains, i.e., *trade losses*. Moreover, massive simulations confirm that such losses are *typical* under high trade costs near autarky (prohibitive trade costs). Further developing these experimental findings, we prove analytically that harm from trade near autarky is *guaranteed* for symmetric countries (Proposition 1). This effect remained unexplored so far because autarky and harmful trade are impossible under CES, most popular specification. Under VES autarky exists, and a simple intuition could suggest, that most beneficial here are the first steps into trade. Instead, we find under VES that the first step is necessarily harmful, while the main hike in gains occurs near free trade! In other words, *globalization needs some critical mass* to become beneficial under monopolistic competition.

Why? The only comparable critical-mass effect that we know is found in oligopoly with homogenous commodity, inBrander and Krugman (1983). As to monopolistic competition, similar effect is found by Chen and Zeng (2014) in footloose capital model but in simple trade harm near autarky looks new and surprising. Our intuitive explanation is based on distortion: market maximizes total revenue but social planner pursues gross utility. Only under CES these goals coincide. When globalization goes under VES, trade becomes profitable for the firm too early from social viewpoint, when we take shrinking variety into consideration. In other words, when trade gradually becomes cheaper, export starts earlier than socially desirable.

Further, we expand our analysis to asymmetric trade of two countries. It worth doing, because welfare can be seriously affected by populations asymmetry and general equilibrium impact on relative wages. Does the bigger country (Home) always have higher wage and welfare? Our answer is "yes." We find it through massive simulations "everywhere" and then prove it analytically in special cases. Specifically, for each value of demand flatness parameter a ($a = 0$ corresponds to CES demand), we study whole "admissible rectangle" of two parameters: trade freeness (from 0 at autarky to 1 at free trade) and countries' asymmetry s.[2] Asymmetry is the share of Home country in the world, varying from 1/2 to 1. "Massive" simulations, by methodology of computational economics Judd (2006), mean that whole domain of admissible parameters is checked by thousands of random tests and the answer for some property P is given in the following form: with confidence level 0.99 "it can be predicted that the area where P does not hold is not more than 0.001 of the admissible domain." With such level of confidence we state that both under VES or CES, the bigger country "always" gets higher wages and higher welfare. This can be proven analytically at the borders of the admissible rectangle: near autarky or free trade, near symmetry or complete asymmetry ($s \approx 1$). Advantage of bigger economy both in wages and welfare is quite robust, that once again explain the agglomeration effects, so noticeable in real world.[3]

However, our findings about "harmful first steps to trade" and lower gains for VES demands—turn out valid in asymmetry also for both countries! Though the big country is better off, the smaller one is more dependent on trade but nevertheless, both of them suffer from first steps to trade and generally gain less than they would under CES (Proposition 5).

Additionally, we find several regularities that helping to explain globalization impact on trade and welfare in asymmetric world with VES: how consumptions, prices, masses of firms (variety) and their sizes behave. It is found that total variety in the world decreases with globalization, whereas firms' size increases and all commodities become cheaper (unlike CES case), because of better exploiting economies of scale. Unlike CES case, both countries practice dumping (incomplete path-through for trade costs), which has nothing to do here with strategic behavior explored in Brander and Krugman (1983). It is just monopolistically-competitive pricing under increasing demand elasticity that yields dumping.

To compare these results to the literature, we first mention (Mrázová and Neary, 2014). Similar Krugman's model, only with symmetric countries, is analyzed analytically. Like in our asymmetric case: (a) dumping

[2]Specifically, in simulations we use AHARA specification of utility: $u(x) \equiv h(a)\left[(a+x)^{\rho} - a^{\rho} - b(a)x\right]$, exploiting some functions $h(a) > 0$, $b(a) \geq 0$, $a \geq 0$, $\rho \in (0, 1)$. It is called also "double-translated CES" (Mrázová and Neary (2013)) because its demand differs from CES one only by horizontal shift a and vertical shift b. In analytical results we use any additive utility.

[3]Trade freeness leads to higher trade volumes between countries. Yet, higher asymmetry in population decreases trade volumes between countries, while increasing import penetration ratio of small country, making it more dependable on trade. This conclusion is supported by welfare analysis showing that small country gains from trade more than bigger country. On the other hand, because bigger country explores economy of scale at higher rates and has higher number of varieties, consumers there are better-off than consumers in smaller country. Surprisingly, departing from CES preferences and allowing pro-competitive effects does not increase welfare gains from trade, but on the contrary, decreases them.

occurs under sub-convex demands, (b) the size of domestic purchase decreases in trade liberalization while import increases, (c) a reduction in trade costs in the neighborhood of free trade reduces firm output and increases the number of domestic firms if and only if demand is subconvex. As to welfare, they provide formulae to estimate gains from trade liberalization under VES but the sign of effect remains unclear even in the neighborhood of free trade, several parameters require calibration. Our study expands (a)–(c) to asymmetry and provides clear welfare results.

Another comparable paper, Bykadorov et al. (2015) studies similar model but only compares free trade and autarky without intermediate stages of liberalization (however, it considers most general additive preferences and non-linear costs). The necessary and sufficient condition on preferences and costs are found for harmful jump from autarky to free trade. This condition is almost impossible to satisfy, in the present setting it is definitely excluded by linear costs and sub-convex demands, so, the jump to free trade here must be beneficial. Our new finding is that the gradual change must be non-monotone.

Finally, similar effect is found by (Chen and Zeng, 2014) in a foot-loose capital model. Unlike us, they fix total number of firms in the economy but consider wage non-equalization and VES, like we do. They prove that: (i) wages are higher in larger countries; (ii) a more-than-proportionate relationship exists between a country's share of world production and its share of world demand; (iii) there is an example of non-monotone welfare response to trade cost, with harmful trade near autarky. So, their HME in wages is similar to ours, though the capital flow between countries seriously changes the mechanism of general equilibrium effects. That is why we are not ready to compare the nature of their effects with ours.

Section 1 outlines the model and describes the AHARA utility class used for massive simulations. In Section 2 we outline and discuss all effects under symmetry to ease exposition and consider K countries. Section 3 performs full-scale analysis of two asymmetric countries: massive simulations and proofs for HME effect, price behavior, trade shares and welfare gains. In Conclusion we summarize our work and Appendix contains most proofs.

1 The Model

The economy consists of two countries $j, k \in \{Home, Foreign\}$ (in Section 2 we extend our analysis to $K+1$ symmetric countries). Countries trade with each other, bearing transportation costs $\tau > 1$ of iceberg-type, meaning that to sell q^{ij} of product from country i to country j, a firm should produce τq^{ij}. The technology of production is linear and the same across countries: $C(Q^j) = cQ^j + f$, where firm's output is $Q^j = q^{jj} + \tau q^{jk}$.

Each country has only one sector producing horizontally differentiated product, using only one factor of production — labor. There is continuum of workers with masses $L^H = sL$ and $L^F = (1-s)L$, when world population is L. When $s \neq \frac{1}{2}$ it means countries' asymmetry and Home will be supposed bigger or same: $s \geq 1/2$. Each worker/consumer inelastically supplies 1 unit of labor and wage in country Foreign is normalized to 1, so that relative wage is $\frac{w^H}{w^F} = w^H \equiv w$. In each country there is a continuum of operating firms with (endogenous) masses N^H and N^F of firms (varieties) respectively.

1.1 Preferences and demand

Consumers are identical. Utility of general-additive type is maximized subject to usual budget constraint:

$$U = \int_0^{N^H} u(x_i^{HH})di + \int_0^{N^F} u(x_i^{FH})di, \quad \int_0^{N^H} p_i^{HH} x_i^{HH} di + \int_0^{N^F} p_i^{FH} x_i^{FH} di \leq w^H$$

with the Lagrange multiplier λ. Then, using the first-order condition (FOC), the inverse demand becomes

$$p(x_i^{jk}) = u'(x_i^{jk})/\lambda^k. \tag{1}$$

Variable x_i^{FH} here means "import from Foreign to Home". The Lagrange multiplier λ^k at the destination-country k, standardly, means marginal utility of income. It becomes the main measure of toughness of competition among the firms in this country.

We assume that elementary utility $u(.)$ is three-times differentiable, strictly increasing, strictly concave. Its concavity means that consumers experience love for variety, a property similar to risk-aversion introduced by Arrow-Pratt: each consumer prefers to consume the whole spectrum of varieties. As in (Zhelobodko et al., 2012), we use Arrow-Pratt *concavity* measure for any function $g(s)$: $r_g(x) \equiv -\frac{x \cdot g''(x)}{g'(x)}$ to measure love for variety. Following (Zhelobodko et al., 2012) and (Mrázová and Neary, 2013)), to make monopolistic pricing possible, we impose standard assumptions $r_u(x) < 1$, $r_{u'}(x) < 2$.

Similarly to concavity, *elasticity* of any function $g(s)$ will be denoted in two ways

$$\mathcal{E}_g(x) \equiv \overset{\varepsilon}{g}(x) \equiv \frac{x \cdot g'(x)}{g(x)},$$

where the variable of differentiation x will be indicated when non-obvious (the concise form $\overset{\varepsilon}{g}$, like concise derivative notation g', helps to squeeze huge expressions). Thereby, by definition,

$$r_g(x) \equiv -\mathcal{E}_{g'}(x).$$

For function u, its concavity r_u will express the (absolute value of) elasticity of the inverse demand. Also, $r_u(x) \equiv \frac{1}{\varepsilon}(x)$ because it is the inverse of the demand elasticity $\varepsilon(p)$ taken at point $p(x)$. As in (Mrázová and Neary, 2013) and (Zhelobodko et al., 2012), the increasing or decreasing behavior of $r_u(x)$ becomes crucial for comparative statics, representing pro- or anti-competitive market effects accordingly.

1.2 Producers

Firms are identical and each produce one unique variety. For a firm to produce Q^k quantity of a product, it is necessary to use fixed costs F and linear variable costs cQ^k, measured in labor. Thus, using the inverse demand from equation (1),

- Home-firm's operational per-consumer profit is

$$\pi(x^{HH}, x^{HF}) = s\frac{x^{HH}u'(x^{HH})}{\lambda_H} + (1-s)\frac{x^{HF}u'(x^{HF})}{\lambda_F} - wc(sx^{HH} + (1-s)x^{HF}\tau) - wF/L;$$

- Foreign-firm's operational per-consumer profit is

$$\pi(x^{FF}, x^{FH}) = (1-s)\frac{x^{FF}u'(x^{FF})}{\lambda_F} + s\frac{x^{FH}u'(x^{FH})}{\lambda_H} - c((1-s)x^{FF} + sx^{FH}\tau) - F/L.$$

1.3 Equilibrium

We take the producer's FOC:

$$\frac{\partial\pi(x^{HH}, x^{HF})}{\partial x^{HH}} = u'(x^{HH})/\lambda_H + x^{HH}u''(x^{HH})/\lambda_H - wc = 0;$$

$$\frac{\partial\pi(x^{HH}, x^{HF})}{\partial x^{HF}} = u'(x^{HF})/\lambda_F + x^{HF}u''(x^{HF})/\lambda_F - wc\tau = 0;$$

$$\frac{\partial\pi(x^{FF}, x^{FH})}{\partial x^{FF}} = u'(x^{FF})/\lambda_F + x^{FF}u''(x^{FF})/\lambda_F - c = 0;$$

$$\frac{\partial\pi(x^{FF}, x^{FH})}{\partial x^{FH}} = u'(x^{FH})/\lambda_H + x^{FH}u''(x^{FH})/\lambda_H - c\tau = 0,$$

and substituting λ_k from consumers' FOC (1), we can express the general monopolistic pricing rule as

$$p(z) = \frac{\tilde{c}}{1 - r_u(z)}, \tag{2}$$

where cost can be $\tilde{c} = cw_k$ or $\tilde{c} = \tau cw_k$. Lerner's markup is the share of profit in price $M = (p - \tilde{c})/p = r_u(z)$.

Thereby the rule of "marginal utilities proportional to prices" can be expressed as:

$$\frac{u'(x^{HH})}{u'(x^{FH})} = \frac{w}{\tau} \cdot \frac{1 - r_u(x^{FH})}{1 - r_u(x^{HH})}, \tag{3}$$

$$\frac{u'(x^{FF})}{u'(x^{HF})} = \frac{1}{w\tau} \cdot \frac{1 - r_u(x^{HF})}{1 - r_u(x^{FF})}. \tag{4}$$

We also use our free entry conditions with substituted λ_k from consumer's FOC (sum of profits from all purchases equals fixed cost):

$$s\frac{r_u(x^{HH})x^{HH}}{1 - r_u(x^{HH})} + (1-s)\tau\frac{r_u(x^{HF})x^{HF}}{1 - r_u(x^{HF})} = \frac{F}{Lc} \tag{5}$$

$$s\tau\frac{r_u(x^{FH})x^{FH}}{1 - r_u(x^{FH})} + (1-s)\frac{r_u(x^{FF})x^{FF}}{1 - r_u(x^{FF})} = \frac{F}{Lc}, \tag{6}$$

labor-market clearing:

$$N^H \left(F + c(sLx^{HH} + \tau(1-s)Lx^{HF})\right) = sL$$

$$N^F \left(F + c((1-s)Lx^{FF} + \tau sLx^{FH})\right) = (1-s)L,$$

and trade balance:

$$N^H(1-s)p^{HF}x^{HF} = N^F sp^{FH}x^{FH}.$$

Trade equilibrium is a vector of variables

$$\{x^{HH}, x^{HF}, x^{FF}, x^{FH}, \lambda^H, \lambda^F, N^H, N^F, w\},$$

that satisfies all these conditions. It implies positive reciprocal trade $x^{HF} > 0$, $x^{FH} > 0$, whereas under very high (finite or infinite) trade costs, the solution may degenerate into *autarky equilibrium*, governed by similar equations but for $x^{HF} = 0 = x^{FH}$.

Trade balance, after substituting expressions from labor-market conditions, monopolistic pricing rule and consumer's FOC become:

$$w \cdot \frac{x^{HF}}{1 - ru(x^{HF})} \cdot \frac{1}{sx^{HH} + (1-s)x^{HF}\tau + F/(cL)} = \frac{x^{FH}}{1 - ru(x^{FH})} \cdot \frac{1}{(1-s)x^{FF} + sx^{FH}\tau + F/(cL)}.$$

After reformulating it, we express the wage differential as:

$$w = \frac{x^{FH}\left(1 - r_u(x^{HF})\right)}{x^{HF}\left(1 - r_u(x^{FH})\right)} \cdot \frac{\frac{F}{cL} + sx^{HH} + \tau(1-s)x^{HF}}{\frac{F}{cL} + (1-s)x^{FF} + \tau sx^{FH}} = \frac{R\left(x^{FH}\right)}{R\left(x^{HF}\right)} \cdot \frac{C(Q^H)}{C(Q^F)}, \tag{7}$$

through denoting $R(z)$ as revenue and $C(Q^k)$ as total costs. We observe the wage differential as a relation of per-purchase revenue to some measure of total cost.

Now, we can reformulate the equilibrium conditions as a system of only 5 equations (3),(4),(5),(6),(7) that depends on 5 variables $\{x^{HH}, x^{HF}, x^{FF}, x^{FH}, w\}$, from which all other equilibrium variables can be recovered.

1.4 AHARA utility for massive simulations

Apparently, our system of equilibrium equation do not allow comprehensive algebraic analysis. Therefore we turn to "massive simulation" method, an approach widely used in computational economics. It means to simulate equilibrium outcome under "all" plausible values of parameters, taken with some grid or randomly from the whole domain. If some property of equilibrium (say, higher wage in bigger country) is observed always among 1 mln samples, we can predict with probability close to one that this property holds "everywhere".[4]

To conduct global simulations we need a parametrized utility class. We choose a rich special class: Augmented Hyperbolic Absolute Risk Aversion (AHARA) utilities:

$$u(x, a, h, b, \gamma) = h\frac{(a+x)^{\gamma} - a^{\gamma}}{\gamma} - bx.$$

In (Mrázová and Neary, 2013) AHARA function is called "Doubly-Translated CES" super family because one can see that related inverse demand u' differs from CES demand with the same power γ only in horizontal shift a and vertical shift b of the demand curve. This flexible parametrized preference class includes "everything" between classical CES-preferences (under $a = 0, b = 0$) and almost-quadratic utility function under big a, b (linear). The nice property of this utility family is that concavity of demand has a simple form:

$$r_{u'} = -\frac{u'''x}{u''} = (2-\gamma)\frac{x}{x+a}.$$

Through increasing a, we make the demand flatter everywhere, thus enforcing pro-competitive effects, and look how equilibrium responds when we gradually move from CES demand to linear demand. Bridging these most popular specifications as a polar cases is our goal.

This utility class has too much degrees of freedom. To better compare CES with "similar" flatter demands, we keep three things unchanged: power γ, slope u'' and level u' of basic demand curve at point $x = 1$, as explained in Fig.1. This requires expressing parameters $h(a), b(a)$ through a, γ in certain way. In each experiment, we fix mark-up at certain point $r_u(x = 1) \equiv m$ during comparative statics w.r.t. a (i.e., we fix power γ) and modify $h(a), b(a)$ in such a way that for all $a > 0$ our AHARA is

$$u(x, a, m) = \frac{(a+1)^{m+1}\left((a+x)^{1-m} - a^{1-m}\right)}{1-m} - ax.$$

One can see that two normalizations result in utility function $u(x, a, m)$ having two degrees of freedom. CES utility is $u(x, 0, m) = \frac{x^{1-m}}{1-m}$, CES demand occurs under $a = 0$ ($r_{u'} = 2 - \gamma$), while linear demand emerges under $a = \infty$ ($r_{u'} = 0$). These two extreme cases are not called AHARA in the sequel; AHARA will mean $0 < a < \infty$, in contrast with CES. Figure 1 illustrates how convexity of demand varies with our parameter a, the slope and size of demand at the middle remaining the same.[5] We suppose that among additive utilities AHARA class covers or approximates practically *all interesting cases*, leaving aside mainly unrealistic ones (decreasingly-elastic demands).

2 Trade Gains and Losses among Symmetric Countries

To lighten the exposition, we first discuss trade gains or losses under symmetric (equal) countries, postponing asymmetry to another section, because the essence of gains/losses is more perceptible under symmetry. We build here on Mrázová and Neary (2014), who study the same symmetric trade model and find some welfare measures. Additionally, Bykadorov et al. (2015) has shown that a jump from autarky to free trade must bring trade gains, except for very peculiar and unrealistic preferences/costs. However, *monotonicity* (or not) of gains from trade along the whole path of increasing trade freeness (decreasing trade cost) remains unstudied. We rise this question now.

We start with an example. In Figure 2, the blue curve shows welfare (measured by compensating variation) as a function of trade cost τ under AHARA utility. We see that for costs above ≈ 2.2 trade stops, i.e., autarky starts.[6] We see also that to the left from this autarky point, welfare first *decreases* with trade freeness, then increases and finally reaches level about 1.20 in situation of free trade $\tau = 1$. Thus, some *critical mass* of freeness is needed for trade gains. Moreover, such critical mass or the interval of decreasing welfare is not

[4] Though methods of computational economics are not widespread so far but increasing complexity of new models makes computation inevitable. The ideology of computational economics one can find in (Judd, 2006) and (Lehtinen and Kuorikoski, 2007).

[5] $m = 0.25$, a :$a = 0$ — red line, $a = 1$ — green line, $a = 10$ — brown line.

[6] Parameters' values are $m = 0.25$ (i.e., $\gamma = 3/4$), $c = 10/3$, $F = 1$, $L = 1$, $s = 0.5$ with $a \in \{0, 1\}$.

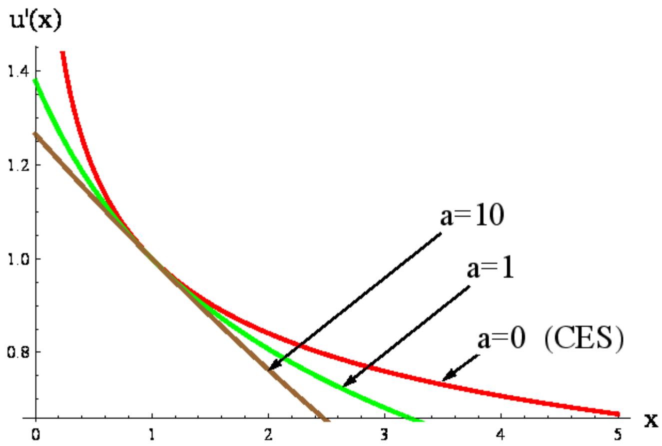

Figure 1: Demand shape varies from CES to linear.

small in this example: almost half of the way from autarky to free trade is harmful (from prohibitive $\bar{\tau} \approx 2.2$ to $\tau \approx 1.6$). Observe also, that trade gains arise mainly near free trade, the closer the stronger. These features look surprising to our intuition.

Further, one can compare these trade gains under AHARA with the red curve generated under CES utility having the same power: $u = \frac{4}{3}x^{3/4}$. Naturally, autarky is absent for CES, trade gains are monotone, but more interesting is that under CES trade gains are *everywhere higher* than under comparable AHARA utility.

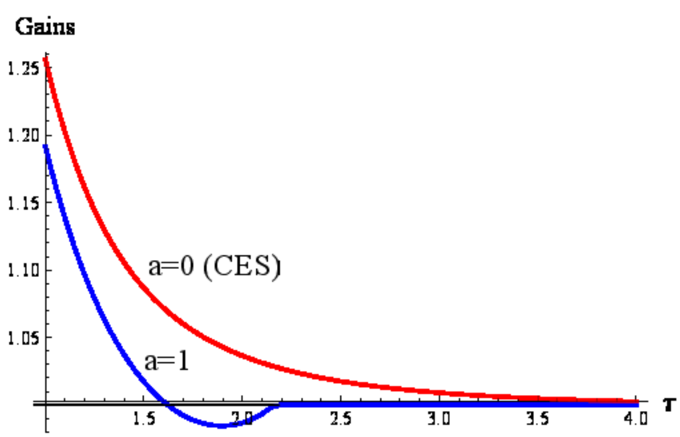

Figure 2: Gains from trade under AHARA or CES utilitites

Such interesting behavior of welfare found in an example rises curiosity: is it guaranteed for *any* AHARA utility? The probabilistic answer is "yes." This computational result below is achieved through massive computer simulations, both for symmetric and asymmetric countries.

Let us introduce one notion from methods of computational economics Judd (2006) to determine how probable some property P (say, welfare decrease) is on domain D. We study any $\mu > 0$ which is hypothetical share of the whole domain of parameters where P does not hold. If the computation shows that property P holds in each case of N random samples from domain D, then we could say: We reject the hypothesis that measure (share of domain) of counter-examples to proposition P exceeds $\varepsilon > 0$ at the confidence level

$$Con(\varepsilon, N) \equiv 1 - (1 - \varepsilon)^N.$$

In particular, when we target confidence level 0.99, our 10000 simulations reporting "yes", mean that no more than 0.001 share of the admissible domain can violate our property. We formulate all our observations below using this approach: we give confidence level 0.99 to share 0.999 of the domain of parameters satisfying the needed property. We restrict the admissible domain to $a \in [0,5]$, $m \in [0,0.6]$, $\tau \in [0,6]$ and take relative market size $\frac{F}{cL} = 0.3$, for the reasons of realism, described in Appendix. It means specific domain

$$D := \{a \in [0,5],\ m \in (0,0.6),\ \tau \in (1,6),\ s \in (0,1),\ \frac{F}{cL} = 0.3\}.$$

For each observations that we mention (including Observation 1), we have explored 10000 random samples out of D, all of them confirmed the property studied, that is why we everywhere report confidence 0.99 to share 0.999.

In addition, for each property studied, we have explored 231000 uniformly distributed samples (regular grid) of each dimension of our domain D consisting of $s \times \tau \times a \times m$ and also observed the property everywhere without any counter-example. We shall report this fact in our observations together with random result.

Observation 1. *Under any AHARA utility, we always (with confidence level 0.99 to share* 0.999 *of domain D through random samples and for each of* 231000 *regular samples) find that: (i) When trade freeness* $1/\tau$ *increases starting from autarky, welfare first decreases (displays losses from trade), then increases. (ii) The power parameter* γ *and other parameters being equal, the flatter demand is (higher a), the smaller are gains from trade.*

Calculation: see Appendix.

Result (i), trade losses in monopolistic competition, up to our knowledge, is absent in the literature, except for similar example of local trade loss in the model with footloose capital by Chen and Zeng (2014), who do not provide any intuitive explanations.

Result (ii) reminds (Arkolakis et al., 2012b) who show that "estimated" gains from trade under VES preferences are smaller than similar gains under CES. However, their result differs from our "predicted" gains. The intuition proposed by (Arkolakis et al., 2012b) is the following: lower trade cost reduces the mark-ups of domestically consumed goods, but it also increases mark-ups of imported goods, that is why "gains from trade liberalization are actually lower than those predicted by standard models with CES utility functions". We would add that transition from CES to VES preferences invokes market misallocation of resources, which is another side of the same coin.

Now we turn to analytical proof of harmful trade near autarky in symmetry, using general arbitrary additive utility that has a choke-price $u'(0) < \infty$. Symmetry ($L^j \equiv L$) allows to consider only symmetric equilibria where domestic consumptions are equal across countries, denoted $x \equiv x^{jj}$, and imports are $z \equiv x^{ij}$. We denote normalized revenue $R(x) \equiv xu'(x)$, normalized marginal revenue $R'(x) \equiv u'(x)\,[1 - r_u(x)]$, marginal utility of income is λ, profit is expressed as

$$\pi(x,z,\lambda) \equiv [R(x)/\lambda - cx + KR(z)/\lambda - K\tau cz]\,L - F.$$

Then the equilibrium equations w.r.t. (x,z,λ,N) become:

$$\pi'_x(x,z,\lambda) = 0 = \pi'_z(x,z,\lambda) \tag{8}$$

$$\pi(x,z,\lambda) = 0, \tag{9}$$

$$N^j = L^j/C(x + K\tau z). \tag{10}$$

To simplify exposition, we shall assume in this section unit labor $L^j \equiv 1$ in each country, i.e., $L = K + 1$. The main three equations can be reformulated as

$$R'(x) = c\lambda,\ \ R'(z) = c\tau\lambda,$$

$$R(x) + KR(z) = C(x + K\tau z)\lambda.$$

Studying these equilibrium equations, we get the needed proposition about harm. Let us present here the main part of the proof which contains intuitions. The idea is that substituting N^j from the labor balance, each consumer's welfare can be presented as utility divided by total cost of any firm:

$$U(\bar{x},\bar{z}) = N^H u(\bar{x}) + N^F K u(\bar{z}) = \frac{u(\bar{x}) + Ku(\bar{z})}{C(\bar{x} + K\tau\bar{z})}, \tag{11}$$

where consumptions $\bar{x} = \bar{x}(\tau)$, $\bar{z} = \bar{z}(\tau)$ are the solutions to (8)–(9). We shall see that when trade cost τ increases, decreasing consumption of import z in the numerator struggles with decreasing cost $C(x + K\tau z)$ in the denominator, and finally the welfare effect of costs can be positive. We denote by $\bar{\tau}$ the autarky point (or infinity when autarky is absent because $u'(0) = \infty$). It is such a prohibitive trade cost $\bar{\tau}$ that a solution to our equations brings exactly $\bar{z}(\bar{\tau}) = 0$. So, $[1, \bar{\tau}) \ni \tau$ is the domain for positive trade.

Proposition 1.[7] *Assume $K + 1$ symmetric countries and additive preferences that can generate autarky ($u'(0) < \infty$). Then equilibrium welfare increases in trade cost τ within some non-empty interval $(\underline{\tau}, \bar{\tau}]$ near autarky point $\bar{\tau}$ (thereby, welfare decreases in trade freeness on $(\underline{\tau}, \bar{\tau}]$).*

Proof. Totally differentiating the equilibrium equations (see Appendix) we find the following responses of equilibrium consumptions and outputs to increasing τ:

Lemma 1. *(i) Domestic consumption x respond to trade cost by increase which slows down: $x'_\tau(\tau) > 0\,(\tau \in (1, \bar{\tau}))$, $x'_\tau(\bar{\tau}) = 0$. (ii) Import z decreases without slowdown: $z'_\tau(\tau) < 0\,(\tau \in (1, \bar{\tau}])$, $z < x$). (iii) Output q decreases near autarky: $q'_\tau(\bar{\tau}) < 0$.*

To find welfare changes, we totally differentiate equilibrium utility (11) in τ at point $\bar{\tau}$, and express the increment in utility U'_τ via increments x'_τ and z'_τ of domestic and imported equilibrium consumptions:

$$U'_\tau = \frac{[u'(x)x'_\tau + Ku'(z)z'_\tau]C(Q) - C'(Q) \cdot Q'_\tau \cdot (u(x) + Ku(z))}{C^2(Q)} =$$

$$\frac{1}{C(Q)}\left[u'(x)x'_\tau + Ku'(z)z'_\tau - \frac{C'(Q)Q}{C(Q)} \cdot \frac{(x'_\tau + Kz + K\tau z'_\tau) \cdot (u(x) + Ku(z))}{Q}\right].$$

Now recall $u(0) = 0$, $u'(0) < \infty$ and simple properties at autarky $x'_\tau = 0$, $z = 0$, $Q = x + K\tau z = x$. Then the expression desired simplifies to

$$U'_\tau\Big|_{\tau=\bar{\tau}} = \frac{1}{C(Q)}\left[Ku'(z)z'_\tau - \overset{\varepsilon}{C}(Q) \cdot \frac{K\tau z'_\tau u(x)}{x}\right] =$$

$$= \frac{Ku(x)z'_\tau}{C(Q)x}\left[\frac{u'(z)x}{u(x)} - \overset{\varepsilon}{C}(Q)\tau\right].$$

To estimate the expression in the brackets at autarky, we use identities

$$\frac{u'(z)}{u'(x)} \equiv \frac{R'(z)(1 - r_{u(x)})}{R'(x)(1 - r_{u(z)})} = \frac{R'(z)(1 - r_{u(x)})}{R'(x)(1 - r_{u(0)})} = \tau(1 - r_{u(x)}) = \tau\overset{\varepsilon}{R}(x)$$

(because $r_u(0) = 0$ and FOC entails $\frac{R'(z)}{R'(x)} = \tau$). Recalling that elasticities of cost and revenue are equal at equilibrium ($\overset{\varepsilon}{C}(x + 0) = \overset{\varepsilon}{R}(x)$) when $z = 0$, we get

$$U'_{\tau=\bar{\tau}} = \left[\frac{u'(z)}{u'(x)}\overset{\varepsilon}{u}(x) - \overset{\varepsilon}{R}(x)\tau\right]\frac{Ku(x)z'_\tau}{C(Q)x} = \left[\tau\overset{\varepsilon}{R}(x)\overset{\varepsilon}{u}(x) - \overset{\varepsilon}{R}(x)\tau\right]\frac{Ku(x)z'_\tau}{C(Q)x} =$$

$$= \tau\overset{\varepsilon}{R}(x)\left[\overset{\varepsilon}{u}(x) - 1\right]\frac{Ku(x)z'_\tau}{C(Q)x} > 0, \tag{12}$$

as we needed (here $\overset{\varepsilon}{u}(x) - 1 < 0$ by concavity and $z'_\tau < 0$, whereas other terms are positive). We expand this conclusion onto some interval $(\underline{\tau}, \bar{\tau}]$ by continuity of all functions involved. **Q.E.D**.

Thus, we have shown that in Krugman's world *a first step away from autarky is always harmful!* This fact may please the protectionists. However, more important is to infer, what is the mechanism of such strange outcome?

Intuitively, when trade cost τ somewhat decreases from its prohibitively high value $\bar{\tau}$, the firms start to trade. They please the customer with new (though expensive) foreign varieties, becoming now available. Everybody behaving voluntarily, how it comes that the customer is finally worse off? We can say that each firm ignores its influence on variety. From the viewpoint of "consumption and variety", the size x of domestic

[7] We expand such proposition onto asymmetric countries in another section.

purchase near autarky remains essentially the same but the mass N of firms shrinks. Such shrink of total variety $2N$ is *insufficiently* compensated by increasing consumption z of the imported good. This explains inequality (12).

The alternative explanation replaces the mass of firms by the labor balance (10) which pins down N. We simply compare through expression (11) costs and benefits of additional consumption z arising from freeness (x remaining almost the same). Expression (11 says that when costs of increasing import z exceed the benefits, then losses from freeness may occur. It is the case near autarky, because, when starting trade, the firm compares its revenue $R(z) \equiv u(z)\overset{\varepsilon}{u}(z)$ with cost $C(x + K\tau z)$. By contrast, the consumer compares cost with utility $u(z)$. One can check that in our model, market maximizes total revenue (see similar effect in Dhingra and Morrow (2012)). That means that function

$$\mathbf{R}(x, z) \equiv \frac{R(x) + KR(z)}{C(x + K\tau z)}$$

comparable to (11) reaches its maximum w.r.t. (x, z) at the equilibrium $(\bar{x}, \bar{z})$. Here DEU (Decreasingly Elastic Utility) property is guaranteed near the autarky point under choke-price, because $\overset{\varepsilon}{u}(0) = 1$, $\overset{\varepsilon}{u}(z) < 1\,(z > 0) \Rightarrow \mathcal{E}\overset{\varepsilon}{u}(z) < 0\,(z \approx 0)$. Knowing that under DEU market provides too many varieties (too small individual consumption x), we compare equilibrium $\bar{x}$ and optimum x^o as $\bar{x} < x^o$. Then one can pose our question about distortion in the following way. Starting from the equilibrium point $(\bar{x}, \bar{z})$ near autarky, should social planner increase both consumptions (x, z) or increase domestic x and decrease foreign consumption $z \approx 0$? The second suggestion should be correct, according to our Proposition 1.

These considerations explain why, during globalization, near $z \approx 0$ *import earlier (too early) becomes profitable for the firm than beneficial for the customer.* In this sense, it is *market distortion* that explains harmful trade under realistic DEU preferences. Under CES such harmful effect is absent, as well as autarky.

These ideas about distortion yielding harm in trade can be generalized. The same discrepancy between utility and revenue remains true in various models, including those with heterogeneous firms, as shown in Dhingra and Morrow (2012). This idea justifies a hypothesis that *harmful trade near autarky should arise from distortion under firms' heterogeneity* as well.

The same consequence of distortion could be true also in other realistic modifications of the Krugman's model. In particular, next section relaxes the symmetry assumption. Populations asymmetry involves an important issue: wage asymmetry. Does it modify our conclusions on welfare? To answer and explain the mechanism of gains, we find how *all* equilibrium variables respond to trade freeness.

3 Complete comparative statics in trade freeness and asymmetry

Confining ourselves to two countries in this section, we shall vary several parameters of our model to see how trade responds to trade freeness and countries' asymmetry. First we look on wage and other HME effects, to compare VES with CES case. Then we consider behavior of mark-ups and prices (pro-competitive effects). Finally, we show trade volumes and welfare gains from trade. For each topic, we first give a numerical confidence-level result for AHARA utility on unrestricted domain, provide intuitions, and then attempt proving each property analytically for more general additive utilities but restricted domain. Thereby, we (supposingly) cover all questions that Krugman's trade model with two countries can answer, up to the limits of tractability.

3.1 Home market effect in wages and firms

Trade literature discusses several kinds of HME (Home market effect). The bigger country can show: (i) higher wage, (ii) disproportionally higher number of firms, (iii) disproportionally higher total physical output (or export), and (iv) disproportionally high total output (or export) value. The latter effect (output value) is equivalent to the first one in our model, because of simple labor balance: GDP is labor force multiplied by wage. Effects (ii) and (iii) are weak and mixed here because of only one sector present: there is no room for intersectoral relocation of resources. So, we focus first on HME in wages.

Through massive simulations (see Appendix), we discover for "all" AHARA utilities the following wage effects.

Observation 2. *Under any AHARA utility, we always (with confidence level 0.99 to share* 0.999 *of domain D through random samples and for each of* 231000 *regular samples) find that: (i) Wage in the bigger country is higher, thus showing positive HME in wages* ($w \equiv \frac{w_H}{w_F} > 1$)*, being bounded as* $w < \tau$*. (ii) An increase in trade freeness leads to lower wage difference. (iii) Higher population asymmetry leads to higher wage difference and higher response of wage difference to trade freeness. (iv) The flatter demand is, the more sensitive wage difference is to asymmetry in population.*

Figure 3-a[8] illustrates such dependence of wage differential w upon countries' asymmetry and trade costs altogether. Such counter-plot here and further displays "double" comparative statics, simultaneously in trade cost τ and country's share s in the world population. From the right this *admissible rectangle* of (τ, s) is bounded by the brown autarky curve, showing $\bar{\tau} = \bar{\tau}(s)$ where trade vanishes (the more symmetric countries are, the lower are their prohibitive trade costs). Figure 3-b shows influence of demand convexity. One can see that Observation 2 is confirmed by both figures.

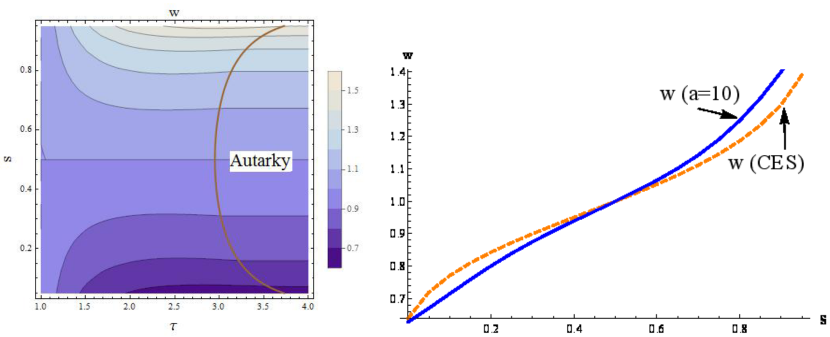

Figure 3: Wage response to trade cost, asymmetry, flatness.

Intuitively, wage effects are simple. The bigger country better exploits increasing returns to scale, which makes its goods cheaper and thus more competitive in trade. This "population" advantage, in turn, rises wages in this country that mitigates this tendency. The smaller τ the smaller is wage differential.

Now we turn to analytical statements of such effect where possible. Under CES specification, one can find that HME in wages is determined via equation

$$(sw^{\frac{\rho}{\rho-1}} - (1-s)) = \tau^{\frac{\rho}{\rho-1}}(sw - (1-s)). \tag{13}$$

Under symmetry ($s = 1/2$) and free trade ($\tau = 1$), equation (13) simplifies to $w^{\frac{\rho}{\rho-1}} = w$, that request $w = 1$. On the other hand, if $s \neq 1/2$, we rewrite equation (13) in the form:

$$\frac{w^{\frac{\rho}{\rho-1}} - \tau^{\frac{\rho}{\rho-1}} w}{(1 - \tau^{\frac{\rho}{\rho-1}})} = \frac{s}{1-s}.$$

The left-hand side is increasing in w. So, when asymmetry ratio $\frac{s}{1-s}$ rises, than relative wage w also rises, being a monotone function of relative country size, and $w > 1$ ($s > 1/2$). In particular, under extreme asymmetry when $s \to 1$, we get $w = \tau^{\rho}$. Thus, relative wage is bounded under CES as:

$$\tau^{-\rho} < w < \tau^{\rho}$$

To get comparative statics in another dimension of our admissible rectangle, equation (13) can be rewritten in such a form:

$$\tau^{\frac{\rho}{\rho-1}} = \frac{(sw^{\frac{\rho}{\rho-1}} - (1-s))}{(sw - (1-s))}.$$

It appears that an increase in transportation costs τ results in an increase of relative wage w, thus $w > 1$ under $\tau > 1$ and asymmetry.

[8] Parameters values are $m = 0.25$, $c = 10/3$, $F = 1$, $L = 1$, $a = 0.1$ for the first graphic and $m = 0.25$, $c = 10/3$, $\tau = 1.8$ and $a \in \{0, 10\}$ for the second graphic.

Proving such wage dependencies like Observation 2 and Figure 3 under arbitrary additive utility is difficult for whole rectangle (τ, s). However, analytics is possible at least at the rectangle borders.

Proposition 2. *For two asymmetric trading countries, their wage differential $w = w^H/w^F$ is bounded as $1/\tau < w < \tau$. It increases in τ and s near all borders: near free trade and near autarky, near symmetry and near complete asymmetry (small open economy). Wage differential near autarky, and at complete asymmetry is greater than one ($w > 1$), whereas at symmetry or free trade $w = 1$.*

Proof: see Appendix.

This proposition supports Observation 2. The intuition behind Observation 2 (that relative wage increases with asymmetry and trade costs) can be explained through the extreme example when one country is infinitesimally small. Since almost whole production in the world is consumed in the bigger country, workers in the smaller country mostly produce for another country. However, transportation costs make their production costly. Therefore, labor in the smaller country appears less efficient and thus cheaper (though dis-proportionally cheaper to τ, because $w < \tau$). The more symmetric countries are, or the smaller trade costs become — the less wage difference becomes.

Mass of firms

Now we turn to HME in terms of mass of firms N^j, to see that it can be absent. It is defined as $N^H/(N^H + N^F) > s$, that is, the bigger country would keep its mass of firms dis-proportionally higher.

Observation 3. *Under any AHARA utility, we always (with confidence level 0.99 to share* 0.999 *of domain D through random samples and for each of* 231000 *regular samples) find that: Home market effect in terms of mass of firms first increases in trade costs τ (being positive), then disappears, then becomes negative. Share of output everywhere changes in the opposite direction to the mass of firms. Flatter demands generate stronger changes. Under CES nothing changes,* HME-*N is absent.*

Analytically, for additive utilities we are able to prove at least

Proposition 3. *Home market effect in terms of mass of firms increases and becomes positive in trade costs τ near free trade under increasingly elastic (flat) demands.*

Figure 4 illustrates this Observation.[9]

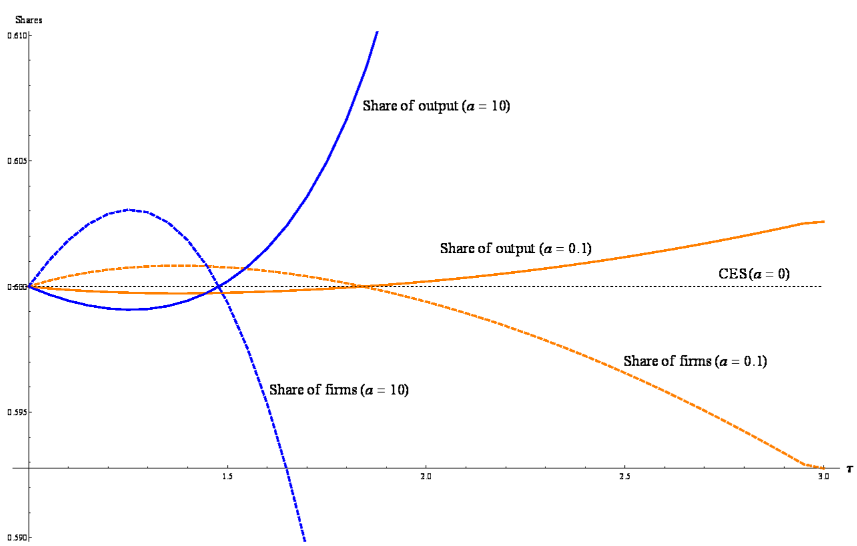

Figure 4: HME in firms and output shares

Indeed, under CES the output of a firm is fixed, while the number of firms is proportional to the market size; this completely excludes HME in firms and outputs. Instead, VES preference allows for HME effect, because of pro-competitive effect. Variable mark-up affects the number of firms and their outputs. In Figure 4, HME variation increases when demand becomes flatter. The HME effects of both kinds are dependent on the value of transportation costs, making HME positive, disappear or even change its sign (reverse HME). Similar indefinite sign of HME was found in (Chen and Zeng, 2014) under VES preferences and foot-loose

[9]Parameters value are $m = 0.25$, $c = 10/3$, $F = 1$, $L = 1$, $s = 0.6$ for $a \in \{0, 0.1, 10\}$

capital. They argue that this behavior fits "mixed" empirical evidence about such HME. We find similar effect even under one factor of production, without capital. Our total number of firms in the economy is not fixed, thus making possible over- or under-provision of variety. HME in firms and HME in outputs always behave oppositely in signs because of the labor balance $N^j \cdot C(q^j) = L^j$ and concave cost $C = cq + f$.

Why non-monotonicity arise? To explain it by example, consider the extreme case: infinitesimally small country and zero transport costs. Here the mass of firms must be proportional to population. Then, a shock in the transport costs makes availability of the foreign market decreasing, so, fewer firms can stay in market. The bigger country is only slightly affected, while the small country is affected a lot. It has to reduce the amount of firms (and wage) by higher rate. This yields HME affect in firms: small country reduces its N_k at high rate. However, further increase in transport costs leads to lower and lower availability of foreign goods, thus countries will have to rely on their own varieties. Now the smaller country starts to specialize in producing many small varieties and restores its N_k. What makes the play is that variety mass itself is costless to trade — only the quantity is costly.

3.2 Prices: pro-competetive effects and dumping

Observation 4. *Under any AHARA utility, we always (with confidence level 0.99 to share* 0.999 *of domain D) find that: (i) an increase in trade freeness leads to price-decreasing competition, i.e.,* $dp^{ij}/d\tau > dp^{ii}/d\tau > 0$*; (ii) higher asymmetry in population results in stronger pro-competitive effect, showing price decrease for growing country and price increase for 'melting' country; (iii) trade shows reciprocal dumping effect, i.e., domestic prices are higher than exported ones corrected for trade costs:* $p^{ij} < p^{ii}\tau\,(\tau > 1)$.

Analytically, for additive utilities we are able to prove

Proposition 4. *Under increasingly elastic (flat) demands all prices decrease with trade freeness* $1/\tau$ *and show dumping. Instead, under CES the path-through is proportional.*

Trade cost impact, which is effect (i) of Observation 4, is reported in Mrázová and Neary (2014), now we expand it to asymmetric countries.

Asymmetry impact, effect (ii), have similar flavor. Unlike CES case, where price is not affected by entry, under increasingly elastic demand an increase in population of the country results into higher number of firms in the market, tougher competition and lower prices, in other words, pro-competitive effect. Prices for domestic and imported goods decrease. Reverse is happening in the 'melting' country, where prices of domestic and imported goods increase.

As to dumping (see Mrázová and Neary (2014)), that is $p^{HH} > \tau p^{HF}$ and $p^{FF} > \tau p^{FH}$, it is very easily explained under VES preferences. Prices for exported goods are higher than for domestically consumed. When $L^H > L^F$, the individual consumptions show such ordering: $x^{FF} > x^{HH} > x^{FH} > x^{HF}$ observed in all simulated examples. The increasing elasticity of demand translates this ordering into corresponding pattern of mark-ups: $r_u(x^{FF}) > r_u(x^{HH}) > r_u(x^{FH}) > r_u(x^{HF})$. That is why prices on exported goods are less than proportionally adjusted to trade costs, that is $p^{HH} > \tau p^{HF}$ and $p^{FF} > \tau p^{FH}$. This dumping has nothing to do with oligopolistic one because no strategic considerations are present. Mis-understanding this, trade regulators can make mistakes in anti-dumping cases thinking of collusion. But in essence, simply markets with pro-competitive effect display freight-absorption or disproportionally low pass-through.

Naturally, an increase in trade costs τ leads to higher domestic consumption and lower exports, thus mark-ups decrease for export and increase for domestic goods, enforcing the reciprocal dumping effect.

Prices go down with trade freeness, but we have seen that welfare increase is not guaranteed. Higher asymmetry leads to higher wage and lower prices in the growing country, unlike the melting country. This means that consumers in growing country get better off, consumers in 'melting' one get worse off, that could suggest a motive for migration. In more detail, welfare gains are addressed in the next section.

3.3 Welfare gains in asymmetric world

We first describe how trade volumes change, then address welfare.

Observation 5. *Under any AHARA utility, we always (with confidence level 0.99 to share* 0.999 *of domain D through random samples and for each of* 231000 *regular samples) find that: (i) An increase in trade freeness* $1/\tau$ *leads to higher trade volumes* $N^H(1-s)x^{HF}, N^F s x^{FH}$ *between countries and lower individual domestic consumption* x^{kk}*, higher share of import to total consumption* $\frac{N^k x^{kj}}{N^k x^{kj} + N^j x^{jj}}$*. (ii) Higher asymmetry* *in population pulls down trade volumes, increases individual domestic consumption x^{HH} for growing country, decreases consumption x^{FF} for 'melting' country (whose share of trade volume in its total production remains higher). (iii) Higher demand flatness a pulls down gains from trade. (iv) There are loses from trade near autarky.*

Analytically, for any additive utilities we get

Proposition 5. *Whenever autarky point exists ($u(0) < \infty$) there is an interval near autarky where welfare decreases with trade freeness for both countries.*

Result (i) of Observation is very intuitive: trade becomes simpler under freeness and consumers tend to diversify their consumption, thus they increase import penetration ratio both in physical terms ($\frac{N^k x^{kj}}{N^k x^{kj}+N^j x^{jj}}$) and value terms $\frac{N^k p^{kj} x^{kj}}{N^k p^{kj} x^{kj}+N^j p^{jj} x^{jj}}$.

The result (ii) is mentioned in (Helpman and Krugman, 1985): countries with similar size trade more. Economically, the higher the asymmetry is, the more varieties are located in bigger country that decrease stimuli to trade with smaller country. The result (ii) shows that small country depends on trade more, because most varieties arise in the bigger country.

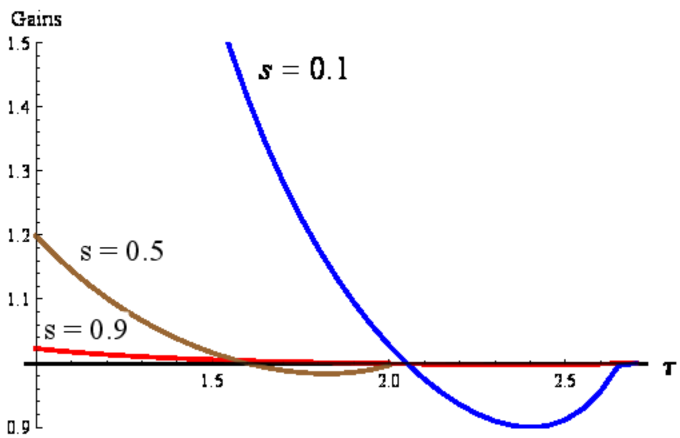

Figure 5: Welfare gains (compensated variation) for small, big or symmetric countries

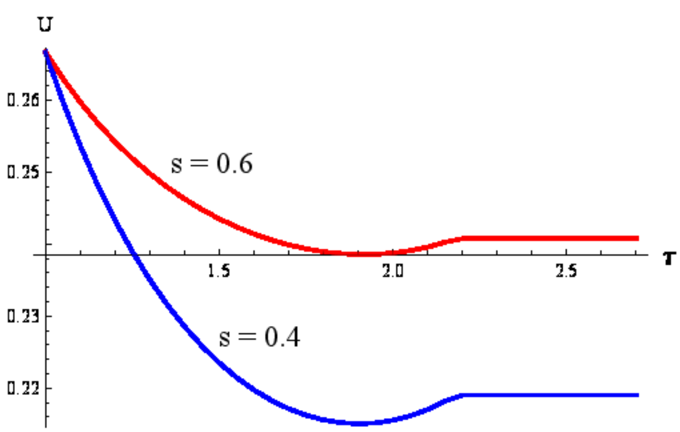

Figure 6: Utility gains for not very different countries

Figure 5[10] represents gains from trade computed via compensation variation and illustrates Observation 5. The smaller country is, the more it gains from trade. On the other hand, as Figure 6[11] shows that welfare in the bigger country is higher: it better exploits economies of scale and love for variety. The left corner

[10]Parameters' values here are $m = 0.25$, $c = 10/3$, $F = 1$, $L = 1$, $a = 5$, $s = 0.1$, 0.5, 0.9.

[11]Similar parameters but $s = 0.6$, 0.4.

represents free trade whereas the right side — autarky (for different market sizes s autarky point differs). Near free trade both countries gain a lot, but first steps (half of the way!) from autarky to free trade is harmful here, as well as for symmetric countries. *Critical mass of freeness is needed* for trade gains.

Returning to discussion of harmful trade (generalized now to asymmetric countries), we can add now that trade liberalization even near autarky *decreases* prices of all goods more than nominal wage can decrease. And still harm occurs both in big and small countries! How can consumers loose from trade? The only dimension of freedom remaining is the number of varieties. Indeed, it can be recovered from our formulas. Facilitated trade brings higher competition from abroad, less firms can continue their business, thus lowering the mass of varieties in the world. Apparently, this decreases utility more than an increase in utility due to an increase in imported consumption, offsetting gains from trade.

This mechanism is missed in CES and other models where the number of varieties is exogenous. However, this effect was also discovered by (Chen and Zeng, 2014) with footloose capital, where quantity and variety don't compete for the same factor of production.

We suppose this new and striking conclusion about losses from trade be economically connected with misallocation of resources and over- or under-provision of varieties broadly discussed by (Dixit and Stiglitz, 1977) and (Dhingra and Morrow, 2012). The core reason of this distortion is that the interests of consumers are not perfectly aligned with interests of producers.

Size of gains

Another important question raised by (Arkolakis et al., 2012a) is: how large are the welfare gains from trade in terms of compensating variation? The astonishingly small estimated gains for United States range from 0.7 to 1.4 percent of GDP. The possible explanation for such small estimates was CES preferences, that do not take into account additional welfare gains from pro-competitive effects. However, as (Arkolakis et al., 2012b) shows, *estimated* gains from trade under VES preferences are even smaller! This fact reminds our result where *derived* gains are also smaller as represented in Graph (7)[12].

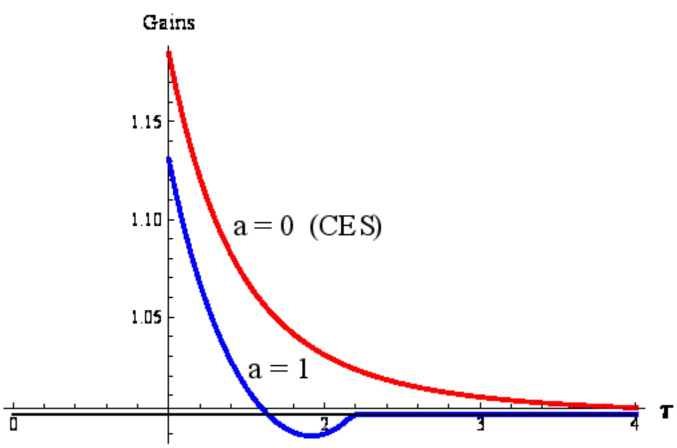

Figure 7: Gains from trade under more concave utilitites

The intuition of smaller gains proposed by (Arkolakis et al., 2012b) states that lower trade costs reduce the mark-ups of domestically consumed goods, but increases mark-ups of imported goods. Then "gains from trade liberalization are actually lower than those predicted by standard models with CES utility functions". Instead, our explanation in Section 2 focus on market misallocation of resources — non-optimal mass of varieties, which is absent under CES. However, it can be that our method of comparing "similar" demands differing only in flatness a plays its role. As one can see in Fig.1, we change a trying to keep the level and derivative of demand unchanged at point $x = 1$, which relates to autarky. However, the potential consumer surplus (the area beneath the demand) appears essentially *higher* for CES than for VES demands. Then, eliminating underexploited consumer surplus by decreasing trade cost — should be bigger where the whole pie was bigger, in CES case. CES advantage could be an artifact of method, this idea deserves further study.

[12]Parameters values are $m = 0.25$, $c = 10/3$, $F = 1$, $s = 0.6$, $L = 1$, with $a \in \{0, 1\}$

Concluding Remarks

We study standard Krugman's trade model with asymmetric countries both through massive simulation and algebra where it is possible. We perform complete comparative statics: market response to trade freeness, countries asymmetry and demand flatness.

The most interesting result is "harmful trade", i.e, essential zone of losses from trade freeness near autarky. This means *critical mass of freeness* needed for trade gains under VES, unlike CES demands. A protectionist would say: if liberalizing, do not do it gradually, do not stay half-way to freeness! Our explanation involves distortion: non-optimal number of variety, offsetting gains from pro-competitive effects.

Another finding is wage advantage of the bigger country (wage-HME) that always increases in trade cost and asymmetry. However, it does not outweigh other sources of welfare: harmful trade remains true for both big and small countries.

Other results include other HME effects, price-decreasing effect, dumping. Trade liberalization fosters trade between countries, increasing import penetration ratio. On the other hand, the more similar countries are in population, the higher is trade between them.

In further study, we would like to check harmful trade effect under heterogenous firms.

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Appendix

Here we describe our calculations and proofs.

A.1 Massive simulations

We now describe more precisely the method of "Massive simulations." The idea is to numerically check if some property P holds in the defined domain D of admissible values of parameters for studied system of equilibrium equations.

For practical simulation, we restrict our domain D of admissible values of parameters. We assume that: (i) $a \in [0,5]$, instead of $a \in [0,\infty)$. The reason is that we observe that under $a = 5$ the demand function already becomes almost linear, so, there is no interest in looking in more "flat" demands than $a = 5$. (ii) We take $\tau \in (1,6)$, instead of $\tau \in [0,\infty)$ because $\tau = 6$ looks for us as highest realistic value for transport costs for usual commodities. (iii) We let the relative population size to take any value $s \in (0,1)$. (iv) The possible mark-up is taken $m \in (0,0.6)$ that sufficiently exceeds typical mark-up $m = 0.25$ for common goods.

To conduct simulations, we also need parameters' values for costs and market size. The measure of size of the world population is normalized here to $L = 1$ because we can always choose the units of measure for labor. For the analysis, only the ratio of costs and market size $\frac{F}{cL}$ is important. We decided to take $\frac{F}{cL} = 0.3$ because $\frac{F}{c} = 0.3$ supposingly represents the typical fraction of capital income to labor income in many economies (according to empirical stylized facts). So, we chose the specific domain

$$D := \{a \in [0,5],\, m \in (0,0.6),\, \tau \in (1,6),\, s \in (0,1),\, \frac{F}{cL} = 0.3\}.$$

We have used two alternative systematic approaches to taking samples in our computational task, and both give the same result.

The first one examines a set of points in D that are located equidistant from each other and therefore make a uniform grid of possible cases. For example, we take fixed step for parameter s to be equal to $step_s = 0.1$ that means $s \in \{0 + 10^{-8},\, 0.1,\, 0.2, ..., 0.9,\, 1 - 10^{-8}\}$ that makes 11 samples. Similarly, $step_a = 0.05$ for $a \in [0,1]$ (because changes in this interval effects equilibrium system most drastically) and increased $step_a = 0.1$ for $a \in [1,5]$ that makes 20+40 samples. We also choose $step_m = 0.1$ that makes 7 samples, and $step_\tau = 0.1$ that makes 50 samples. Thus, in overall we have verified about 231000 numerical examples. This method enables to check if proposition P "probably" holds everywhere in the domain D, meaning that we verified "many" representative points in each direction. This uniform-grid method of sampling has advantage that guarantees the definite size of rectangles explored. However, regularity of points' locations can be viewed as a shortcoming if property P could be violated with some regularity. For this reason, we practiced random sampling also.

Second approach is based on random sampling and can be called Monte Carlo method. Relying on (Judd (2006)) we explain it as follows. Suppose one drew sample of N points from D and at each point property P holds. What then can be said about the probability that P holds everywhere in D? Assume that there is a part of domain D where property P doesn't hold, its normalized measure we denote as $\varepsilon \in [0,1]$. Then, assuming uniform distribution of samples and their independence, the probability that one drew N points and observe property P holding at *each* case—equals $(1-\varepsilon)^N$. More precisely, $(1-\varepsilon)^N$ is also the probability that the part of domain D without P—exceeds ε. That is why we can say that "At confidence level $1-(1-\varepsilon)^N$, we reject the hypothesis that part of domain D where property P doesn't hold (domain of counterexamples) exceeds ε." This method allows formulating the computational results in quite compact way. For our purposes and taking into account the computational constraints we drew 10000 examples to check each property, that results into confidence level more than 0,99 with measure $\varepsilon = 0.001$.

In some sense, we get a higher confidence level from our uniform sampling with 231000 examples but we formulate it in non-probabilistic terms, rather "maximal rectangles." Namely, this uniform-grid approach formulates the maximal size of a cube where counterexamples are not excluded. If we observe property P holding at each point of our grid, than the smaller is the cube whose interiority we have not explored—the "stronger" is our belief in property P. Specifically, we have explored the maximum cube not bigger than $0.1 \times 0.1 \times 0.1 \times 0.1$ of each dimension of our domain D consisting of $s \times \tau \times a \times m$.

A.2 The basic Lemma for the case of $K + 1$ symmetric countries

Lemma 1.*[13] *In case of trade among $K + 1$ symmetric countries with linear costs, in the symmetric equilibrium the following holds.*

(i) Domestic consumption x respond to trade cost by increase which slows down: $x'_\tau(\tau) > 0$ $(\tau \in (1, \bar{\tau}))$, $x'_\tau(\bar{\tau}) = 0$.

(ii) Import z decreases without slowdown: $z'_\tau(\tau) < 0$ $(\tau \in (1, \bar{\tau}))$.

(iii) Domestic consumption x is larger than Import $z : z(\tau) < x(\tau)$.

(iv) Marginal utility of income λ responds to trade cost by decrease which slows down: $\lambda'_\tau(\tau) < 0$ $(\tau \in (1, \bar{\tau}))$, $\lambda'_\tau(\bar{\tau}) = 0$.

(v) Output q decreases near autarky: $q'_\tau(\bar{\tau}) < 0$ but under IED it increases near free trade: $q'_\tau(1) > 0 \Longleftrightarrow r'_u(1) > 0$.

Proof. The linear cost $C(x + K\tau z) = c \cdot (x + K\tau z) + F$ is a function of each firm's output $q = x + K\tau z$. Denote normalized revenue $R(\xi) = u'(\xi) \cdot \xi$, normalized marginal revenue $R'(\xi) = u'(\xi) \cdot (1 - r_u(\xi))$, marginal utility of income λ, profit

$$\pi = \pi(x, z, \lambda) = \frac{R(x)}{\lambda} + K \cdot \frac{R(z)}{\lambda} - C(q).$$

Then the equilibrium equations w.r.t. (x, z, λ) become

$$\pi'_x(x, z, \lambda) = 0 \tag{14}$$

$$\pi'_z(x, z, \lambda) = 0 \tag{15}$$

$$\pi(x, z, \lambda) = 0$$

i.e.

$$R'(x) = c\lambda \tag{16}$$

$$R'(z) = c\tau\lambda \tag{17}$$

$$R(x) + K \cdot R(z) = \lambda \cdot C(q) \tag{18}$$

Note that function $R'(\xi)$ decreases, since $R''(\xi) < 0$ due to Second Order Condition. Moreover, $R'(x) < R'(z)$ when $\tau > 1$, due to (16) and (17). Therefore, $z < x$. So part (iii) is proved.

We totally differentiate in τ equations (16)-(18). Using Envelope theorem (i.e., (14) and (15)) we get total derivatives $x' \equiv \frac{d}{d\tau}x$, $z' \equiv \frac{d}{d\tau}z$, $\lambda' \equiv \frac{d}{d\tau}\lambda$ of (h, z, λ) and their signs:

$$R''(x) \cdot x' - c \cdot \lambda' = 0$$

$$R''(z) \cdot z' - c\tau \cdot \lambda' = c\lambda$$

$$-C(q) \cdot \lambda' = Kc\lambda z$$

which entails (because $\lambda > 0$)

$$\lambda' = -\frac{Kcz}{C(q)} \cdot \lambda \leq 0$$

[13] This lemma is similar but more broad than Lemma 1 used in the main text.

$$x' = \frac{c}{R''(x)} \cdot \lambda' \geq 0$$

$$z' = c \cdot \frac{\tau \cdot \lambda' + \lambda}{R''(z)} = c \cdot \frac{-\tau \cdot Kcz + C(q)}{R''(z) \cdot C(q)} \cdot \lambda = c \cdot \frac{cx + F}{R''(z) \cdot C(q)} \cdot \lambda < 0$$

Hence parts (i),(ii) and (iv) is proved.

Further,

$$q' \equiv \frac{d}{d\tau} q = x' + K\tau \cdot z' + Kz =$$

$$= \frac{c}{R''(x)} \cdot \lambda' + K\tau c \cdot \frac{\tau \cdot \lambda' + \lambda}{R''(z)} + K \cdot z =$$

$$= c \cdot \left(\frac{1}{R''(x)} + K \cdot \tau \cdot \frac{\tau}{R''(z)} \right) \cdot \lambda' + K \cdot \left(\tau \cdot \frac{c\lambda}{R''(z)} + z \right).$$

Hence

$$q'(\bar{\tau}) = c \cdot \left(\frac{1}{R''(x)} + K\tau \cdot \frac{\tau}{R''(z)} \right) \cdot \lambda'(\bar{\tau}) + K \cdot \left(\tau \cdot \frac{c\lambda}{R''(z)} + z(\bar{\tau}) \right) =$$

$$= K \cdot \tau \cdot \frac{c \cdot \lambda}{R''(z)} < 0.$$

As to the behavior of $q(\tau)$ near free trade ($\tau \approx 1$), note that

$$x(1) = z(1) = x, \quad q(1) = (K+1) \cdot x.$$

Moreover, (16) and (18) give us

$$R'(x) = c\lambda(1) \tag{19}$$

$$(K+1) \cdot R(x) = \lambda(1) \cdot C(q(1)) \tag{20}$$

In particular,

$$(K+1) \cdot \frac{c}{C(q(1))} = \frac{R'(x)}{R(x)} \tag{21}$$

Hence

$$q'(1) = c \cdot \left(\frac{1}{R''(x)} + K \cdot \frac{1}{R''(x)} \right) \cdot \lambda'(1) + K \cdot \left(\frac{c \cdot \lambda(1)}{R''(x)} + x \right) =$$

$$= c \cdot (K+1) \cdot \frac{1}{R''(x)} \cdot \lambda'(1) + K \cdot \left(\frac{c \cdot \lambda(1)}{R''(x)} + x \right) =$$

$$= -\frac{c}{R''(x)} \cdot (K+1) \cdot \frac{K \cdot c \cdot x}{C(q(1))} \cdot \lambda(1) + K \cdot \left(\frac{c \cdot \lambda(1)}{R''(x)} + x \right) =$$

(because of (21))

$$= -\frac{c}{R''(x)} \cdot K \cdot x \cdot \frac{R'(x)}{R(x)} \cdot \lambda(1) + K \cdot \left(\frac{c \cdot \lambda(1)}{R''(x)} + x \right) =$$

$$= K \cdot \left(-\frac{c}{R''(x)} \cdot x \cdot \frac{R'(x)}{R(x)} \cdot \lambda(1) + \frac{c \cdot \lambda(1)}{R''(x)} + x \right) =$$

$$= K \cdot \left(\frac{c \cdot \lambda(1)}{R''(x)} \cdot \left(1 - x \cdot \frac{R'(x)}{R(x)} \right) + x \right) =$$

(because of (19))

$$= K \cdot \left(\frac{R'(x)}{R''(x)} \cdot \left(1 - x \cdot \frac{u'(x) \cdot (1 - r_u(x))}{R(x)} \right) + x \right) =$$

$$= K \cdot \left(\frac{R'(x)}{R''(x)} \cdot r_u(x) + x \right) =$$

$$= K \cdot \left(\frac{u'(x) \cdot (1 - r_u(x))}{u''(x) \cdot (2 - r_{u'}(x))} \cdot r_u(x) + x \right) =$$

$$= K \cdot \frac{u'(x) \cdot (1 - r_u(x)) \cdot r_u(x) + u''(x) \cdot (2 - r_{u'}(x)) \cdot x}{u''(x) \cdot (2 - r_{u'}(x))} =$$

$$= K \cdot \frac{u'(x) \cdot \left((1 - r_u(x)) \cdot r_u(x) + \frac{u''(x)}{u'(x)} \cdot (2 - r_{u'}(x)) \cdot x \right)}{u''(x) \cdot (2 - r_{u'}(x))} =$$

$$= K \cdot \frac{r_u(x) \cdot (1 - r_u(x) - (2 - r_{u'}(x)))}{\frac{u''(x)}{u'(x)} \cdot (2 - r_{u'}(x))} =$$

$$= K \cdot \frac{r_u(x) \cdot (1 + r_u(x) - r_{u'}(x)) \cdot x}{r_u(x) \cdot (2 - r_{u'}(x))} =$$

$$= K \cdot \frac{r'_u(x) \cdot x^2}{r_u(x) \cdot (2 - r_{u'}(x))} .$$

Hence (v) holds. This complete the proof.

A.3 The case of two asymmetric countries

Consumer's problem in country H is:

$$\int\limits_{N^H} u\left(x_i^{HH}\right) di + \int\limits_{N^F} u\left(x_i^{FH}\right) di \to max$$

$$\int\limits_{N^H} p_i^{HH} x_i^{HH} di + \int\limits_{N^F} p_i^{FH} x_i^{FH} di \leqslant w$$

Consumer's problem in country F is:

$$\int\limits_{N^F} u\left(x_i^{FF}\right) di + \int\limits_{N^H} u\left(x_i^{HF}\right) di \to max$$

$$\int\limits_{N^F} p_i^{FF} x_i^{FF} di + \int\limits_{N^H} p_i^{HF} x_i^{HF} di \leqslant 1$$

In an equilibrium all similar firms behave similarly (as guaranteed by concave profit function) and we drop firm's index:

$$p^{km} = \frac{u'\left(x^{km}\right)}{\lambda^m}, k \in \{H, F\}, m \in \{H, F\}$$

$$q^{km} := L^m \cdot x^{km}, k \in \{H, F\}, m \in \{H, F\}$$

$$Q^H := q^{HH} + \tau \cdot q^{HF} = L^H \cdot x^{HH} + \tau \cdot L^F \cdot x^{HF}$$

$$Q^F := q^{FF} + \tau \cdot q^{FH} = L^F \cdot x^{FF} + \tau \cdot L^H \cdot x^{FH}$$

Using "normalized" revenue ($R(z) := u'(z) \cdot z$), profits are:

$$\pi^H = \pi^H\left(x^{HH}, x^{HF}, \lambda^H, \lambda^F, w, \tau\right) = L^H \cdot \frac{R\left(x^{HH}\right)}{\lambda^H} + L^F \cdot \frac{R\left(x^{HF}\right)}{\lambda^F} - w \cdot C\left(Q^H\right)$$

$$\pi^F = \pi^F\left(x^{FF}, x^{FH}, \lambda^F, \lambda^H, w, \tau\right) = L^F \cdot \frac{R\left(x^{FF}\right)}{\lambda^F} + L^H \cdot \frac{R\left(x^{FH}\right)}{\lambda^H} - C\left(Q^F\right)$$

Comparative statics of equilibrium variables $\left(x^{HH}, x^{HF}, x^{FF}, x^{FH}, \lambda^H, \lambda^F, w\right)$ w.r.t. τ goes through total differentiation in τ of the equilibrium system

$$\frac{\partial \pi^H}{\partial x^{HH}} = 0, \quad \frac{\partial \pi^H}{\partial x^{HF}} = 0, \quad \frac{\partial \pi^F}{\partial x^{FF}} = 0, \quad \frac{\partial \pi^F}{\partial x^{FH}} = 0,$$

$$\pi^H = 0, \quad \pi^F = 0, \quad EE = 0,$$

where notation EE means Trade Balance:

$$EE \equiv \frac{R\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^H\right)} - \frac{R\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)}$$

Total differentiation gives seven equations linear in derivatives needed:

$$\frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial w} \cdot \frac{dw}{d\tau} = -\frac{\partial^2 \pi^H}{\partial x^{HH} \partial \tau}$$

$$\frac{\partial^2 \pi^H}{\partial x^{HF} \partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial w} \cdot \frac{dw}{d\tau} = -\frac{\partial^2 \pi^H}{\partial x^{HF} \partial \tau}$$

$$\frac{\partial^2 \pi^F}{\partial x^{FF} \partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FF} \partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FF} \partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = -\frac{\partial^2 \pi^F}{\partial x^{FF} \partial \tau}$$

$$\frac{\partial^2 \pi^F}{\partial x^{FH} \partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FH} \partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FH} \partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} = -\frac{\partial^2 \pi^F}{\partial x^{FH} \partial \tau}$$

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial \pi^H}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} + \frac{\partial \pi^H}{\partial w} \cdot \frac{dw}{d\tau} = -\frac{\partial \pi^H}{\partial \tau}$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = -\frac{\partial \pi^F}{\partial \tau}$$

$$\frac{\partial EE}{\partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial EE}{\partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial EE}{\partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial EE}{\partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial EE}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial EE}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = -\frac{\partial EE}{\partial \tau}$$

Due to linearity of cost function $C(Q) = cQ + F$, one has

$$\frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HF}} = \frac{\partial^2 \pi^H}{\partial x^{HH} \partial \tau} = \frac{\partial^2 \pi^H}{\partial x^{FF} \partial x^{FH}} = \frac{\partial^2 \pi^F}{\partial x^{FF} \partial \tau} = 0.$$

Hence our 7 equations are simplified as

$$\frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial w} \cdot \frac{dw}{d\tau} = 0$$

$$\frac{\partial^2 \pi^H}{\partial x^{HF} \partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial w} \cdot \frac{dw}{d\tau} = -\frac{\partial^2 \pi^H}{\partial x^{HF} \partial \tau}$$

$$\frac{\partial^2 \pi^F}{\partial x^{FF} \partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FF} \partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = 0$$

$$\frac{\partial^2 \pi^F}{\partial x^{FH} \partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FH} \partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} = -\frac{\partial^2 \pi^F}{\partial x^{FH} \partial \tau}$$

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial \pi^H}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} + \frac{\partial \pi^H}{\partial w} \cdot \frac{dw}{d\tau} = -\frac{\partial \pi^H}{\partial \tau}$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = -\frac{\partial \pi^F}{\partial \tau}$$

$$\frac{\partial EE}{\partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial EE}{\partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial EE}{\partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial EE}{\partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial EE}{\partial \lambda^H} \cdot \frac{d\lambda^H}{d\tau} + \frac{\partial EE}{\partial \lambda^F} \cdot \frac{d\lambda^F}{d\tau} = -\frac{\partial EE}{\partial \tau}$$

Let us use the following notation for elasticities of λ^H, λ^F, w w.r.t. τ:

$$\mathcal{E}_{\lambda^H} \equiv \mathcal{E}_{\lambda^H/\tau} = \frac{d\lambda^H}{d\tau} \cdot \frac{\tau}{\lambda^H}, \quad \mathcal{E}_{\lambda^F} \equiv \mathcal{E}_{\lambda^F/\tau} = \frac{d\lambda^F}{d\tau} \cdot \frac{\tau}{\lambda^F}, \quad \mathcal{E}_w \equiv \mathcal{E}_{w/\tau} = \frac{dw}{d\tau} \cdot \frac{\tau}{w}.$$

Lemma 2. *In trade between two asymmetric countries with linear costs, the equilibrium x^{ij} responds to trade costs in such a way that*

$$\frac{dx^{HH}}{d\tau} = \frac{R'\left(x^{HH}\right)}{R''\left(x^{HH}\right)} \cdot \frac{1}{\tau} \cdot \left(\mathcal{E}_{\lambda^H} + \mathcal{E}_w\right), \quad \frac{dx^{HF}}{d\tau} = \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot \frac{1}{\tau} \cdot \left(\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1\right),$$

$$\frac{dx^{FF}}{d\tau} = \frac{R'\left(x^{FF}\right)}{R''\left(x^{FF}\right)} \cdot \frac{1}{\tau} \cdot \mathcal{E}_{\lambda^F}, \quad \frac{dx^{FH}}{d\tau} = \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot \frac{1}{\tau} \cdot \left(\mathcal{E}_{\lambda^H} + 1\right),$$

where $\mathcal{E}_{\lambda^H}, \mathcal{E}_{\lambda^F}, \mathcal{E}_w$ satisfy the conditions

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^H}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{w}{\tau} \cdot \mathcal{E}_w = -\frac{\partial \pi^H}{\partial \tau},$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = -\frac{\partial \pi^F}{\partial \tau},$$

$$\frac{\partial EE}{\partial x^{HH}} \cdot \frac{R'\left(x^{HH}\right)}{R''\left(x^{HH}\right)} \cdot \left(\mathcal{E}_{\lambda^H} + \mathcal{E}_w\right) + \frac{\partial EE}{\partial x^{HF}} \cdot \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot \left(\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1\right) +$$

$$+ \frac{\partial EE}{\partial x^{FF}} \cdot \frac{R'\left(x^{FF}\right)}{R''\left(x^{FF}\right)} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial EE}{\partial x^{FH}} \cdot \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot \left(\mathcal{E}_{\lambda^H} + 1\right) +$$

$$+ \frac{\partial EE}{\partial \lambda^H} \cdot \left(\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F}\right) = -\frac{\partial EE}{\partial \tau} \cdot \tau.$$

Proof. We rewrite our differentiated equations in elasticities:

$$\frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial^2 \pi^H}{\partial x^{HH} \partial w} \cdot \frac{w}{\tau} \cdot \mathcal{E}_w = 0$$

$$\frac{\partial^2 \pi^H}{\partial x^{HF} \partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial^2 \pi^H}{\partial x^{HF} \partial w} \cdot \frac{w}{\tau} \cdot \mathcal{E}_w = -\frac{\partial^2 \pi^H}{\partial x^{HF} \partial \tau}$$

$$\frac{\partial^2 \pi^F}{\partial x^{FF} \partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FF} \partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = 0$$

$$\frac{\partial^2 \pi^F}{\partial x^{FH} \partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} + \frac{\partial^2 \pi^F}{\partial x^{FH} \partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} = -\frac{\partial^2 \pi^F}{\partial x^{FH} \partial \tau}$$

We again use Envelope theorem for derivative of profits:

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^H}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial \pi^H}{\partial w} \cdot \frac{w}{\tau} \cdot \mathcal{E}_w = -\frac{\partial \pi^H}{\partial \tau}$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = -\frac{\partial \pi^F}{\partial \tau}$$

$$\frac{\partial EE}{\partial x^{HH}} \cdot \frac{dx^{HH}}{d\tau} + \frac{\partial EE}{\partial x^{HF}} \cdot \frac{dx^{HF}}{d\tau} + \frac{\partial EE}{\partial x^{FF}} \cdot \frac{dx^{FF}}{d\tau} + \frac{\partial EE}{\partial x^{FH}} \cdot \frac{dx^{FH}}{d\tau} +$$

$$+ \frac{\partial EE}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial EE}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = -\frac{\partial EE}{\partial \tau}.$$

Note that, due to equilibrium equations,

We plug particular expressions (equilibrium equations) in this general equation.

$$\frac{\partial^2 \pi^H}{\partial x^{HH} \partial \lambda^H} \cdot \frac{\lambda^H}{\tau} = -\frac{L^H}{\tau} \cdot \frac{R'\left(x^{HH}\right)}{\lambda^H} = -\frac{L^H}{\tau} \cdot w \cdot C'\left(Q^H\right) = \frac{\partial^2 \pi^H}{\partial x^{HH} \partial w} \cdot \frac{w}{\tau}$$

$$\frac{\partial^2\pi^H}{\partial x^{HF}\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}=-\frac{L^F}{\tau}\cdot\frac{R'\left(x^{HF}\right)}{\lambda^F}=-L^F\cdot w\cdot C'\left(Q^H\right)=\frac{\partial^2\pi^H}{\partial x^{HF}\partial w}\cdot\frac{w}{\tau}=\frac{\partial^2\pi^H}{\partial x^{HF}\partial\tau}$$

$$\frac{\partial^2\pi^F}{\partial x^{FH}\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}=-\frac{L^H}{\tau}\cdot\frac{R'\left(x^{FH}\right)}{\lambda^H}=-L^H\cdot C'\left(Q^F\right)=\frac{\partial^2\pi^F}{\partial x^{FH}\partial\tau}$$

$$\frac{\partial EE}{\partial\lambda^H}\cdot\lambda^H=\frac{R\left(x^{FH}\right)}{\lambda^H\cdot C\left(Q^F\right)}=\frac{R\left(x^{HF}\right)}{\lambda^F\cdot C\left(Q^H\right)}=-\frac{\partial EE}{\partial\lambda^F}\cdot\lambda^F.$$

Hence

$$\frac{\partial^2\pi^H}{\partial x^{HH}\partial x^{HH}}\cdot\frac{dx^{HH}}{d\tau}+\frac{\partial^2\pi^H}{\partial x^{HH}\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}+\mathcal{E}_w\right)=0$$

$$\frac{\partial^2\pi^H}{\partial x^{HF}\partial x^{HF}}\cdot\frac{dx^{HF}}{d\tau}+\frac{\partial^2\pi^H}{\partial x^{HF}\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\left(\mathcal{E}_{\lambda^F}+\mathcal{E}_w+1\right)=0$$

$$\frac{\partial^2\pi^F}{\partial x^{FF}\partial x^{FF}}\cdot\frac{dx^{FF}}{d\tau}+\frac{\partial^2\pi^F}{\partial x^{FF}\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F}=0$$

$$\frac{\partial^2\pi^F}{\partial x^{FH}\partial x^{FH}}\cdot\frac{dx^{FH}}{d\tau}+\frac{\partial^2\pi^F}{\partial x^{FH}\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}+1\right)=0$$

$$\frac{\partial\pi^H}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\mathcal{E}_{\lambda^H}+\frac{\partial\pi^H}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F}+\frac{\partial\pi^H}{\partial w}\cdot\frac{w}{\tau}\cdot\mathcal{E}_w=-\frac{\partial\pi^H}{\partial\tau}$$

$$\frac{\partial\pi^F}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\mathcal{E}_{\lambda^H}+\frac{\partial\pi^F}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F}=-\frac{\partial\pi^F}{\partial\tau}$$

$$\frac{\partial EE}{\partial x^{HH}}\cdot\frac{dx^{HH}}{d\tau}+\frac{\partial EE}{\partial x^{HF}}\cdot\frac{dx^{HF}}{d\tau}+\frac{\partial EE}{\partial x^{FF}}\cdot\frac{dx^{FF}}{d\tau}+\frac{\partial EE}{\partial x^{FH}}\cdot\frac{dx^{FH}}{d\tau}+$$

$$+\frac{\partial EE}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}-\mathcal{E}_{\lambda^F}\right)=-\frac{\partial EE}{\partial\tau}.$$

So,

$$\frac{dx^{HH}}{d\tau}=-\frac{\frac{\partial^2\pi^H}{\partial x^{HH}\partial\lambda^H}}{\frac{\partial^2\pi^H}{\partial x^{HH}\partial x^{HH}}}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}+\mathcal{E}_w\right),\quad\frac{dx^{HF}}{d\tau}=-\frac{\frac{\partial^2\pi^H}{\partial x^{HF}\partial\lambda^F}}{\frac{\partial^2\pi^H}{\partial x^{HF}\partial x^{HF}}}\cdot\frac{\lambda^F}{\tau}\cdot\left(\mathcal{E}_{\lambda^F}+\mathcal{E}_w+1\right),$$

$$\frac{dx^{FF}}{d\tau}=-\frac{\frac{\partial^2\pi^F}{\partial x^{FF}\partial\lambda^F}}{\frac{\partial^2\pi^F}{\partial x^{FF}\partial x^{FF}}}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F},\quad\frac{dx^{FH}}{d\tau}=-\frac{\frac{\partial^2\pi^F}{\partial x^{FH}\partial\lambda^H}}{\frac{\partial^2\pi^F}{\partial x^{FH}\partial x^{FH}}}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}+1\right),$$

We also use $\frac{\partial\pi^H}{\partial w}\cdot\frac{w}{\tau}=\frac{\partial\pi^H}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}+\frac{\partial\pi^H}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}$ and get

$$\frac{\partial\pi^H}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}+\mathcal{E}_w\right)+\frac{\partial\pi^H}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\left(\mathcal{E}_{\lambda^F}+\mathcal{E}_w\right)=-\frac{\partial\pi^H}{\partial\tau},$$

$$\frac{\partial\pi^F}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\mathcal{E}_{\lambda^H}+\frac{\partial\pi^F}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F}=-\frac{\partial\pi^F}{\partial\tau},$$

$$\frac{\partial EE}{\partial x^{HH}}\cdot\frac{dx^{HH}}{d\tau}+\frac{\partial EE}{\partial x^{HF}}\cdot\frac{dx^{HF}}{d\tau}+\frac{\partial EE}{\partial x^{FF}}\cdot\frac{dx^{FF}}{d\tau}+\frac{\partial EE}{\partial x^{FH}}\cdot\frac{dx^{FH}}{d\tau}+$$

$$+\frac{\partial EE}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\left(\mathcal{E}_{\lambda^H}-\mathcal{E}_{\lambda^F}\right)=-\frac{\partial EE}{\partial\tau}.$$

In particular, substituting $\dfrac{dx^{km}}{d\tau}$ from the first four equations into the last equation, we obtain three linear equations in elasticities $\mathcal{E}_{\lambda^H},\mathcal{E}_{\lambda^F},\mathcal{E}_w$:

$$\frac{\partial\pi^H}{\partial\lambda^H}\cdot\frac{\lambda^H}{\tau}\cdot\mathcal{E}_{\lambda^H}+\frac{\partial\pi^H}{\partial\lambda^F}\cdot\frac{\lambda^F}{\tau}\cdot\mathcal{E}_{\lambda^F}+\frac{\partial\pi^H}{\partial w}\cdot\frac{w}{\tau}\cdot\mathcal{E}_w=-\frac{\partial\pi^H}{\partial\tau},$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = -\frac{\partial \pi^F}{\partial \tau},$$

$$-\frac{\partial EE}{\partial x^{HH}} \cdot \frac{\frac{\partial^2 \pi^H}{\partial x^{HH} \partial \lambda^H}}{\frac{\partial^2 \pi^H}{\partial x^{HH} \partial x^{HH}}} \cdot \frac{\lambda^H}{\tau} \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) - \frac{\partial EE}{\partial x^{HF}} \cdot \frac{\frac{\partial^2 \pi^H}{\partial x^{HF} \partial \lambda^F}}{\frac{\partial^2 \pi^H}{\partial x^{HF} \partial x^{HF}}} \cdot \frac{\lambda^F}{\tau} \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1) -$$

$$-\frac{\partial EE}{\partial x^{FF}} \cdot \frac{\frac{\partial^2 \pi^F}{\partial x^{FF} \partial \lambda^F}}{\frac{\partial^2 \pi^F}{\partial x^{FF} \partial x^{FF}}} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} - \frac{\partial EE}{\partial x^{FH}} \cdot \frac{\frac{\partial^2 \pi^F}{\partial x^{FH} \partial \lambda^H}}{\frac{\partial^2 \pi^F}{\partial x^{FH} \partial x^{FH}}} \cdot \frac{\lambda^H}{\tau} \cdot (\mathcal{E}_{\lambda^H} + 1) +$$

$$+\frac{\partial EE}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot (\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F}) = -\frac{\partial EE}{\partial \tau}.$$

Finally, note that

$$\frac{\frac{\partial^2 \pi^i}{\partial x^{ij} \partial \lambda^j}}{\frac{\partial^2 \pi^i}{\partial x^{ij} \partial x^{ij}}} \cdot \lambda^j = -\frac{R'(x^{ij})}{R''(x^{ij})}, i \in \{H, F\}, j \in \{H, F\},$$

which completes the proof of Lemma 2.

Now we use these expressions of the total derivatives at particular points like autarky or free trade.

The case of autarky

Denote the autarky point $\bar{\tau}$, it is such that exports $x^{HF}(\bar{\tau}) = 0 = x^{FH}(\bar{\tau})$. Then the following is true at autarky under additive utilities and linear costs.

Lemma 3. *In trade between two asymmetric countries, at the autarky point $\bar{\tau}$ the elasticities and derivatives of the equilibrium variables can be expressed as*

$$\mathcal{E}_{\lambda^H} = -\mathcal{E}_w, \qquad \mathcal{E}_{\lambda^F} = 0, \qquad \mathcal{E}_w = \frac{\frac{R''(x^{HF})}{\lambda^F \cdot C(Q^F)} - w^2 \cdot \frac{R''(x^{FH})}{\lambda^H \cdot C(Q^H)}}{\frac{R''(x^{HF})}{\lambda^F \cdot C(Q^F)} + w^2 \cdot \frac{R''(x^{FH})}{\lambda^H \cdot C(Q^H)}},$$

$$\frac{dx^{HH}}{d\tau} = \frac{dx^{FF}}{d\tau} = 0,$$

$$\frac{dx^{HF}}{d\tau} = \frac{\frac{2}{\tau} \cdot \frac{R'(x^{HF})}{\lambda^F \cdot C(Q^F)}}{\frac{R''(x^{HF})}{\lambda^F \cdot C(Q^F)} + w^2 \cdot \frac{R''(x^{FH})}{\lambda^H \cdot C(Q^H)}} < 0,$$

$$\frac{dx^{FH}}{d\tau} = w \cdot \frac{C(Q^F)}{C(Q^H)} \cdot \frac{dx^{HF}}{d\tau} < 0,$$

$$\frac{dQ^H}{d\tau} = \tau \cdot L^F \cdot \frac{dx^{HF}}{d\tau} < 0, \quad \frac{dQ^F}{d\tau} = \tau \cdot L^H \cdot \frac{dx^{FH}}{d\tau} < 0,$$

$$\frac{dU^H}{d\tau} = -L^F \cdot \frac{R'(x^{FH})}{C(Q^F)} \cdot \frac{1 - \mathcal{E}_u(x^{HH})}{\mathcal{E}_u(x^{HH})} \cdot \frac{dx^{FH}}{d\tau} > 0,$$

$$\frac{dU^F}{d\tau} = -L^H \cdot \frac{R'(x^{HF})}{C(Q^H)} \cdot \frac{1 - \mathcal{E}_u(x^{FF})}{\mathcal{E}_u(x^{FF})} \cdot \frac{dx^{HF}}{d\tau} > 0.$$

Proof. At autarky zero exports $x^{HF} = 0 = x^{FH}$ bring zero revenues:

$$R(x^{HF}) = R(x^{FH}) = 0.$$

Plugging these into previous expressions we get

$$\frac{\partial \pi^H}{\partial \lambda^F} \cdot \lambda^F = -L^F \cdot \frac{R\left(x^{HF}\right)}{\lambda^F} = 0 = -L^H \cdot \frac{R\left(x^{FH}\right)}{\lambda^H} = \frac{\partial \pi^F}{\partial \lambda^H} \cdot \lambda^H,$$

$$\frac{\partial \pi^H}{\partial \tau} = -L^F x^{HF} \cdot w \cdot C'\left(Q^H\right) = 0 = -L^H x^{FH} \cdot C'\left(Q^F\right) = \frac{\partial \pi^F}{\partial \tau},$$

$$\frac{\partial EE}{\partial x^{HH}} = -L^H \cdot \frac{R\left(x^{HF}\right) \cdot C'\left(Q^H\right)}{\lambda^F \cdot \left(C\left(Q^H\right)\right)^2} = 0 = L^F \cdot \frac{R\left(x^{FH}\right) \cdot C'\left(Q^F\right)}{\lambda^F \cdot \left(C\left(Q^F\right)\right)^2} = \frac{\partial EE}{\partial x^{FF}},$$

$$\frac{\partial EE}{\partial \lambda^H} \cdot \lambda^H = \frac{R\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)} = 0,$$

$$\frac{\partial EE}{\partial \tau} = -L^F x^{HF} \cdot \frac{R\left(x^{HF}\right) \cdot C'\left(Q^H\right)}{\lambda^F \cdot \left(C\left(Q^H\right)\right)^2} + L^H x^{FH} \cdot \frac{R\left(x^{FH}\right) \cdot C'\left(Q^F\right)}{\lambda^F \cdot \left(C\left(Q^F\right)\right)^2} = 0,$$

$$\frac{\partial EE}{\partial x^{HF}} = \frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^H\right)} - \tau \cdot L^F \cdot \frac{R\left(x^{HF}\right) \cdot C'\left(Q^H\right)}{\lambda^F \cdot \left(C\left(Q^H\right)\right)^2} = \frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^H\right)},$$

$$\frac{\partial EE}{\partial x^{FH}} = -\frac{R'\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)} + \tau \cdot L^H \cdot \frac{R\left(x^{FH}\right) \cdot C'\left(Q^F\right)}{\lambda^F \cdot \left(C\left(Q^F\right)\right)^2} = -\frac{R'\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)}.$$

Therefore, $\mathcal{E}_{\lambda^H}, \mathcal{E}_{\lambda^F}, \mathcal{E}_w$ satisfy the conditions

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \left(\mathcal{E}_{\lambda^H} + \mathcal{E}_w\right) = 0,$$

$$\frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = 0,$$

$$\frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^H\right)} \cdot \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot \left(\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1\right) - \frac{R'\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)} \cdot \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot \left(\mathcal{E}_{\lambda^H} + 1\right) = 0.$$

So,

$$\mathcal{E}_{\lambda^H} + \mathcal{E}_w = 0,$$

$$\mathcal{E}_{\lambda^F} = 0,$$

$$\frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^H\right)} \cdot \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot \left(\mathcal{E}_w + 1\right) - \frac{R'\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^F\right)} \cdot \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot \left(1 - \mathcal{E}_w\right) = 0.$$

Therefore (here we use the linearity of costs $C : C'\left(Q^H\right) = C'\left(Q^F\right) = c$),

$$\mathcal{E}_{\lambda^H} = -\mathcal{E}_w, \qquad \mathcal{E}_{\lambda^F} = 0,$$

$$\mathcal{E}_w = \frac{\dfrac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} - w^2 \cdot \dfrac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}}{\dfrac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \dfrac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}}.$$

To see how domestic consumptions at autarky stabilize, we simplify the expression from Lemma 2 at the autarky point to

$$\frac{dx^{HH}}{d\tau} = \frac{dx^{FF}}{d\tau} = 0.$$

As to export consumptions, they decrease:

$$\frac{dx^{HF}}{d\tau} = \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot \frac{1}{\tau} \cdot \left(\mathcal{E}_w + 1\right) = \frac{\dfrac{2}{\tau} \cdot \dfrac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)}}{\dfrac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \dfrac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}} < 0,$$

$$\frac{dx^{FH}}{d\tau} = \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot \frac{1}{\tau} \cdot (1 - \mathcal{E}_w) = \frac{\frac{2}{\tau} \cdot w^2 \cdot \frac{R'\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}}{\frac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \frac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}} =$$

$$= \frac{\frac{2}{\tau} \cdot \frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)}}{\frac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \frac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}} \cdot w^2 \cdot \frac{R'\left(x^{FH}\right)}{R'\left(x^{HF}\right)} \cdot \frac{\lambda^F \cdot C\left(Q^F\right)}{\lambda^H \cdot C\left(Q^H\right)} =$$

$$= \frac{\frac{2}{\tau} \cdot \frac{R'\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)}}{\frac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \frac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}} \cdot w^2 \cdot \frac{1}{w} \cdot \frac{\lambda^H}{\lambda^F} \cdot \frac{\lambda^F \cdot C\left(Q^F\right)}{\lambda^H \cdot C\left(Q^H\right)} =$$

$$= w \cdot \frac{C\left(Q^F\right)}{C\left(Q^H\right)} \cdot \frac{dx^{HF}}{d\tau} < 0.$$

As to outputs,

$$\frac{dQ^H}{d\tau} = L^H \cdot \frac{dx^{HH}}{d\tau} + L^F \cdot \left(\tau \cdot \frac{dx^{HF}}{d\tau} + x^{HF}\right) = L^F \cdot \tau \cdot \frac{dx^{HF}}{d\tau} =$$

$$= \frac{2 \cdot L^F \cdot \frac{R'\left(x^{HF}\right)}{C\left(Q^F\right)}}{\frac{R''\left(x^{HF}\right)}{\lambda^F \cdot C\left(Q^F\right)} + w^2 \cdot \frac{R''\left(x^{FH}\right)}{\lambda^H \cdot C\left(Q^H\right)}} < 0,$$

$$\frac{dQ^F}{d\tau} = L^F \cdot \frac{dx^{FF}}{d\tau} + L^H \cdot \left(\tau \cdot \frac{dx^{FH}}{d\tau} + x^{FH}\right) = L^H \cdot \tau \cdot \frac{dx^{FH}}{d\tau} < 0.$$

Consider now the welfare functions of each country:

$$U^H = L^H \cdot \left(N^H \cdot u\left(x^{HH}\right) + N^F \cdot u\left(x^{FH}\right)\right) = L^H \cdot \left(\frac{L^H}{C\left(Q^H\right)} \cdot u\left(x^{HH}\right) + \frac{L^F}{C\left(Q^F\right)} \cdot u\left(x^{FH}\right)\right),$$

$$U^F = L^F \cdot \left(N^F \cdot u\left(x^{FF}\right) + N^H \cdot u\left(x^{HF}\right)\right) = L^F \cdot \left(\frac{L^F}{C\left(Q^F\right)} \cdot u\left(x^{FF}\right) + \frac{L^H}{C\left(Q^H\right)} \cdot u\left(x^{HF}\right)\right).$$

The total derivative of individual welfare is

$$\frac{dU^H}{d\tau} \cdot \frac{1}{L^H} =$$

$$= L^H \cdot \left(\frac{u'\left(x^{HH}\right)}{C\left(Q^H\right)} \cdot \frac{dx^{HH}}{d\tau} - \frac{C'\left(Q^H\right) \cdot u\left(x^{HH}\right)}{\left(C\left(Q^H\right)\right)^2} \cdot \frac{dQ^H}{d\tau}\right) +$$

$$+ L^F \cdot \left(\frac{u'\left(x^{FH}\right)}{C\left(Q^F\right)} \cdot \frac{dx^{FH}}{d\tau} - \frac{C'\left(Q^F\right) \cdot u\left(x^{FH}\right)}{\left(C\left(Q^F\right)\right)^2} \cdot \frac{dQ^F}{d\tau}\right) =$$

$$= -L^H \cdot \frac{C'\left(Q^H\right) \cdot u\left(x^{HH}\right)}{\left(C\left(Q^H\right)\right)^2} \cdot \frac{dQ^H}{d\tau} + L^F \cdot \frac{u'\left(x^{FH}\right)}{C\left(Q^F\right)} \cdot \frac{dx^{FH}}{d\tau} =$$

$$= -L^H \cdot \frac{C'\left(Q^H\right) \cdot u\left(x^{HH}\right)}{\left(C\left(Q^H\right)\right)^2} \cdot L^F \cdot \tau \cdot \frac{dx^{HF}}{d\tau} + L^F \cdot \frac{u'\left(x^{FH}\right)}{C\left(Q^F\right)} \cdot \frac{dx^{FH}}{d\tau}.$$

Using the relation between derivatives of export consumptions: $\frac{dx^{FH}}{d\tau}=w\cdot\frac{C\left(Q^{F}\right)}{C\left(Q^{H}\right)}\cdot\frac{dx^{HF}}{d\tau}$ we simplify the latter expression as

$$\frac{dU^{H}}{d\tau}\cdot\frac{1}{L^{H}}=-L^{H}\cdot\frac{C'\left(Q^{H}\right)\cdot u\left(x^{HH}\right)}{\left(C\left(Q^{H}\right)\right)^{2}}\cdot L^{F}\cdot\tau\cdot\frac{1}{w}\cdot\frac{C\left(Q^{H}\right)}{C\left(Q^{F}\right)}\cdot\frac{dx^{FH}}{d\tau}+L^{F}\cdot\frac{u'\left(x^{FH}\right)}{C\left(Q^{F}\right)}\cdot\frac{dx^{FH}}{d\tau}=$$

$$=\frac{L^{F}}{C\left(Q^{F}\right)}\cdot\left(-L^{H}\cdot\frac{C'\left(Q^{H}\right)\cdot u\left(x^{HH}\right)}{C\left(Q^{H}\right)}\cdot\tau\cdot\frac{1}{w}+u'\left(x^{FH}\right)\right)\cdot\frac{dx^{FH}}{d\tau}=$$

(using the fact that $R\left(x^{HH}\right)\equiv u(x^{HH})\mathcal{E}_{u}\left(x^{HH}\right)$)

$$=\frac{L^{F}}{C\left(Q^{F}\right)}\cdot\left(-L^{H}\cdot\frac{C'\left(Q^{H}\right)\cdot R\left(x^{HH}\right)}{C\left(Q^{H}\right)\cdot\mathcal{E}_{u}\left(x^{HH}\right)}\cdot\tau\cdot\frac{1}{w}+u'\left(x^{FH}\right)\right)\cdot\frac{dx^{FH}}{d\tau}=$$

(simplifying this due to Zero-Profit condition at autarky: $R\left(x^{HH}\right)=C\left(Q^{H}\right)w$)

$$=\frac{L^{F}}{C\left(Q^{F}\right)}\cdot\left(-\frac{C'\left(Q^{H}\right)}{\mathcal{E}_{u}\left(x^{HH}\right)}\cdot\tau\cdot\lambda^{H}+u'\left(x^{FH}\right)\right)\cdot\frac{dx^{FH}}{d\tau}=$$

(because at autarky $R'\left(x^{FH}\right)=(1-r_{u}(x^{FH})u'\left(x^{FH}\right))=(1-r_{u}(0)u'\left(x^{FH}\right))=u'\left(x^{FH}\right)$)

$$=\frac{L^{F}}{C\left(Q^{F}\right)}\cdot\left(-\frac{C'\left(Q^{F}\right)}{\mathcal{E}_{u}\left(x^{HH}\right)}\cdot\tau\cdot\lambda^{H}+R'\left(x^{FH}\right)\right)\cdot\frac{dx^{FH}}{d\tau}=$$

(due to producer's FOC at autarky $C'\left(Q^{F}\right)=R'\left(x^{FH}\right)$)

$$=\frac{L^{F}}{C\left(Q^{F}\right)}\cdot\left(-\frac{R'\left(x^{FH}\right)}{\mathcal{E}_{u}\left(x^{HH}\right)}+R'\left(x^{FH}\right)\right)\cdot\frac{dx^{FH}}{d\tau}=$$

$$=-L^{F}\cdot\frac{R'\left(x^{FH}\right)}{C\left(Q^{F}\right)}\cdot\frac{1-\mathcal{E}_{u}\left(x^{HH}\right)}{\mathcal{E}_{u}\left(x^{HH}\right)}\cdot\frac{dx^{FH}}{d\tau}>0\,.$$

Analogously, the derivative in Foreign country is

$$\frac{dU^{F}}{d\tau}\cdot\frac{1}{L^{F}}=$$

$$=L^{F}\cdot\left(\frac{u'\left(x^{FF}\right)}{C\left(Q^{F}\right)}\cdot\frac{dx^{FF}}{d\tau}-\frac{C'\left(Q^{F}\right)\cdot u\left(x^{FF}\right)}{\left(C\left(Q^{F}\right)\right)^{2}}\cdot\frac{dQ^{F}}{d\tau}\right)+$$

$$+L^{H}\cdot\left(\frac{u'\left(x^{HF}\right)}{C\left(Q^{H}\right)}\cdot\frac{dx^{HF}}{d\tau}-\frac{C'\left(Q^{H}\right)\cdot u\left(x^{HF}\right)}{\left(C\left(Q^{H}\right)\right)^{2}}\cdot\frac{dQ^{H}}{d\tau}\right)=$$

$$=-L^{F}\cdot\frac{C'\left(Q^{F}\right)\cdot u\left(x^{FF}\right)}{\left(C\left(Q^{F}\right)\right)^{2}}\cdot L^{H}\cdot\tau\cdot\frac{dx^{FH}}{d\tau}+L^{H}\cdot\frac{u'\left(x^{HF}\right)}{C\left(Q^{H}\right)}\cdot\frac{dx^{HF}}{d\tau}=$$

$$=-L^{F}\cdot\frac{C'\left(Q^{F}\right)\cdot u\left(x^{FF}\right)}{\left(C\left(Q^{F}\right)\right)^{2}}\cdot L^{H}\cdot\tau\cdot w\cdot\frac{C\left(Q^{F}\right)}{C\left(Q^{H}\right)}\cdot\frac{dx^{HF}}{d\tau}+L^{H}\cdot\frac{u'\left(x^{HF}\right)}{C\left(Q^{H}\right)}\cdot\frac{dx^{HF}}{d\tau}=$$

$$=\frac{L^{H}}{C\left(Q^{H}\right)}\cdot\left(-L^{F}\cdot\frac{C'\left(Q^{F}\right)\cdot u\left(x^{FF}\right)}{C\left(Q^{F}\right)}\cdot\tau\cdot w+u'\left(x^{HF}\right)\right)\cdot\frac{dx^{HF}}{d\tau}=$$

$$= \frac{L^H}{C\left(Q^H\right)} \cdot \left(-L^F \cdot \frac{C'\left(Q^F\right) \cdot R\left(x^{FF}\right)}{C\left(Q^F\right) \cdot \mathcal{E}_u\left(x^{FF}\right)} \cdot \tau \cdot w + u'\left(x^{HF}\right)\right) \cdot \frac{dx^{HF}}{d\tau} =$$

$$= \frac{L^H}{C\left(Q^H\right)} \cdot \left(-\lambda^F \cdot \frac{C'\left(Q^F\right)}{\mathcal{E}_u\left(x^{FF}\right)} \cdot \tau \cdot w + u'\left(x^{HF}\right)\right) \cdot \frac{dx^{HF}}{d\tau} =$$

$$= \frac{L^H}{C\left(Q^H\right)} \cdot \left(-\lambda^F \cdot \frac{C'\left(Q^H\right)}{\mathcal{E}_u\left(x^{FF}\right)} \cdot \tau \cdot w + u'\left(x^{HF}\right)\right) \cdot \frac{dx^{HF}}{d\tau} =$$

$$= \frac{L^H}{C\left(Q^H\right)} \cdot \left(-\frac{R'\left(x^{HF}\right)}{\mathcal{E}_u\left(x^{FF}\right)} + R'\left(x^{HF}\right)\right) \cdot \frac{dx^{HF}}{d\tau} =$$

$$= -L^H \cdot \frac{R'\left(x^{HF}\right)}{C\left(Q^H\right)} \cdot \frac{1 - \mathcal{E}_u\left(x^{FF}\right)}{\mathcal{E}_u\left(x^{FF}\right)} \cdot \frac{dx^{HF}}{d\tau} > 0.$$

To complete the proof, let us note that if function $u(\xi)$ is strictly concave and twice differentiable, with $u(0) = 0,\ u(\xi) > 0\ \forall \xi > 0$, then $\mathcal{E}_u(\xi) < 1\ \forall \xi > 0$.[14] This completes the proof.

The case of free trade

Consider the situation near $\tau = 1$ (free trade). Then, trivially, all consumptions are equal:

$$x^{HH} = x^{HF} = x^{FF} = x^{FH} = x, \qquad Q^H = Q^F = Q = \left(L^H + L^F\right) \cdot x = L \cdot x.$$

Moreover, marginal utilities are equal:

$$\lambda^H = \lambda^F = \lambda, \qquad w = 1.$$

This makes trade balance trivial:

$$EE = 0 \Longleftrightarrow \frac{R(x)}{\lambda \cdot C(Q)} = \frac{R(x)}{\lambda \cdot C(Q)}.$$

Hence, at free trade the equilibrium equations simplify to

$$L \cdot \frac{R(x)}{\lambda} = C(Q), \qquad \frac{R'(x)}{\lambda} = C'(Q),$$

$$N^H = \frac{L^H}{C(Q)}, \qquad N^F = \frac{L^F}{C(Q)}.$$

Under similar arguments we simplify notations:

$$R = R(x), \qquad R' = R'(x), \qquad R'' = R''(x),$$

$$C = C(Q), \qquad C' = C'(Q),$$

$$\mathcal{E}_R = \mathcal{E}_R(x) = \frac{R'(x) \cdot x}{R(x)} = 1 - r_u(x) = 1 - r_u, \qquad r'_u = r'_u(x).$$

Lemma 4. *In the trade between two asymmetric countries with linear costs, in free trade one has*

$$\mathcal{E}_w = \mathcal{E}_{w/\tau} = \frac{L^H - L^F}{L} \cdot \mathcal{E}_R,$$

$$\mathcal{E}_{\lambda^H} = \mathcal{E}_{\lambda^H/\tau} = -\frac{L^F}{L} \cdot \left(\frac{L^H}{L^F} - \frac{L^H - L^F}{L} \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R}\right) \cdot \mathcal{E}_R,$$

[14] Indeed, for every $\forall \xi > 0$, $\mathcal{E}_{\mathrm{u}}(\xi) < 1 \Longleftrightarrow u'(\xi) \cdot \xi - u(\xi) < 0 \quad \forall \xi > 0$. Consider the function $g(\xi) = u'(\xi) \cdot \xi - u(\xi)$. One has $g'(\xi) \equiv u''(\xi) \cdot \xi < 0\ \forall \xi > 0$ due to strictly concavity of u. But $g(0) = u(0) = 0$. Hence $g(\xi) < 0\ \forall \xi > 0$, i.e., $u'(\xi) \cdot \xi - u(\xi) < 0\ \forall \xi > 0$.

$$\mathcal{E}_{\lambda^F} = \mathcal{E}_{\lambda^F/\tau} = -\frac{L^H}{L} \cdot \left(1 + \frac{L^H - L^F}{L} \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R}\right) \cdot \mathcal{E}_R \,,$$

$$\mathcal{E}_{x^{HH}} = \mathcal{E}_{x^{HH}/\tau} = -\frac{\mathcal{E}_{\lambda^H} + \mathcal{E}_w}{r_R} \,, \qquad \mathcal{E}_{x^{HF}} = \mathcal{E}_{x^{HF}/\tau} = -\frac{\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1}{r_R} \,,$$

$$\mathcal{E}_{x^{FF}} = \mathcal{E}_{x^{FF}/\tau} = -\frac{\mathcal{E}_{\lambda^F}}{r_R} \,, \qquad \mathcal{E}_{x^{FH}} = \mathcal{E}_{x^{FH}/\tau} = -\frac{\mathcal{E}_{\lambda^H} + 1}{r_R} \,,$$

$$\mathcal{E}_{Q^H} = \mathcal{E}_{Q^H/\tau} = \frac{L^F}{L} \cdot \frac{r_u' \cdot x}{r_R \cdot \mathcal{E}_R} \,, \qquad \mathcal{E}_{Q^F} = \mathcal{E}_{Q^F/\tau} = \frac{L^H}{L} \frac{r_u' \cdot x}{r_R \cdot \mathcal{E}_R} \,,$$

$$\frac{dU^H}{d\tau} = -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(A^H \cdot \mathcal{E}_u + B^H \cdot r_u' \cdot x\right) \,,$$

$$\frac{dU^F}{d\tau} = -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(A^F \cdot \mathcal{E}_u + B^F \cdot r_u' \cdot x\right) \,,$$

where

$$A^H = \frac{\left(L^H - L^F\right) \cdot r_u + 2 \cdot L^F}{L \cdot (\mathcal{E}_R + r_R)} \cdot r_R > 0, \qquad B^H = 2 \cdot L^H \cdot \frac{r_u' \cdot x + (1 - \mathcal{E}_u) \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \,,$$

$$A^F = \frac{\left(L^H - L^F\right) \cdot r_R \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R)} + r_u > 0, \qquad B^F = \frac{\left(L^H - L^F\right) \cdot \mathcal{E}_u}{L \cdot (\mathcal{E}_R + r_R)} + 2 \cdot \frac{L^F}{L} > 0.$$

Hence if in free trade $r_u' > 0$, then in every country Welfare decreases in τ.

Moreover,

$$\frac{dU^H}{d\tau} + \frac{dU^F}{d\tau} = -\frac{2 \cdot L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (\mathcal{E}_{r_u} + \mathcal{E}_u) \cdot r_u = -\frac{2 \cdot L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (2 - r_{u'} - \mathcal{E}_{\mathcal{E}_u}) \cdot r_u \,.$$

Hence if in free trade $r_u' > 0$ or $\mathcal{E}_u' < 0$, then the total Welfare decreases in τ.

Proof. One has

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \lambda = \frac{\partial \pi^F}{\partial \lambda^H} \cdot \lambda = -L^H \cdot \frac{R}{\lambda} \,, \qquad \frac{\partial \pi^H}{\partial \lambda^F} \cdot \lambda = \frac{\partial \pi^F}{\partial \lambda^F} \cdot \lambda = -L^F \cdot \frac{R}{\lambda} \,,$$

$$-\frac{\partial \pi^H}{\partial \tau} = L^F \cdot x \cdot C' = L^F \cdot x \cdot \frac{R'}{\lambda} \,, \qquad -\frac{\partial \pi^F}{\partial \tau} = L^H \cdot x \cdot C' = L^H \cdot x \cdot \frac{R'}{\lambda} \,,$$

$$\frac{\partial EE}{\partial x^{HH}} = -L^H \cdot \frac{R \cdot C'}{\lambda \cdot C^2} = -\frac{L^H}{(L)^2} \cdot \frac{R'}{R} \,, \qquad \frac{\partial EE}{\partial x^{HF}} = \frac{R'}{\lambda \cdot C} - L^F \cdot \frac{R \cdot C'}{\lambda \cdot C^2} = \frac{L^H}{(L)^2} \cdot \frac{R'}{R} \,,$$

$$\frac{\partial EE}{\partial x^{FF}} = L^F \cdot \frac{R \cdot C'}{\lambda \cdot C^2} = \frac{L^F}{(L)^2} \cdot \frac{R'}{R} \,, \qquad \frac{\partial EE}{\partial x^{FH}} = -\frac{R'}{\lambda \cdot C} + L^H \cdot \frac{R \cdot C'}{\lambda \cdot C^2} = -\frac{L^F}{(L)^2} \cdot \frac{R'}{R} \,,$$

$$\frac{\partial EE}{\partial \lambda^H} \cdot \lambda = \frac{R}{\lambda \cdot C} = \frac{1}{L} \,, \qquad -\frac{\partial EE}{\partial \tau} = -\left(L^H - L^F\right) \cdot x \cdot \frac{R \cdot C'}{\lambda \cdot C^2} = -\frac{L^H - L^F}{(L)^2} \cdot x \cdot \frac{R'}{R} \,.$$

Hence, the conditions

$$\frac{\partial \pi^H}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^H}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial \pi^H}{\partial w} \cdot \frac{w}{\tau} \cdot \mathcal{E}_w = -\frac{\partial \pi^H}{\partial \tau} \,,$$

$$\frac{\partial \pi^F}{\partial \lambda^H} \cdot \frac{\lambda^H}{\tau} \cdot \mathcal{E}_{\lambda^H} + \frac{\partial \pi^F}{\partial \lambda^F} \cdot \frac{\lambda^F}{\tau} \cdot \mathcal{E}_{\lambda^F} = -\frac{\partial \pi^F}{\partial \tau} \,,$$

$$\frac{\partial EE}{\partial x^{HH}} \cdot \frac{R'\left(x^{HH}\right)}{R''\left(x^{HH}\right)} \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) + \frac{\partial EE}{\partial x^{HF}} \cdot \frac{R'\left(x^{HF}\right)}{R''\left(x^{HF}\right)} \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1) +$$

$$+ \frac{\partial EE}{\partial x^{FF}} \cdot \frac{R'\left(x^{FF}\right)}{R''\left(x^{FF}\right)} \cdot \mathcal{E}_{\lambda^F} + \frac{\partial EE}{\partial x^{FH}} \cdot \frac{R'\left(x^{FH}\right)}{R''\left(x^{FH}\right)} \cdot (\mathcal{E}_{\lambda^H} + 1) +$$

$$+\frac{\partial EE}{\partial \lambda^H} \cdot (\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F}) = -\frac{\partial EE}{\partial \tau} \cdot \tau$$

have the form

$$-L^H \cdot \frac{R}{\lambda} \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) - L^F \cdot \frac{R}{\lambda} \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w) = L^F \cdot x \cdot \frac{R'}{\lambda},$$

$$-L^H \cdot \frac{R}{\lambda} \cdot \mathcal{E}_{\lambda^H} - L^F \cdot \frac{R}{\lambda} \cdot \mathcal{E}_{\lambda^F} = L^H \cdot x \cdot \frac{R'}{\lambda},$$

$$-\frac{L^H}{L} \cdot \frac{(R')^2}{R \cdot R''} \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) + \frac{L^H}{L} \cdot \frac{(R')^2}{R \cdot R''} \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1) +$$

$$+\frac{L^F}{L} \cdot \frac{(R')^2}{R \cdot R''} \cdot \mathcal{E}_{\lambda^F} - \frac{L^F}{L} \cdot \frac{(R')^2}{R \cdot R''} \cdot (\mathcal{E}_{\lambda^H} + 1) +$$

$$+\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F} = -\frac{L^H - L^F}{L} \cdot x \cdot \frac{R'}{R},$$

i.e.,

$$-L^H \cdot \mathcal{E}_{\lambda^H} - L^F \cdot \mathcal{E}_{\lambda^F} - L \cdot \mathcal{E}_w = L^F \cdot \mathcal{E}_R,$$

$$-L^H \cdot \mathcal{E}_{\lambda^H} - L^F \cdot \mathcal{E}_{\lambda^F} = L^H \cdot \mathcal{E}_R,$$

$$\frac{L^H}{L} \cdot \frac{\mathcal{E}_R}{r_R} \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) - \frac{L^H}{L} \cdot \frac{\mathcal{E}_R}{r_R} \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1) -$$

$$-\frac{L^F}{L} \cdot \frac{\mathcal{E}_R}{r_R} \cdot \mathcal{E}_{\lambda^F} + \frac{L^F}{L} \cdot \frac{\mathcal{E}_R}{r_R} \cdot (\mathcal{E}_{\lambda^H} + 1) +$$

$$+\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F} = -\frac{L^H - L^F}{L} \cdot \mathcal{E}_R,$$

i.e.,

$$\mathcal{E}_w = \frac{L^H - L^F}{L} \cdot \mathcal{E}_R,$$

$$-L^H \cdot \mathcal{E}_{\lambda^H} - L^F \cdot \mathcal{E}_{\lambda^F} = L^H \cdot \mathcal{E}_R,$$

$$\mathcal{E}_{\lambda^H} - \mathcal{E}_{\lambda^F} = \frac{L^H - L^F}{L} \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_R.$$

One has

$$\mathcal{E}_w = \frac{L^H - L^F}{L} \cdot \mathcal{E}_R > 0,$$

$$\mathcal{E}_{\lambda^H} = -\frac{L^F}{L} \cdot \left(\frac{L^H}{L^F} - \frac{L^H - L^F}{L} \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R}\right) \cdot \mathcal{E}_R,$$

$$\mathcal{E}_{\lambda^F} = -\frac{L^H}{L} \cdot \left(1 + \frac{L^H - L^F}{L} \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R}\right) \cdot \mathcal{E}_R.$$

Further,

$$\frac{dQ^H}{d\tau} = L^H \cdot \frac{dx^{HH}}{d\tau} + L^F \cdot \frac{dx^{HF}}{d\tau} + L^F \cdot x^{HF} = L^H \cdot \frac{dx^{HH}}{d\tau} + L^F \cdot \frac{dx^{HF}}{d\tau} + L^F \cdot x =$$

$$= \left(L^H \cdot \mathcal{E}_{x^{HH}} + L^F \cdot \mathcal{E}_{x^{HF}} + L^F\right) \cdot x =$$

$$= \left(-L^H \cdot \frac{\mathcal{E}_{\lambda^H} + \mathcal{E}_{w/\tau}}{r_R} - L^F \cdot \frac{\mathcal{E}_{\lambda^F} + \mathcal{E}_{w/\tau} + 1}{r_R} + L^F\right) \cdot x =$$

(we use the condition $L^H \cdot \mathcal{E}_{\lambda^H} + L^F \cdot \mathcal{E}_{\lambda^F} = -L^H \cdot \mathcal{E}_R$)

$$= \left(-\frac{-L^H \cdot \mathcal{E}_R + \left(L^H - L^F\right) \cdot \mathcal{E}_R + L^F}{r_R} + L^F\right) \cdot x =$$

$$= \frac{L^F \cdot x}{r_R} \cdot (\mathcal{E}_R + r_R - 1) = L^F \cdot \frac{r_u' \cdot x^2}{r_R \cdot \mathcal{E}_R},$$

$$\frac{dQ^F}{d\tau} = L^F \cdot \frac{dx^{FF}}{d\tau} + L^H \cdot \frac{dx^{FH}}{d\tau} + L^H \cdot x^{FH} = L^F \cdot \frac{dx^{FF}}{d\tau} + L^H \cdot \frac{dx^{FH}}{d\tau} + L^H \cdot x =$$

$$= \left(L^F \cdot \mathcal{E}_{x^{FF}} + L^H \cdot \mathcal{E}_{x^{FH}} + L^H\right) \cdot x = \left(-L^F \cdot \frac{\mathcal{E}_{\lambda^F}}{r_R} - L^H \cdot \frac{\mathcal{E}_{\lambda^H} + 1}{r_R} + L^H\right) \cdot x =$$

$$= \frac{L^H \cdot x}{r_R} \cdot (\mathcal{E}_R + r_R - 1) = L^H \cdot \left(\frac{x}{r_R}\right)^2 \cdot r_u'(x).$$

Consider now welfare functions

$$U^H = L^H \cdot \left(N^H \cdot u\left(x^{HH}\right) + N^F \cdot u\left(x^{FH}\right)\right) = L^H \cdot \left(\frac{L^H}{C\left(Q^H\right)} \cdot u\left(x^{HH}\right) + \frac{L^F}{C\left(Q^F\right)} \cdot u\left(x^{FH}\right)\right),$$

$$U^F = L^F \cdot \left(N^F \cdot u\left(x^{FF}\right) + N^H \cdot u\left(x^{HF}\right)\right) = L^F \cdot \left(\frac{L^F}{C\left(Q^F\right)} \cdot u\left(x^{FF}\right) + \frac{L^H}{C\left(Q^H\right)} \cdot u\left(x^{HF}\right)\right).$$

One has

$$\frac{dU^H}{d\tau} \cdot \frac{1}{L^H} = L^H \cdot \frac{u' \cdot \frac{dx^{HH}}{d\tau} \cdot C - u \cdot C' \cdot \frac{dQ^H}{d\tau}}{C^2} + L^F \cdot \frac{u' \cdot \frac{dx^{FH}}{d\tau} \cdot C - u \cdot C' \cdot \frac{dQ^F}{d\tau}}{C^2} =$$

$$= \frac{u}{C} \cdot \left(L^H \cdot \left(\frac{u'}{u} \cdot \frac{dx^{HH}}{d\tau} - \frac{C'}{C} \cdot \frac{dQ^H}{d\tau}\right) + L^F \cdot \left(\frac{u'}{u} \cdot \frac{dx^{FH}}{d\tau} - \frac{C'}{C} \cdot \frac{dQ^F}{d\tau}\right)\right) =$$

$$= \frac{u}{C} \cdot \left(L^H \cdot \left(\mathcal{E}_u \cdot \mathcal{E}_{x^{HH}} - \mathcal{E}_C \cdot \mathcal{E}_{Q^H}\right) + L^F \cdot \left(\mathcal{E}_u \cdot \mathcal{E}_{x^{FH}} - \mathcal{E}_C \cdot \mathcal{E}_{Q^F}\right)\right) =$$

$$= \frac{u}{C} \cdot \left(\mathcal{E}_u \cdot \left(L^H \cdot \mathcal{E}_{x^{HH}} + L^F \cdot \mathcal{E}_{x^{FH}}\right) - \mathcal{E}_C \cdot \left(L^H \cdot \mathcal{E}_{Q^H} + L^F \cdot \mathcal{E}_{Q^F}\right)\right) =$$

$$= \frac{u}{C} \cdot \left(\mathcal{E}_u \cdot \left(L^H \cdot \mathcal{E}_{x^{HH}} + L^F \cdot \mathcal{E}_{x^{FH}}\right) - \mathcal{E}_R \cdot \left(L^H \cdot \mathcal{E}_{Q^H} + L^F \cdot \mathcal{E}_{Q^F}\right)\right) =$$

$$= \frac{u}{C} \cdot \left(-\mathcal{E}_u \cdot \left(L^H \cdot \frac{\mathcal{E}_{\lambda^H} + \mathcal{E}_w}{r_R} + L^F \cdot \frac{\mathcal{E}_{\lambda^H} + 1}{r_R}\right) - 2 \cdot \frac{L^H \cdot L^F}{L} \cdot \frac{r_u'(x) \cdot x}{r_R}\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(L^H \cdot (\mathcal{E}_{\lambda^H} + \mathcal{E}_w) + L^F \cdot (\mathcal{E}_{\lambda^H} + 1)\right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u'(x) \cdot x\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(L \cdot \mathcal{E}_{\lambda^H} + L^H \cdot \mathcal{E}_w + L^F\right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u'(x) \cdot x\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-L^F \cdot \left(\frac{L^H}{L^F} \cdot \mathcal{E}_R - \frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w\right) + L^H \cdot \mathcal{E}_w + L^F\right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u' \cdot x\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-L^H \cdot \mathcal{E}_R + \left(L^F \cdot \frac{1 - r_R}{\mathcal{E}_R + r_R} + L^H\right) \cdot \mathcal{E}_w + L^F\right) + 2 \cdot 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u' \cdot x\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-L^H \cdot (\mathcal{E}_R - \mathcal{E}_w) + L^F \cdot \left(\frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + 1\right)\right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u' \cdot x\right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-2 \cdot \frac{L^H \cdot L^F}{L} \cdot \mathcal{E}_R + L^F \cdot \left(\frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + 1\right)\right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u' \cdot x\right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-2 \cdot \frac{L^H}{L} \cdot \mathcal{E}_R + \frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + 1\right) + 2 \cdot \frac{L^H}{L} \cdot r_u' \cdot x\right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \frac{-2 \cdot \frac{L^H}{L} \cdot r_u' \cdot x + \left(2 \cdot \frac{L^F}{L} + \frac{L^H - L^F}{L} \cdot r_u\right) \cdot r_R}{\mathcal{E}_R + r_R} + 2 \cdot \frac{L^H}{L} \cdot r_u' \cdot x \right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\frac{2 \cdot \frac{L^F}{L} + \frac{L^H - L^F}{L} \cdot r_u}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_u \cdot r_R + 2 \cdot \frac{L^H}{L} \cdot \left(1 - \frac{\mathcal{E}_u}{\mathcal{E}_R + r_R}\right) \cdot r_u' \cdot x \right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\frac{2 \cdot \frac{L^F}{L} + \frac{L^H - L^F}{L} \cdot r_u}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_u \cdot r_R + 2 \cdot \frac{L^H}{L} \cdot \frac{\mathcal{E}_R + r_R - 1 + 1 - \mathcal{E}_u}{\mathcal{E}_R + r_R} \cdot r_u' \cdot x \right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\frac{2 \cdot \frac{L^F}{L} + \frac{L^H - L^F}{L} \cdot r_u}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_u \cdot r_R + 2 \cdot \frac{L^H}{L} \cdot \frac{\frac{r_u' \cdot x}{\mathcal{E}_R} + 1 - \mathcal{E}_u}{\mathcal{E}_R + r_R} \cdot r_u' \cdot x \right) =$$

$$= -L^F \cdot \frac{u}{C \cdot r_R} \cdot \left(\frac{2 \cdot \frac{L^F}{L} + \frac{L^H - L^F}{L} \cdot r_u}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_u \cdot r_R + 2 \cdot \frac{L^H}{L} \cdot \frac{r_u' \cdot x + (1 - \mathcal{E}_u) \cdot \mathcal{E}_R}{(\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r_u' \cdot x \right).$$

Analogously,

$$\frac{dU^F}{d\tau} \cdot \frac{1}{L^F} = L^F \cdot \frac{u' \cdot \frac{dx^{FF}}{d\tau} \cdot C - u \cdot C' \cdot \frac{dQ^F}{d\tau}}{C^2} + L^H \cdot \frac{u' \cdot \frac{dx^{HF}}{d\tau} \cdot C - u \cdot C' \cdot \frac{dQ^H}{d\tau}}{C^2} =$$

$$= \frac{u}{C} \cdot \left(L^F \cdot \left(\frac{u'}{u} \cdot \frac{dx^{FF}}{d\tau} - \frac{C'}{C} \cdot \frac{dQ^F}{d\tau} \right) + L^H \cdot \left(\frac{u'}{u} \cdot \frac{dx^{HF}}{d\tau} - \frac{C'}{C} \cdot \frac{dQ^H}{d\tau} \right) \right) =$$

$$= \frac{u}{C} \cdot \left(L^F \cdot \left(\mathcal{E}_u \cdot \mathcal{E}_{x^{FF}} - \mathcal{E}_C \cdot \mathcal{E}_{Q^F} \right) + L^H \cdot \left(\mathcal{E}_u \cdot \mathcal{E}_{x^{HF}} - \mathcal{E}_C \cdot \mathcal{E}_{Q^H} \right) \right) =$$

$$= \frac{u}{C} \cdot \left(\mathcal{E}_u \cdot \left(L^F \cdot \mathcal{E}_{x^{FF}} + L^H \cdot \mathcal{E}_{x^{HF}} \right) - \mathcal{E}_C \cdot \left(L^F \cdot \mathcal{E}_{Q^F} + L^H \cdot \mathcal{E}_{Q^H} \right) \right) =$$

$$= \frac{u}{C} \cdot \left(\mathcal{E}_u \cdot \left(L^F \cdot \mathcal{E}_{x^{FF}} + L^H \cdot \mathcal{E}_{x^{HF}} \right) - \mathcal{E}_R \cdot \left(L^F \cdot \mathcal{E}_{Q^F} + L^H \cdot \mathcal{E}_{Q^H} \right) \right) =$$

$$= \frac{u}{C} \cdot \left(-\mathcal{E}_u \cdot \left(L^F \cdot \frac{\mathcal{E}_{\lambda^F}}{r_R} + L^H \cdot \frac{\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1}{r_R} \right) - 2 \cdot \frac{L^H \cdot L^F}{L} \cdot \frac{r_u'(x) \cdot x}{r_R} \right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(L^F \cdot \mathcal{E}_{\lambda^F} + L^H \cdot (\mathcal{E}_{\lambda^F} + \mathcal{E}_w + 1) \right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u'(x) \cdot x \right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(L \cdot \mathcal{E}_{\lambda^F} + L^H \cdot (\mathcal{E}_w + 1) \right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u'(x) \cdot x \right) =$$

$$= -\frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-L^H \cdot \left(\mathcal{E}_R + \frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w \right) + L^H \cdot (\mathcal{E}_w + 1) \right) + 2 \cdot \frac{L^H \cdot L^F}{L} \cdot r_u'(x) \cdot x \right) =$$

$$= -L^H \cdot \frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-\mathcal{E}_R - \frac{1 - r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + \mathcal{E}_w + 1 \right) + 2 \cdot \frac{L^F}{L} \cdot r_u'(x) \cdot x \right) =$$

$$= -L^H \cdot \frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(-\mathcal{E}_R + \frac{r_R}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + 1 \right) + \mathcal{E}_u \cdot \frac{\mathcal{E}_R + r_R - 1}{\mathcal{E}_R + r_R} \cdot \mathcal{E}_w + 2 \cdot \frac{L^F}{L} \cdot r_u'(x) \cdot x \right) =$$

$$= -L^H \cdot \frac{u}{C \cdot r_R} \cdot \left(\mathcal{E}_u \cdot \left(\frac{L^H - L^F}{L} \cdot \frac{r_R \cdot \mathcal{E}_R}{\mathcal{E}_R + r_R} + 1 - \mathcal{E}_R \right) + \frac{\mathcal{E}_u \cdot \mathcal{E}_w}{\mathcal{E}_R + r_R} \cdot \frac{r'_u(x) \cdot x}{\mathcal{E}_R} + 2 \cdot \frac{L^F}{L} \cdot r'_u(x) \cdot x \right).$$

One has

$$\frac{dU^H}{d\tau} = -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(A^H \cdot \mathcal{E}_u + B^H \cdot r'_u \cdot x \right),$$

$$\frac{dU^F}{d\tau} = -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(A^F \cdot \mathcal{E}_u + B^F \cdot r'_u \cdot x \right),$$

where

$$A^H = \frac{\left(L^H - L^F\right) \cdot r_u + 2 \cdot L^F}{L \cdot (\mathcal{E}_R + r_R)} \cdot r_R > 0, \qquad B^H = 2 \cdot L^H \cdot \frac{r'_u \cdot x + (1 - \mathcal{E}_u) \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R},$$

$$A^F = \frac{\left(L^H - L^F\right) \cdot r_R \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R)} + r_u > 0, \qquad B^F = \frac{\left(L^H - L^F\right) \cdot \mathcal{E}_u}{L \cdot (\mathcal{E}_R + r_R)} + 2 \cdot \frac{L^F}{L} > 0.$$

Hence

$$\frac{dU^H}{d\tau} + \frac{dU^F}{d\tau} = -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(A^H \cdot \mathcal{E}_u + B^H \cdot r'_u \cdot x + A^F \cdot \mathcal{E}_u + B^F \cdot r'_u \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\left(A^H + A^F\right) \cdot \mathcal{E}_u + \left(B^H + B^F\right) \cdot r'_u \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{\left(L^H - L^F\right) \cdot r_u + 2 \cdot L^F}{L \cdot (\mathcal{E}_R + r_R)} \cdot r_R + \frac{\left(L^H - L^F\right) \cdot r_R \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R)} + r_u \right) \cdot \mathcal{E}_u -$$

$$-\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(2 \cdot L^H \cdot \frac{r'_u \cdot x + (1 - \mathcal{E}_u) \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} + \frac{\left(L^H - L^F\right) \cdot \mathcal{E}_u}{L \cdot (\mathcal{E}_R + r_R)} + 2 \cdot \frac{L^F}{L} \right) \cdot r'_u \cdot x =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{\left(L^H - L^F\right) \cdot r_u \cdot r_R + 2 \cdot L^F \cdot r_R + \left(L^H - L^F\right) \cdot r_R \cdot \mathcal{E}_R + L \cdot (\mathcal{E}_R + r_R) \cdot r_u}{L \cdot (\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u -$$

$$-\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{2 \cdot L \cdot r'_u \cdot x + 2 \cdot L^F \cdot \mathcal{E}_R + 2 \cdot L^H \cdot (1 - \mathcal{E}_u) \cdot \mathcal{E}_R + \left(L^H - L^F\right) \cdot \mathcal{E}_u \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{\left(L^H - L^F\right) \cdot (r_u + \mathcal{E}_R) \cdot r_R + 2 \cdot L^F \cdot r_R + L \cdot (\mathcal{E}_R + r_R) \cdot r_u}{L \cdot (\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u -$$

$$-\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{2 \cdot L \cdot r'_u \cdot x + 2 \cdot L \cdot \mathcal{E}_R - 2 \cdot L^H \cdot \mathcal{E}_u \cdot \mathcal{E}_R + \left(L^H - L^F\right) \cdot \mathcal{E}_u \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{\left(L^H - L^F\right) \cdot r_R + 2 \cdot L^F \cdot r_R + L \cdot (\mathcal{E}_R + r_R) \cdot r_u}{L \cdot (\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u -$$

$$-\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{2 \cdot L \cdot r'_u \cdot x + 2 \cdot L \cdot \mathcal{E}_R - L \cdot \mathcal{E}_u \cdot \mathcal{E}_R}{L \cdot (\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{L \cdot r_R + L \cdot (\mathcal{E}_R + r_R) \cdot r_u}{L \cdot (\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u -$$

$$-\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \frac{2 \cdot r'_u \cdot x + 2 \cdot \mathcal{E}_R - \mathcal{E}_u \cdot \mathcal{E}_R}{(\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{r_R + (\mathcal{E}_R + r_R) \cdot r_u}{(\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u + \frac{2 \cdot r'_u \cdot x + (-\mathcal{E}_u + 2) \cdot \mathcal{E}_R}{(\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{r_R + (\mathcal{E}_R + r_R) \cdot r_u - r'_u \cdot x}{(\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u + 2 \cdot \frac{r'_u \cdot x + \mathcal{E}_R}{(\mathcal{E}_R + r_R) \cdot \mathcal{E}_R} \cdot r'_u \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{r_R + (\mathcal{E}_R + r_R) \cdot r_u - (\mathcal{E}_R + r_R - 1) \cdot \mathcal{E}_R}{(\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u + 2 \cdot r'_u \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{r_R + \mathcal{E}_R + (\mathcal{E}_R + r_R) \cdot (r_u - \mathcal{E}_R)}{(\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u + 2 \cdot r_u' \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot \left(\frac{(\mathcal{E}_R + r_R) \cdot (r_u - \mathcal{E}_R + 1)}{(\mathcal{E}_R + r_R)} \cdot \mathcal{E}_u + 2 \cdot r_u' \cdot x \right) =$$

$$= -\frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (2 \cdot r_u \cdot \mathcal{E}_u + 2 \cdot r_u' \cdot x) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot r_u \cdot \left(\mathcal{E}_u + \frac{r_u' \cdot x}{r_u} \right) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot r_u \cdot (\mathcal{E}_u + \mathcal{E}_{r_u}) .$$

Moreover

$$\frac{dU^H}{d\tau} + \frac{dU^F}{d\tau} = -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (r_u \cdot \mathcal{E}_u + r_u' \cdot x) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (r_u \cdot (1 + r_u - r_{u'}) + r_u \cdot \mathcal{E}_u) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (r_u \cdot (2 - r_{u'}) - r_u \cdot (1 - r_u) + r_u \cdot \mathcal{E}_u) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot (r_u \cdot (2 - r_{u'}) - r_u \cdot (1 - r_u - \mathcal{E}_u)) =$$

$$= -2 \cdot \frac{L^H \cdot L^F \cdot u}{C \cdot r_R} \cdot r_u \cdot ((2 - r_{u'}) - \mathcal{E}_{\mathcal{E}_u}) .$$

This completes the proof.