

Characteristic classes of affine varieties and Plucker formulas for affine morphisms

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Abstract

An enumerative problem on a variety V is usually solved by reduction to intersection theory in the cohomology of a compactification of V . However, if the problem is invariant under a “nice” group action on V (so that V is spherical), then many authors suggested a better home for intersection theory: the direct limit of the cohomology rings of all equivariant compactifications of V . We call this limit the *affine cohomology* of V and construct *affine characteristic classes* of subvarieties of a complex torus, taking values in the affine cohomology of the torus.

This allows us to make the first steps in computing *affine Thom polynomials*. Classical Thom polynomials count how many fibers of a generic proper map of a smooth variety have a prescribed collection of singularities, and our affine version addresses the same question for generic polynomial maps of affine algebraic varieties. This notion is also motivated by developing an intersection-theoretic approach to tropical correspondence theorems: they can be reduced to the computation of affine Thom polynomials, because the fundamental class of a variety in the affine cohomology is encoded by the tropical fan of this variety.

The first concrete answer that we obtain is the affine version of what were, historically speaking, the first three Thom polynomials – the Plucker formulas for the degree and the number of cusps and nodes of a projectively dual curve. This, in particular, classifies toric varieties, whose projective dual is a hypersurface, computes the tropical fan of the variety of double tangent hyperplanes to a toric variety, and describes the Newton polytope of the hypersurface of non-Morse polynomials of a given degree. We also make a conjecture on the general form of affine Thom polynomials – a key ingredient is the n -ary fan, generalizing the secondary polytope.

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1 Introduction

1.1 Affine Plucker formulas

Thom polynomials count how many fibers of a generic proper morphism of a smooth variety have a prescribed collection of singularities. More generally, they count the cohomology class of a multisingularity stratum (the closure of the set of all points, whose preimages have a prescribed collection of singularities). Historically, the first Thom polynomials are the Plucker formulas, expressing the degree and the number of cusps and double points of the curve \mathcal{C}^\vee projectively dual to a generic plane curve \mathcal{C} in terms of the degree of \mathcal{C} . To see these expressions as Thom polynomials, note that \mathcal{C}^\vee is the discriminant of the tautological projection

$$\pi : \{(l, x) \in \mathbb{C}\mathbb{P}^{2\vee} \times \mathbb{C}\mathbb{P}^2 \mid x \in l \cap \mathcal{C}\} \rightarrow \mathbb{C}\mathbb{P}^{2\vee}, \quad (*)$$

i.e. the closure of all points $l \in \mathbb{C}\mathbb{P}^{2\vee}$, whose fiber under π has one singularity of local degree 2 (A_1 singularity). Similarly, the cusps and double points of \mathcal{C}^\vee are the images of the fibers with one singularity of local degree 3 (A_2 singularity) or two A_1 singularities respectively. So the dual curve \mathcal{C}^\vee , the set of its cusps and the set of its double points are the multisingularity strata of the projection π .

The contemporary versions of the Plucker formulas are the Thom polynomials, expressing the fundamental classes of the multisingularity strata for a generic map $\pi : M \rightarrow N$ of arbitrary compact smooth surfaces in terms of the characteristic classes of these surfaces: that is, the fundamental classes of the three multisingularity strata in $H(N)$ equal the direct images of c_1 , $c_1^2 + c_2$ and $-2(2c_1^2 + c_2)$ respectively, where c_i are the characteristic classes of the virtual vector bundle $\pi^*TN - TM$. We give [Kn03] as a general reference for this fact and all subsequent appeals to Thom polynomials and multisingularity theory. In the special case of the tautological projection $(*)$, the

characteristic classes are functions of the degree d of the given curve \mathcal{C} , the fundamental class of the stratum \mathcal{C}^\vee in $H_1(\mathbb{C}\mathbb{P}^2; \mathbb{Z}) = \mathbb{Z}$ equals $d(d-1)$ and counts its degree, and the fundamental classes of the 0-dimensional multisingularity strata in $H_0(\mathbb{C}\mathbb{P}^2; \mathbb{Z}) = \mathbb{Z}$ equal $3d(d-2)$ and $(d-3)(d-2)d(d+3)/2$ and count their cardinality.

We shall make the first steps towards computation of *affine Thom polynomials* that count how many fibers of a generic polynomial map of an affine algebraic variety have a prescribed collection of singularities. An example of the affine version of the Plucker formulas is as follows.

EXAMPLE 1.1. Let c_i be a generic polynomial of a given degree. For how many values of the parameter $t \in \mathbb{C}$, is the polynomial $f_t(x) = c_1(t)x + \dots + c_n(t)x^n + x^{n+1}$ not Morse?

To answer this question, let π be the restriction of the projection $(\mathbb{C} \setminus 0)^3 \rightarrow (\mathbb{C} \setminus 0)^2$, $(x, y, t) \mapsto (x, t)$, to the graph $M = \{(x, y, t) \mid y = f_t(x)\}$, and let A_1 , A_2 and $2A_1$ be its multisingularity strata. The sought number of non-Morse polynomials equals the cardinality $|2A_1| + |A_2|$, which can be found from the following three equations. Denote the Euler characteristics by e and define $a : \{1, \dots, n+1\} \rightarrow \mathbb{R}$ to be the minimal concave non-negative function such that $a(i) \geq \deg c_i$ for every $i = 1, \dots, n$. In this notation, the three equations are

$$e(A_1) + |2A_1| + |A_2| = a(1) - 2 \sum_i a(i),$$

$$e(A_1) + |2A_1| = 3a(1) - 4 \sum_i a(i),$$

$$e(A_1) - |2A_1| - 2|A_2| = a(1) - 2 \sum_i (3i - 2)a(i).$$

REMARK 1.2. All three equations can be deduced from certain variations of the Kouchnirenko-Bernstein-Khovanskii formula [Kh77b] for slightly degenerate complete intersections: we should count the Euler characteristics of M , the set of critical points of π , and the set of critical values of π respectively. Note that we could not reduce this problem to classical Thom polynomials by compactifying the ambient tori $(\mathbb{C} \setminus 0)^3 \supset M$ and $(\mathbb{C} \setminus 0)^2$: any such compactification would give a highly non-generic projection of the closure of M , to which classical Thom polynomials are not applicable.

1.2 Affine cohomology ring

In order to obtain an affine version of the Plucker formula for degree, we have to choose a cohomology-like ring, in which we shall compute the fundamental class of the discriminant of π . Remark 1.2 suggests that the Chow ring of any particular compactification of $(\mathbb{C} \setminus 0)^2$ is not a good choice. For this reason we shall switch to the direct limit of the cohomology of all toric compactifications of a complex torus $(\mathbb{C} \setminus 0)^n$. We shall call this graded ring the *affine cohomology* and denote it by C . The ring C governs all equivariant enumerative problems in $(\mathbb{C} \setminus 0)^n$ in the sense that the subvarieties $M, N \subset (\mathbb{C} \setminus 0)^n$ have the same fundamental class in C if and only if every subvariety of complementary codimension, shifted by a generic element of the torus, has the same intersection number with M as with N . This fact was first noticed in [DCP85] for certain homogeneous spaces different from complex tori, but all of

the arguments in this important paper are actually applicable to arbitrary spherical homogeneous spaces, including $(\mathbb{C} \setminus 0)^n$ (see also [B89] and [FMSS95] for details on the spherical setting).

The affine cohomology ring of the complex torus is isomorphic to the polytope algebra, which had been introduced in [M89] and extensively studied ([M93], [PK93]) by the time when these two rings were identified in [FS94] and [B96]. In particular, computation of the fundamental class of an algebraic hypersurface $H \subset (\mathbb{C} \setminus 0)^n$ in the affine cohomology of $(\mathbb{C} \setminus 0)^n$ amounts to computation of the Newton polytope of the defining equation of H .

EXAMPLE 1.3. Denoting the Newton polytope of M from Example 1.1 by N , the fundamental class of the discriminant A_1 (i.e. the Newton polygon of this plane curve) is the fiber polytope of N , see [G00]. As explained above, one can see this equality as the affine version of the Plucker formula for degree in the setting of Example 1.1.

Recall that Minkowski summation of polytopes $A + B = \{a + b \mid a \in A, b \in B\}$ obviously extends to the notion of Riemann integral of a polyhedral-valued function, and the *fiber polytope* of $S \subset \mathbb{R}^3$ in a plane $L \subset \mathbb{R}^3$ is the Riemann integral $\int_I S(t) dt$, where the segment $I \subset \mathbb{R}^3/L$ is the projection of S along L , the volume form dt on the line \mathbb{R}^3/L is induced by the lattice \mathbb{Z}^3/L , and $S(\cdot)$ is a polygon-valued function on I that sends every t to the plane section $S \cap (L + t)$ (see [BS92] for details).

REMARK 1.4. More generally, computation of the affine fundamental class of an arbitrary subvariety $H \subset (\mathbb{C} \setminus 0)^n$ in the affine cohomology of $(\mathbb{C} \setminus 0)^n$ amounts to computation of the *tropical fan* of H (see [K99] and [K03], [ST08], [GKM07]). However, we prefer to think here in terms of affine cohomology classes, because some of our constructions and reasonings work well in the context of arbitrary spherical spaces, to which the notion of tropical fan has not been generalized so far.

1.3 Affine multisingularity theory

The setting in Example 1.1 admits many variations: we can consider more complicated $M \subset (\mathbb{C} \setminus 0)^3$, increase the dimensions of the complex tori, replace the projection of tori with a generic polynomial map, look for the fundamental classes of more complicated singularity strata and so on. However, all of these variations can be reduced to the universal Problem 1.5 which follows. The reduction is outlined in Section 3.6 and will be treated in detail in a separate paper.

Choose a finite set A in the character lattice \mathbb{Z}^n of the torus $(\mathbb{C} \setminus 0)^n$, and consider the space \mathbb{C}^A of all linear combinations of the characters from A . For every finite collection S of isolated hypersurface singularities, consider the *universal S -multisingularity stratum* $\{S\} \subset \mathbb{C}^A$, i.e. the set of all $f \in \mathbb{C}^A$, such that the singularities of the hypersurface $f = 0$ in $(\mathbb{C} \setminus 0)^n$ are in one-to-one correspondence with the singularities of S and are equivalent to them; see Section 3.1 for a more precise definition.

PROBLEM 1.5. Compute the fundamental class of the universal S -multisingularity stratum $\{S\} \subset \mathbb{C}^A$ in the affine cohomology of the torus $(\mathbb{C} \setminus 0)^A$.

Many recent works can be seen as solutions of this problem for different A and S , see Section 3.1 for an overview. We now describe our contribution and its immediate applications.

Corollary 3.18 solves Problem 1.5 for the universal codimension 1 stratum: it describes the Newton polytope of the A -discriminant. Although this Newton polytope has been extensively studied ([GKZ94], [G00], [DFS07], [E10], [CCdRDS13], [DEK14], [AT12], etc.), even in this special case our approach yields something new: Corollary 3.18 is a positive formula for the Newton polytope of the A -discriminant, different from the known one [DFS07] and sufficient to classify non dual defective toric varieties.

Recall that a projective variety is called dual defective, if its projectively dual variety is not a hypersurface. Classification of dual defective varieties is a classical problem, and even the toric case remains unsolved (smooth dual defective toric varieties are classified in [DR06], see also [CC07] and [CDR08] for further results). We classify non dual defect toric varieties. A set $B \subset \mathbb{Z}^n$ is called a *circuit*, if it consists of $n + 2$ points, any $n + 1$ of whom are affinely independent. It is called an *iterated circuit*, if there exists a flag

$$\emptyset = L_0 \subset \{0\} = L_1 \subset L_2 \subset \dots \subset L_k = \mathbb{Z}^n$$

such that, for every $i = 1, \dots, k$, the set $B \cap L_i \setminus L_{i-1}$ consists of $\dim L_i - \dim L_{i-1} + 1$ points, whose images under the projection $L_i \rightarrow L_i/L_{i-1}$ together with 0 form a circuit.

THEOREM 1.6. *(Corollary 3.18.2) An A -toric variety is not dual defective if and only if, up to a translation, A contains an iterated circuit.*

Problem 1.5 for the universal codimension 2 strata A_2 and $2A_1$ is solved by Theorems 3.23, 3.24 and 3.28, which generalize the three equations from Example 1.1. The answer is illustrated for a particular simple $A \subset \mathbb{Z}^2$ in Section 3.9, see also [K09] for another example.

In particular, this allows the description of the Newton polytope of the hypersurface $H \subset (\mathbb{C} \setminus 0)^A$ of all polynomials $f : (\mathbb{C} \setminus 0)^n \rightarrow \mathbb{C}$ that are not Morse functions (i.e. have two equal critical values or a degenerate critical value), see Section 3.6. At the level of degrees (i.e. when $A = d \cdot$ (standard simplex), and we are computing the degrees of the sets A_2 , $2A_1$ and H in terms of d), these problems were solved in [A98] and [LZ04] (for $n = 1$) respectively.

1.4 Affine characteristic classes

Multisingularity theory is based on characteristic classes, and the proof of the aforementioned results will be based on *affine characteristic classes* of subvarieties of a complex torus, taking values in the affine cohomology ring \mathcal{C} .

THEOREM 1.7. *For every algebraic set $V \subset (\mathbb{C} \setminus 0)^n$, the Schwarz-MacPherson characteristic classes of V in the toric compactifications of $(\mathbb{C} \setminus 0)^n$ converge to a certain class in \mathcal{C} : there exists a class in \mathcal{C} , whose restriction to the cohomology of every sufficiently rich toric compactification X equals the Schwarz-MacPherson characteristic class of V in X .*

See the original papers [M74] and [Sch82] and e.g. a survey [A05] for the background on Schwarz-MacPherson classes.

DEFINITION 1.8. The limit in Theorem 1.7 is called the *affine characteristic class* of V and will be denoted by $[V]$.

This class inherits nice functorial properties of the Schwarz-MacPherson classes: e.g. $[U \cap V] + [U \cup V] = [U] + [V]$ and $[U \cap gV] = [U][V]$ for a generic $g \in (\mathbb{C} \setminus 0)^n$. Denoting the codimension i component of $[V]$ by $[V]_i$, we have $[V]_n = e(V)$, $[V]_i = 0$ for $i < \text{codim } V$, and $[V]_{\text{codim } V}$ is the fundamental class of V in \mathcal{C} .

EXAMPLE 1.9. The formal rational function $\frac{N_1 \dots N_k}{(N_1+1) \dots (N_k+1)}$ of the Newton polytopes of the polynomials f_1, \dots, f_k , which was considered in [Kh77b] and in subsequent literature in relation to the nondegenerate complete intersection $f_1 = \dots = f_k = 0$ in a complex torus, turns out to equal the affine characteristic class of this complete intersection (when evaluated in the affine cohomology ring), see Examples 2.8 and 2.13 for more details.

REMARK 1.10. Affine characteristic classes seem to be indispensable in solving Problem 1.5, because they appear of their own accord even in the answer for the universal strata A_2 and $2A_1$ of codimension 2. More specifically, the answer given by Theorems 3.23, 3.24 and 3.28 is a system of three independent linear equations on the fundamental classes of A_2 and $2A_1$ and one more unknown class, which turns out to be the second affine characteristic class of the discriminant A_1 . Note the similarity to Example 1.1, where the third unknown $e(A_1)$ can also be interpreted as the second affine characteristic class of the discriminant.

REMARK 1.11. Note that, in Theorem 1.7, we take the Schwarz-MacPherson classes of V itself, and not the closure of V in X . The statement of the theorem obviously generalizes to an arbitrary spherical homogeneous space, however I have no constructive proof in this case. For the toric case, the proof occupies Section 2.6 and is indirect: firstly, affine characteristic classes are constructed in a different way and shown to have functorial properties, similar to Schwarz-MacPherson classes. Secondly, these properties are shown to imply that the restriction of the affine characteristic class to the cohomology ring of a sufficiently rich compactification equals the Schwarz-MacPherson class.

1.5 Relation to tropical enumeration

The classical approach to enumerative geometry is through intersection theory, characteristic classes and Thom polynomials. Answers to enumerative questions can usually be given by certain Thom-like polynomials, see e.g. [G98] and [T10] for the problem of counting rational curves passing through given generic points of a given surface. In the last decade, a fundamentally new class of enumerative results was invented by Mikhalkin in [M05]. Results of this type are seemingly unrelated to the aforementioned classical approach to enumerative geometry, and are referred to as tropical correspondence theorems. Such theorems state that the answers to certain enumerative questions coincide over \mathbb{C} and over the tropical semifield $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Since enumerative geometry over \mathbb{T} is combinatorics, this gives a new combinatorial answer to classical problems of enumerative geometry over \mathbb{C} .

Proofs of tropical correspondence theorems usually rely upon some kind of deformation theory technique. Every such proof, besides establishing the desired equality between the number of complex and tropical objects, implicitly contains an algorithm of how to reconstruct the whole 1-parametric family of complex objects being deformed to a given tropical one. E. g. for a given 1-parametric family of points $P(t)$ in the

projective plane and a given rational tropical curve T , passing through the points of the tropicalization of $P(t)$, the deformation-theoretic proof of Mikhalkin's correspondence theorem ([S06], [SN06], [T12]) provides an algorithm of how to compute (up to an arbitrary power of t) the 1-parametric family of rational curves, passing through $P(t)$ and tending to T . One might hope that another approach to the proof, which does not inevitably provide such excessive information, would appear to be more universally applicable.

One such intersection theoretic approach was recently developed in [G14] and references therein. It is oriented towards enumeration of rational curves.

We suggest yet another approach, oriented towards curves and hypersurfaces, given by implicit equations. It turns out that once we know the fundamental class of a universal S -multisingularity stratum in the affine cohomology, we can obtain tropical correspondence theorems for hypersurfaces with multisingularity S and various incidence conditions. So tropical enumeration can be regarded as the classical approach to enumerative geometry, in which the cohomology, the characteristic classes and the Thom polynomials are replaced by their affine versions. This is because the affine fundamental class of a subvariety of a torus is encoded by its *tropicalization* ([K99] and [K03], [ST08], [GKM07]). See Section 3.2 for details, and a simple illustration below: we deduce Mikhalkin's correspondence theorem for curves with one node from the description of the Newton polyhedron of the A -discriminant (i.e. the affine fundamental class of the universal singularity stratum of codimension 1. We start with the definition of tropical numbers.

Let $\mathbb{T} = \mathbb{R} \sqcup \{-\infty\}$ be the tropical semifield with the operations

$$a \cdot_{\mathbb{T}} b = a +_{\mathbb{R}} b \quad \text{and} \quad a +_{\mathbb{T}} b = \begin{cases} \max(a, b) & \text{if } a \neq b \\ [-\infty, a] & \text{if } a = b \end{cases}$$

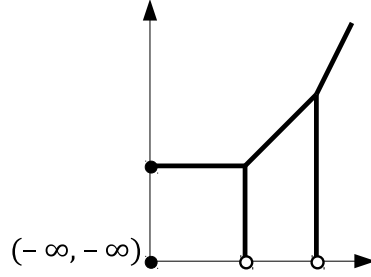
REMARK 1.12. Multivalued tropical summation that we use is a little more convenient than the conventional one (see e.g. the subsequent definition) and is also more natural: the tropical summation and multiplication should simulate the behaviour of the degree of the sum and the product of two polynomials, whereas $\deg(f + g)$ can attain any value within $[-\infty, \deg f]$ if $\deg f = \deg g$.

In what follows, $\mathbf{1} \in \mathbb{T}$ always stands for $0 \in \mathbb{R}$, and $\mathbf{o} \in \mathbb{T}$ for $-\infty$. Moreover, due to multivaluedness, “ $= \mathbf{o}$ ” in the tropical context always means “ $\ni -\infty$ ”.

REMARK 1.13. We shall denote tropical objects in Fraktur, because we think of the tropical torus $\mathbb{T} \setminus \mathbf{o}$ as the Lie algebra of the complex torus $\mathbb{C} \setminus 0$.

DEFINITION 1.14. A *tropical hypersurface* is a set of the form $S = \{x \mid f(x) = \mathbf{o}\}$, where f is a tropical polynomial. In a small neighborhood of a generic point $x \in S$, the polynomial f can be represented as $g \cdot h^k$, where $g(x) \neq \mathbf{o}$ and h is irreducible. The integer k is called the *multiplicity* of the hypersurface S at the point x .

For example, the following picture shows the graph of $f(x) = x^3 + x^2 + 6x + 8$ in bold lines, and the set $\{f = \mathbf{o}\} = \{2, 3\}$ with the multiplicities 1 and 2 respectively as white dots.



Tropical hypersurfaces H_1, \dots, H_n in the tropical torus $\mathbb{R}^n = (\mathbb{T} \setminus 0)^n$ are said to intersect *transversally*, if they intersect by finitely many points, and, in a small neighborhood of every such point p , every H_i is given by a tropical equation of the form $x^{a_{p,i}} + p^{a_{p,i}} = \mathfrak{o}$ for some $a_{p,i} \in \mathbb{Z}^n$. The *intersection number* of H_1, \dots, H_n is then defined as the sum $\sum_p \det(a_{p,1}, \dots, a_{p,n})$. The tropical Kouchnirenko–Bernstein formula states that the intersection number equals the mixed volume of the Newton polytopes of H_1, \dots, H_n .

EXAMPLE 1.15 (Mikhalkin’s correspondence for curves with one node). For a given $A \subset \mathbb{Z}^2$, $0 \in A$, we wish to count polynomials $f \in \mathbb{C}^A$ such that the curve $f = 0$ has one singularity and passes through a given collection of generic $q = |A| - 2$ points $p_1, \dots, p_q \in (\mathbb{C} \setminus 0)^2$. In other words, we wish to count the intersection number I of the following hypersurfaces in \mathbb{C}^A :

- the incidence conditions H_1, \dots, H_q , where $H_i = \{f \mid f(p_i) = 0\}$;
- the normalization $H_0 = \{f \mid f(0) = 1\}$;
- the A -discriminant $S = \{f \mid f = 0 \text{ is not regular}\}$.

Let us tropicalize these objects: choose points $\mathfrak{p}_1, \dots, \mathfrak{p}_q \in (\mathbb{T} \setminus 0)^2$ and define the tropical hypersurfaces $\mathfrak{H}_i = \{f \mid f(\mathfrak{p}_i) = \mathfrak{o}\}$ in \mathbb{T}^A and $\mathfrak{H}_0 = \{f \mid f(\mathfrak{o}) = \mathfrak{1}\}$. Choose the tropical polynomial \mathfrak{D} with unit coefficients and the same Newton polytope as the A -discriminant S , and define the tropical discriminant \mathfrak{S} by the equation $\mathfrak{D} = \mathfrak{o}$. For generic choice of $\mathfrak{p}_1, \dots, \mathfrak{p}_q$, the tropical hypersurfaces $\mathfrak{H}_0, \dots, \mathfrak{H}_q, \mathfrak{S}$ are transversal, and we denote their intersection number by \mathfrak{J} . Then we have

$$I = \mathfrak{J},$$

because both parts equal the mixed volume of the Newton polytopes of H_0, \dots, H_q and S by the Kouchnirenko–Bernstein formula over \mathbb{C} and \mathbb{T} respectively.

The description of \mathfrak{S} (see Section 3.3) shows that the intersection points $\mathfrak{f} \in \mathfrak{H}_0 \cap \dots \cap \mathfrak{H}_q \cap \mathfrak{S}$ are exactly the equations of the tropical curves with one node in the sense of Mikhalkin, passing through $\mathfrak{p}_1, \dots, \mathfrak{p}_q$. Moreover, the intersection multiplicity of $\mathfrak{H}_0, \dots, \mathfrak{H}_q$ and \mathfrak{S} at every such intersection point \mathfrak{f} equals the multiplicity, assigned to the curve $\mathfrak{f} = \mathfrak{o}$ by Mikhalkin’s correspondence theorem.

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2 Affine characteristic classes

Affine intersection theory and affine characteristic classes are introduced for arbitrary spherical homogeneous spaces in Section 2.1 and specialized to the toric case in Sections 2.2 and 2.3 respectively. The rest of Section 2 is more technical: Sections 2.4—2.6 are devoted to the proof of existence of affine characteristic classes in the toric setting, and Section 2.7 is devoted to characteristic classes of degenerate complete intersections of a certain type, whose importance will become clear in Section 3.

2.1 Affine intersection theory and characteristic classes

Let G be a reductive algebraic group (i. e. the complexification of a real compact connected group), and let X be an n -dimensional spherical homogeneous G -space (i. e. a space with a transitive action of G , such that some Borel subgroup of G has a dense orbit in X). For algebraic subsets P and $Q \subset X$ of complementary dimensions $p + q = n$, the set $gQ = \{gq \mid q \in Q\}$, the shifting of Q by a generic element $g \in G$, meets P at finitely many points, and we denote the number of these points by $|P \cdot Q|$.

Denote the space of linear combinations of irreducible codimension k algebraic subsets of X by Z^k , then the aforementioned pairing extends to a pairing $|\cdot| : Z^p \times Z^q \rightarrow \mathbb{Z}$ by linearity, and the space $C = \bigoplus_k C_k = \bigoplus_k Z_k / \{P \mid \forall Q \mid |P \cdot Q| = 0\}$ has a natural ring structure, compatible with this pairing (see [DCP85],[B89], [FMSS95]): for algebraic subsets R and $S \subset X$ of arbitrary dimensions, the intersection of R and gS represents the same element of C for almost all $g \in G$, and this element is called the *product* $R \cdot S$. We shall call this ring the **affine cohomology**, or, following [DCP85], the **ring of conditions** of X .

REMARK 2.1. Recall that it is not the case for non-spherical spaces: for example, if $G = X = \mathbb{C}^3$, then two lines represent the same class in C if and only if they are parallel, however the intersection of the surface $z = xy$ and the plane $x = 0$, shifted by $g \in \mathbb{R}^3$, is a line, whose direction depends on g .

DEFINITION 2.2. The **affine characteristic class** is a mapping that sends every algebraic subset $V \subset X$ to an element $[V] = [V]_0 + \dots + [V]_n \in C$, $[V]_i \in C_i$, with the following properties:

(1) If $V \subset X$ has codimension k , then $[V]_i = 0$ for $i < k$, $[V]_k$ is the equivalence class of V in C_k , and $[V]_n \in C_0 = \mathbb{Z}$ is the Euler characteristic $e(V)$.

(2) For any U and $V \subset X$ and generic $g \in G$, we have $[U \cap gV] = [U][V]/[X]$. Note that $[X]$ is invertable by property (1).

(3) For any spherical homogeneous spaces X and Y and any algebraic subsets $U \subset X$ and $V \subset Y$, we have $[U \times V] = [U] \times [V]$.

(4) The mapping that assigns $[V]$ to the characteristic function of V , extends by linearity to the space of all constructible functions $X \rightarrow \mathbb{Z}$, i.e. $[U \cap V] + [U \cup V] = [U] + [V]$. Recall that a *constructible function* is a linear combination of the characteristic functions of algebraic sets.

(5) For a morphism $p : X \rightarrow Y$ of spherical homogeneous spaces and an algebraic subset $V \subset X$, we have $p_*[V] = [p_*V]$. Here $p_*V : Y \rightarrow \mathbb{Z}$ is the *MacPherson direct image* of V , whose value at every $y \in Y$ is defined as $e(p^{-1}(y) \cap V)$.

(6) For a smooth equivariant compactification $\overline{X} \supset X$, such that the affine characteristic class $[V]$ is contained in the cohomology $H^\bullet(\overline{X}) \subset C$, this class is Poincaré dual to the *Schwarz-MacPherson class* of V in \overline{X} , see [M74] and [Sch82]. Recall that the natural inclusions $H^\bullet(\overline{X}) \subset C$ are induced by the fact that C is the direct limit of $H^\bullet(\overline{X})$ over all equivariant compactifications $\overline{X} \supset X$.

Note that the affine characteristic class is uniquely defined by property (6) and by properties (1–5).

THEOREM 2.3. *In the toric case $G = X = (\mathbb{C} \setminus 0)^n$, the affine characteristic class exists, and property (2) reads as $[U \cap gV] = [U][V]$, because $[X] = 1$.*

The proof is given in Section 2.6 and is based on an explicit combinatorial model for the ring of conditions C of a complex torus. This combinatorial model is widely known as the **ring of tropical fans** ([ST08], [GKM07]), although they are essentially the same thing as Minkowski weights from [FS94] and c-fans from [K03]. The affine characteristic class, considered as an element of the ring of tropical fans, will be referred to as the **tropical characteristic class**.

REMARK 2.4. I cannot constructively prove the existence of the affine characteristic class for an arbitrary spherical space X , because I do not know an algebro-geometric construction for \overline{X} in terms of V in (6). Moreover, this is not the only difficulty with affine characteristic classes for arbitrary X . For instance, the class $[X]$, which is especially important due to property (2), has already been considered in [Kir06] under the name of the *non-compact characteristic class of X* (see also [BK05] and [BJ08] for its equivariant version), but it is not yet computed even for $X = SL_n$ with large n .

So, from now on, we restrict our attention to the toric case, recalling the notion of a tropical fan (we actually need the slightly more general notion of tropical fan with polynomial weights), and develop the technique for computation of tropical characteristic classes to the extent that we need in this paper.

2.2 Toric intersection theory and tropical fans

The complex torus $G = X = (\mathbb{C} \setminus 0)^n$ is the only spherical homogeneous space for which the structure of the ring of conditions is completely understood. More specifically, the definition of the intersection product as the class of $U \cap gV$ for generic $g \in G$ is not constructive because of the word ‘generic’. However, for $G = X = (\mathbb{C} \setminus 0)^n$, we know how to check explicitly if the intersection number of algebraic subsets U and $V \subset (\mathbb{C} \setminus 0)^n$ of complementary dimension equals the product of the classes of U and V in the ring of conditions. In what follows, we write this equality as $[U \cap V] = [U][V]$ in accordance with our notation for characteristic classes.

Let L be the character lattice of the complex torus, and let $I, J \subset \mathbb{C}[L]$ be the radical ideals, defining U and V . For a linear function $\gamma : L \rightarrow \mathbb{Z}$, define $\text{in}_\gamma I$ as the ideal, generated by the initial terms of the polynomials from I in the sense of the partial ordering γ on the lattice of monomials. This ideal $\text{in}_\gamma I$ defines a variety denoted by $\text{in}_\gamma U$.

PROPOSITION 2.5 ([K03]). *If $\text{in}_\gamma U \cap \text{in}_\gamma V$ is empty for every non-zero $\gamma \in L^*$, then $[U \cap V] = [U][V]$.*

In particular, this leads to a combinatorial representation of C as the space of certain fans, based on the fact that the set $\{\gamma \mid \text{in}_\gamma U \neq \emptyset\}$ is a polyhedral fan. We recall this combinatorial representation in the form of [E12p].

Let L be an n -dimensional integer lattice. Consider a pair (P, φ) , where

1) $P \subset L \otimes \mathbb{R}$ is a union of finitely many disjoint rational convex relatively open codimension k polyhedral cones, so that P coincides with a codimension k plane in a neighborhood of every its point p (this plane is denoted by $T_p P$);

2) The function $\varphi : P \rightarrow \mathbb{R}$ equals a rational polynomial $\varphi_p : T_p P \rightarrow \mathbb{R}$ in a small neighborhood of every $p \in P$.

Two such pairs (P, φ) and (Q, ψ) are said to be **equivalent**, if $\varphi(p) = \psi(p)$ for every $p \in P \cap Q$ such that $T_p P = T_p Q$, and $\varphi = 0$ on $P \setminus \bar{Q}$ and $\psi = 0$ on $Q \setminus \bar{P}$.

For a rational k -dimensional plane $R \subset L \otimes \mathbb{R}$ and a point $x \in L \otimes \mathbb{R}$, the affine plane $R + x$ is said to be **transversal** to (P, φ) , if it meets P at finitely many points and does not meet $\bar{P} \setminus P$. The **tropical intersection number** $(R + x) \cdot (P, \varphi)$ is then defined as

$$\sum_{p \in (R+x) \cap P} \varphi(p) \left| \frac{L}{(R \cap L) + (T_p P \cap L)} \right|.$$

This intersection number is a locally polynomial function of x , defined on an open dense subset of $L \otimes \mathbb{R}$. If this function extends to a continuous function $i_R : L \otimes \mathbb{R} \rightarrow \mathbb{R}$ for every R , then the pair (P, φ) is said to be **tropical**, and the **tropical intersection number** $(R + x) \cdot (P, \varphi)$ is defined to be $i_R(x)$ for every x by continuity, even if $R + x$ is not transversal to (P, φ) .

Equivalence classes of codimension k tropical pairs are called tropical fans and form the set $\mathcal{K}_k(L)$. For any two fans \mathcal{P} and \mathcal{Q} , there exists a unique (up to equivalence) fan \mathcal{S} , such that $\mathcal{P} \cdot R + \mathcal{Q} \cdot R = \mathcal{S} \cdot R$ for every affine plane R . This fan \mathcal{S} is called the sum $\mathcal{P} + \mathcal{Q}$, see [E12p] for a more constructive definition.

The set $\mathcal{K}_k(L)$ is a \mathbb{Q} -vector space with respect to this summation. It splits into the sum $\bigoplus_d \mathcal{K}_k^d(L)$, where $\mathcal{K}_k^d(L)$ consists of pairs $(P, \varphi) \in \mathcal{K}_k(L)$ such that φ is locally a homogeneous polynomial of degree d . The direct sum $\bigoplus_k \mathcal{K}_k(L)$ is denoted by $\mathcal{K}(L)$ and will be called the **space of tropical fans with polynomial weights**.

For tropical pairs $(P, \varphi) \in \mathcal{K}(L)$ and $(Q, \psi) \in \mathcal{K}(M)$, their **Cartesian product** is defined as $(P \times Q, \varphi + \psi) \in \mathcal{K}(L \oplus M)$. For an epimorphism of lattices $f : L \rightarrow M$ of dimension n and m respectively, the **push-forward** $f_*(\mathcal{P})$ of a tropical fan $\mathcal{P} \in \mathcal{K}_k(L)$, $k \geq n - m$, is defined as the unique $\mathcal{S} \in \mathcal{K}_{k-n+m}(M)$ such that $\mathcal{P} \cdot f^{-1}(R) = \mathcal{S} \cdot R$ for every affine $(k - n + m)$ -dimensional plane $R \subset M \otimes \mathbb{Q}$. A more explicit description is as follows (see [E12p]).

PROPOSITION 2.6. *The image of the tropical fan \mathcal{P} that consists of $(n - k)$ -dimensional cones $C_i \subset L \otimes \mathbb{R}$ with weights $m_i : C_i \rightarrow \mathbb{R}$ is a tropical fan $f_* \mathcal{P}$ that consists of all $(n - k)$ -dimensional cones $f(C_i)$ with multiplicities $(m_i \circ f^{-1}) \cdot |L / (C_i + \ker f)|$.*

The vector space $\mathcal{K}(L)$ has the natural differential ring structure: the **intersection product** of $\mathcal{P} \in \mathcal{K}_k(L)$ and $\mathcal{Q} \in \mathcal{K}_m(L)$ is the unique $\mathcal{S} \in \mathcal{K}_{k+m}(L)$, such that $\mathcal{S} \cdot R = (\mathcal{P} \times \mathcal{Q} \times R) \cdot (\text{diagonal})$ in $(L \oplus L \oplus L) \otimes \mathbb{Q}$ for every $(k + m)$ -dimensional affine plane $R \subset L \otimes \mathbb{Q}$. The **corner locus derivation** $\delta : \mathcal{K}_k^d(L) \rightarrow \mathcal{K}_{k+1}^{d-1}(L)$ is the unique derivation such that $\delta(\mathcal{P}) \in \mathcal{K}_1^0(L)$ is the corner locus of a continuous piecewise linear function $\mathcal{P} \in \mathcal{K}_0^1(L)$, see [E12p] for more constructive definitions.

Denote the subrings $\bigoplus_k \mathcal{K}_k^0(L)$ and $\bigoplus_d \mathcal{K}_0^d(L)$ of $\mathcal{K}(L)$ by $\mathcal{K}^0(L)$ and $\mathcal{K}_0(L)$ respectively. The former one is also known as the **ring of tropical fans** (see e.g. [FS94], [K03], [ST08], [GKM07] for a more explicit description of the sum and the product in this ring), and the latter one is the ring of all continuous piecewise polynomial functions on $L \otimes \mathbb{Q}$ (with the conventional sum and product). The subrings $\mathcal{K}_0(L^*)$ and $\mathcal{K}^0(L^*)$ provide well known combinatorial models for the ring of conditions C of the complex torus $(\mathbb{C} \setminus 0)^n$ with the character lattice L :

$$C = \mathcal{K}^0(L^*) = \mathcal{K}_0(L^*)/I,$$

where I is the ideal, generated by linear functions on \mathbb{Q}^n . The first of these models was found in [FS94], [M96] and [K03], and the second one in [S87] and [B96]. The isomorphism between the two models is established by the maps $\delta^k : \mathcal{K}_0^k(L^*) \rightarrow \mathcal{K}_k^0(L^*)$ (see also [B97] and [KP08] for other descriptions of this isomorphism). The element of $\mathcal{K}^0(L^*)$, corresponding to the class of a subvariety $V \subset (\mathbb{C} \setminus 0)^n$ in the ring of conditions C , is called the **tropical fan** $\text{Trop } V$, and admits the following description:

PROPOSITION 2.7 ([K03]). *Let V be given by a radical ideal $I \in R$, then its tropical fan is represented by the pair $(P, \varphi) \in \mathcal{K}^0(L^*)$, $P \subset \mathbb{Q}^n$, $\varphi : P \rightarrow \mathbb{Q}$, where $P = \{\gamma \mid \dim_{\mathbb{C}} \mathbb{C}[L]/\text{in}_{\gamma} I \text{ is finite and positive}\}$, and $\varphi(\gamma) = \dim_{\mathbb{C}} \mathbb{C}[L]/\text{in}_{\gamma} I$.*

EXAMPLE 2.8. If N is the Newton polytope of a Laurent polynomial on the complex torus $(\mathbb{C} \setminus 0)^n$, then the tropical fan of the hypersurface $f = 0$ equals the corner locus of the support function of N . We denote this fan by $[N] \in \mathcal{K}_1^0(\mathbb{Z}^n)$ and call it the **dual fan** of N .

REMARK 2.9. This isomorphism between $\mathcal{K}^0(L^*)$ and C respects epimorphisms of tori ([K03], [ST08]): it sends the epimorphism of the rings of conditions, induced by an epimorphism $(\mathbb{C} \setminus 0)^n \rightarrow (\mathbb{C} \setminus 0)^m$, to the epimorphism $\mathcal{K}^0(L^*) \rightarrow \mathcal{K}^0(M^*)$, induced by the corresponding epimorphism of the dual character lattices $L \rightarrow M$.

2.3 Tropical characteristic classes

DEFINITION 2.10. The element of the ring of tropical fans $\mathcal{K}^0(L^*)$, corresponding to the affine characteristic class of a subvariety $V \subset (\mathbb{C} \setminus 0)^n$, is called the **tropical characteristic class** of V .

REMARK 2.11. Here are some natural interesting questions about tropical characteristic classes that we do not address in this paper:

1. Construct “higher Newton polytopes” of V , i.e. codimension 1 tropical fans B_1, \dots, B_n such that $[V]_i = B_1 \dots B_i$. Are they dual to convex polytopes?
2. Express $[V]$ in terms of an arbitrary (not necessarily toric!) compactification $C \supset (\mathbb{C} \setminus 0)^n$, such that the closure of V is transversal to the strata of C .
3. Define and study the “generalized Severi variety”, whose points parameterize all subvarieties $V \subset (\mathbb{C} \setminus 0)^n$ with a given tropical characteristic class. Classical Severi varieties are such spaces for plane curves.
4. Obtain a linear-algebraic description for the tropical characteristic class of a variety in terms of its defining ideal, similarly to Proposition 2.7.
5. Try to apply tropical characteristic classes to the conjectures in [HS13].

We now compute $[V]$ under the assumption that V is **schön**, i.e. there exists a fan Σ such that the closure of V in the corresponding toric compactification $X_\Sigma \supset (\mathbb{C} \setminus 0)^n$ is smooth and intersects the orbits of X_Σ transversally (see for instance [Kh77a] and [D78] for background on toric varieties and their relation to Newton polyhedra). Note that our definition of Schön varieties is a bit more restrictive than in the original paper [T07].

For every cone $\Gamma \in \Sigma$, denote the intersection of the closure of V with the Γ -orbit of X_Σ by V_Γ .

PROPOSITION 2.12. *If V is schön, then the class $[V]_i \in \mathcal{K}_i^0(L^*)$ is represented by the pair (P, φ) , $P \subset \mathbb{Q}^n$, $\varphi : P \rightarrow \mathbb{Q}$, such that P is the union of all codimension i cones in Σ , and the value of φ at every such cone Γ is the Euler characteristic $e(V_\Gamma)$.*

The proof will be given in Section 2.6, because it comes as a byproduct when we prove the existence of affine characteristic classes.

EXAMPLE 2.13. In particular, if V is a generic hypersurface with the Newton polytope Δ , then, counting $e(V_\Gamma)$ by the Kouchnirenko formula, we obtain $[V] = \frac{[\Delta]}{1+[\Delta]}$, or $[V]_i = -(-[\Delta])^i$. Recall that $[B]$ is the **dual fan** of the polytope B , see Example 2.8.

Further, if V_1, \dots, V_k are generic hypersurfaces with the Newton polytopes $\Delta_1, \dots, \Delta_k$, then, by the multiplicativity of affine characteristic classes, we have

$$[V_1 \cap \dots \cap V_k] = \frac{[\Delta_1] \dots [\Delta_k]}{(1 + [\Delta_1]) \dots (1 + [\Delta_k])}. \quad (1)$$

In particular, this gives Khovanskii's formula [Kh77b] for the Euler characteristic of a nondegenerate complete intersection and assigns geometrical meaning to the right hand side of (1) that appeared in [Kh77b] and subsequent literature as a formal expression.

It would be interesting to determine if the characteristic class of every schön variety equals $(*)$ for some (may be virtual) polytopes $\Delta_1, \dots, \Delta_k$.

2.4 Polynomial functions of polytopes

This section prepares polyhedral-geometric tools to prove the existence of tropical characteristic classes. Let $P(L)$ be the semigroup of convex polytopes in an l -dimensional lattice L , and let $[\cdot] : P(L) \rightarrow \mathcal{K}_1^0(L^*)$ be the inclusion, sending a polytope to its dual fan (see Example 2.8). In accordance with this notation, we denote the integer mixed volume of the polytopes B_1, \dots, B_n in \mathbb{Q}^n by $[B_1] \cdot \dots \cdot [B_n]$.

Let e_0, \dots, e_m be the vertices of the standard simplex in \mathbb{Q}^m , let B_0, \dots, B_m and A_1, \dots, A_k be polytopes in \mathbb{Q}^n , and let B be the convex hull of the union $\bigcup B_i \times \{e_i\}$ in \mathbb{Q}^m . The volume of the Cayley polytope B can be expressed in terms of mixed volumes of B_0, \dots, B_m as follows (setting $k = 0$):

LEMMA 2.14. *For any polytopes $A_1, \dots, A_k \subset \mathbb{Q}^m$, we have*

$$[A_1] \dots [A_k] \sum_{\substack{b_0 + \dots + b_m = n - k \\ b_0 \geq 0, \dots, b_m \geq 0}} [B_0]^{b_0} \dots [B_m]^{b_m} = [A_1] \dots [A_k] [B]^{n+m-k}.$$

See e.g. Lemma 1.7 in [E10] for a proof, based on the Kouchnirenko-Bernstein-Khovanskii formula, or Theorem 24 in [E12m] for a combinatorial proof.

DEFINITION 2.15. We say that a function $f : P(L) \rightarrow \mathbb{Q}$ is a polynomial starting from $S \in P(L)$, if there exist tropical fans $F_i \in \mathcal{K}_i^0(L^*)$, such that $f(B) = \sum_i F_i[B]^{n-i}$ for every B that contains S .

REMARK 2.16. The coefficients of the polynomial are uniquely determined by the values of the function f .

Let C be the standard m -dimensional simplex in \mathbb{Q}^m . For a polytope $B \subset \mathbb{Q}^n$, denote the convex hull of the union of $B \times \{0\}$ and $\{0\} \times C$ by $C(B) \subset \mathbb{Q}^n \times \mathbb{Q}^m$. Let $A \in \mathcal{K}_{n-k}^0(\mathbb{Z}^n \times \mathbb{Z}^m)$ be a tropical fan, $k \geq 0$.

PROPOSITION 2.17. *The number $A[C(B)]^{m+k}$, $k \geq 0$, is a polynomial $P_A(B) = \sum_i P_{A,i}[B]^{n-i}$, $P_{A,i} \in \mathcal{K}_i^0(\mathbb{Z}^n)$, starting from some S . This polynomial is multiplicative in A : if $A \in K^0(\mathbb{Z}^n \times \mathbb{Z}^m)$ and $A' \in K^0(\mathbb{Z}^{n'} \times \mathbb{Z}^{m'})$, then $P_{A \times A',i} = \sum_j P_{A,j} \times P_{A',i-j}$.*

PROOF. Every fan A can be represented as a linear combination of complete intersections of the form $[A_1] \dots [A_{n-k}]$ for polytopes $A_i \subset \mathbb{Q}^n \times \mathbb{Q}^m$, so we assume with no loss in generality that $A = [A_1] \dots [A_{n-k}]$. Let T_i be the projection of A_i to \mathbb{Q}^m . For every face F of $T = \sum_i T_i$, represent F as $\sum_i \tilde{F}_i$ of faces $\tilde{F}_i \subset T_i$, and denote the preimage of \tilde{F}_i under the projection $A_i \rightarrow T_i$ by F_i .

Let C_I be the standard I -dimensional simplex in $\mathbb{Q}^I \subset \mathbb{Q}^m$, where \mathbb{Q}^I is the coordinate plane, defined by vanishing of the i -th coordinates for $i \notin I \subset \{1, \dots, m\}$. Let C_I^0 be its facet, disjoint from 0, and let $C_I(B) \subset C(B)$ be the convex hull of B and C_I . For every face $F \subset T$, let I_F be the maximal I such that the faces $C_I^0 \subset C$ and $F \subset T$ are compatible (i.e. $F + C_I^0$ is a face of $T + C$).

Choose any $i_1 < \dots < i_p$ and j , then the interiors of the sums $A_F^{\bar{i},j} = F_{i_1} + \dots + F_{i_p} + jC_{I_F}(B)$ for any two faces $F \subset T$ do not intersect. The key observation is as follows: there exists $S \subset \mathbb{Q}^n$, such that for $B \supset S$ the union of $A_F^{\bar{i},j}$ over all faces $F \subset T$ equals $A_{i_1} + \dots + A_{i_p} + jC(B)$, so the volume of $A_{i_1} + \dots + A_{i_p} + jC(B)$ equals the sum of the volumes of $A_F^{\bar{i},j}$. Combining this observation with the formula $[B_1] \dots [B_q] = \sum_{i_1 < \dots < i_p} (-1)^p \text{Vol}(B_{i_1} + \dots + B_{i_p})$ for any polytopes $B_1, \dots, B_k \subset \mathbb{R}^q$, we conclude that $[A_1] \dots [A_{n-k}][C(B)]^{m+k}$ equals the sum of $[F_1] \dots [F_{n-k}][C_{I_F}(B)]^{m+k}$ over all faces $F \subset T$ for $B \supset S$.

Each of the summands $[F_1] \dots [F_{n-k}][C_{I_F}(B)]^{m+k}$ is a polynomial of B , because we can compute it by Lemma 2.14 with $B_0 = B$, $B_1 = C_{I_F}^0$ and $B_{0,1} = C_{I_F}(B)$. This gives an explicit polynomial formula $\sum_i P_{A_1, \dots, A_{n-k}, i}[B]^i$ for $[A_1] \dots [A_{n-k}][C(B)]^{m+k}$. Also by Lemma 2.14, one can check that $P_{A_1, \dots, A_{n-k}, A'_1, \dots, A'_{n'-k'}, i} = \sum_j P_{A_1, \dots, A_{n-k}, j} \times P_{A'_1, \dots, A'_{n'-k'}, i-j}$ for $A_1, \dots, A_{n-k} \subset \mathbb{Q}^n \times \mathbb{Q}^m$ and $A'_1, \dots, A'_{n'-k'} \subset \mathbb{Q}^{n'} \times \mathbb{Q}^{m'}$. \square

2.5 Base points at infinity

In this section, we discuss families of subvarieties $B_s \subset (\mathbb{C} \setminus 0)^n$ that are flat at infinity in the following sense: for every variety V of a complementary codimension, almost all B_s “do not intersect V at infinity” (i.e. V and B_s satisfy the assumption of Proposition 2.5).

Let A and B be irreducible algebraic varieties of the dimension a and b , and $X \subset A \times B$ be a codimension k irreducible subvariety, whose projections $\pi_A : X \rightarrow A$ and $\pi_B : X \rightarrow B$ are surjective. For every $s \in A$, denote $\pi_B \pi_A^{-1}(s)$ by B_s .

DEFINITION 2.18. The variety X is called the family of subvarieties $B_s \subset B$ (members of the family), parameterised by $s \in A$. A point $y \in B$ is said to be a base point of the family, if $\dim A_y > a - k$.

The following fact is obvious.

LEMMA 2.19. *If the family $X \subset A \times B$ has no base points, then every subvariety $V \subset B$, whose dimension is less than k , has empty intersection with a generic member of the family.*

We are interested in the following special case: $A = (\mathbb{C} \setminus 0)^a$ and $B = (\mathbb{C} \setminus 0)^b$.

DEFINITION 2.20. The family $X \subset A \times B$ is said to have no base points at infinity, if, for any non-zero $\gamma \in \{0\} \times \mathbb{Q}^b \subset \mathbb{Q}^a \times \mathbb{Q}^b$, the family $\text{in}_\gamma X \subset A \times B$ has no base points (see the paragraph before Proposition 2.5 for the meaning of in_γ).

Denote the intersection product of the tropical fan of X and $\{0\} \times \mathbb{Q}^b \subset \mathbb{Q}^a \times \mathbb{Q}^b$ by L_X .

PROPOSITION 2.21. 1) *The tropical fan of a generic member of the family $X \subset A \times B$ equals L_X .*

2) *If $X \subset A \times B$ has no base points at infinity, and $V \subset B$ is a k -dimensional subvariety, then the intersection number of V with a generic member of the family is well defined and equals the tropical intersection number of the tropical fan of V and L_X .*

PROOF. Part (1) is the definition of the product in the ring of conditions for X and a fiber of π_A . If X has no base points at infinity, then, for generic $s \in A$ and any non-zero $\gamma \in \mathbb{Q}^b$, the set $\text{in}_\gamma B_s$ is either empty (for $\gamma \notin L_X$), or does not intersect $\text{in}_\gamma V$ by Lemma 2.19 (for $\gamma \in L_X$, because X has no base points at infinity). Thus, Proposition 2.5 is applicable to B_s and V . \square

We are especially interested in the following two special cases. Recall that, for a finite subset A in the character lattice of the complex torus $(\mathbb{C} \setminus 0)^n$, we denote the space of linear combinations of the characters from A by \mathbb{C}^A .

EXAMPLE 2.22. Obviously, the family of planes $\{f \in \mathbb{C}^A \mid f(g) = 0\}$ for all $g \in (\mathbb{C} \setminus 0)^n$ has no base points at infinity.

DEFINITION 2.23. The a -logarithmic differential of $f \in \mathbb{C}^A$, $A \subset \mathbb{Z}^n$, $a \in \mathbb{R}^n$, is the map $\text{dlog}_a f = (x_1 \frac{\partial f}{\partial x_1} - a_1 f, \dots, x_n \frac{\partial f}{\partial x_n} - a_n f)$.

LEMMA 2.24. *The graphs of $\text{dlog}_a f$ for $a \in \mathbb{C}^n \setminus$ (the union of affine spans of the facets of A) and $f \in \mathbb{C}^A$ form a family in $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^n$ with no base points at infinity.*

PROOF. For any $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ and any non-trivial face $F \subset A$, the dimension of the set of all pairs $(a = (a_1, \dots, a_n), f = \sum_{b \in A} c_b x^b)$, satisfying the equalities $y_i = \sum_{b \in F} (b_i - a_i) c_b x^b$, $i = 1, \dots, n$, does not depend on (x, y) . \square

2.6 Existence of characteristic classes

We refer to e.g. [Kh77a] and [D78] for the background on toric compactifications. For an arbitrary bounded subset $B \subset \mathbb{R}^n$, the space of Laurent polynomials of the form $\sum_{b \in B} c_b x^b$ will be denoted by \mathbb{C}^B instead of $\mathbb{C}^{B \cap \mathbb{Z}^n}$ by a small abuse of notation.

Let $V \subset (\mathbb{C} \setminus 0)^n$ be a smooth subvariety.

DEFINITION 2.25. The **logarithmic conormal variety** $\mathcal{V} \subset (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^n \subset T^*(\mathbb{C} \setminus 0)^n$ of V is the set of all (x, l) such that $x \in V$, and the restriction of $\sum l_i \frac{dx_i}{x_i}$ to $T_x V$ equals 0.

Our idea is to prove that the coefficients of the polynomial $P_{\text{Trop } \mathcal{V}}$ (see Proposition 2.17 for this notation) satisfy the properties of tropical characteristic classes of V . The proof starts from the observation that Proposition 2.21 and Lemma 2.24 imply the following.

LEMMA 2.26. *The intersection number of \mathcal{V} and the graph of $\text{dlog}_a f$ in $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^n$ equals the tropical intersection number of $[\mathcal{V}]$ and $[C(B)]^n$.*

DEFINITION 2.27. Denote the Euler characteristic $e(V \cap \{f_1 = \dots = f_i = 0\})$ for generic $f_1 \in \mathbb{C}^{B_1}, \dots, f_i \in \mathbb{C}^{B_i}$ by $e(V, B_1, \dots, B_i)$.

A sufficient explicit genericity condition for f_1, \dots, f_i is as follows (the proof is standard).

LEMMA 2.28. *Choose a smooth toric compactification of $(\mathbb{C} \setminus 0)^n$, whose fan is compatible with the lattice polytopes B_1, \dots, B_k in \mathbb{Q}^n , so that the closure \overline{M} of M intersects the orbits properly (see e.g. [K99] or [T07]). Let \mathbf{M} be a Whitney stratification of \overline{M} , compatible with the orbit stratification of the ambient toric variety. The tuples $(f_1, \dots, f_k) \in \mathbb{C}^{B_1} \oplus \dots \oplus \mathbb{C}^{B_k}$, such that the closure of $f_{i_1} = \dots = f_{i_p} = 0$ is smooth and transversal to all strata of \mathbf{M} for every sequence $i_1 < \dots < i_p$, is a non-empty Zariski open subset in $\mathbb{C}^{B_1} \oplus \dots \oplus \mathbb{C}^{B_k}$. The Euler characteristic $e(M \cap \{f_1 = \dots = f_k = 0\})$ is the same for all tuples (f_1, \dots, f_k) in this Zariski open set.*

The quantities $e(V, B_1, \dots, B_i)$ satisfy the following higher additivity property (cf. higher additivity for Newton polytopes of discriminants in [E10]).

LEMMA 2.29. *We have $e(M, B_0 + B'_0, B_1, \dots, B_k) + 2e(M, B_0, B'_0, B_1, \dots, B_k) = e(M, B_0, B_1, \dots, B_k) + e(M, B'_0, B_1, \dots, B_k) + e(M, B_0 + B'_0, B_0, B'_0, B_1, \dots, B_k)$.*

PROOF. Compare the Euler characteristic of $M \cap \{gh = f_1 = \dots = f_k = 0\}$ and its smoothening $M \cap \{gh + \varepsilon f = f_1 = \dots = f_k = 0\}$ for generic $g \in \mathbb{C}^{B_0}$, $h \in \mathbb{C}^{B'_0}$, $f \in \mathbb{C}^{B_0 + B'_0}$. \square

PROPOSITION 2.30. *If \mathcal{V} is not empty, then there exists a polytope $S \subset \mathbb{Q}^n$, such that, for any n -dimensional polytope $B \supset S$, we have $e(V, B) = e(V) - (-1)^{\dim V} P_{\text{Trop } \mathcal{V}}(B) = e(V) - (-1)^{\dim V} \sum_j P_{\text{Trop } \mathcal{V}, j}[B]^{n-j}$.*

PROOF. For a generic $f \in \mathbb{C}^B$ in the sense of Lemma 2.28 and a in the interior of B , by the standard Morse theory argument for the function $h : V \rightarrow \mathbb{R}$, $h(x) = |f(x)/x^a|$, we have $e(V, B) = e(V) - (-1)^{\dim V} \cdot$ (the number of critical points of h). The key observation is that the number of critical points equals the intersection number of \mathcal{V} and the graph of $\text{dlog}_a f$. The latter, by Lemma 2.26, equals the tropical intersection number of $[\mathcal{V}]$ and $[C(B)]^n$, which equals $P_{\text{Trop } \mathcal{V}}(B)$ starting from some polytope by Proposition 2.17. \square

We can drop the assumption of non-emptiness of the conormal variety as follows.

PROPOSITION 2.31. 1) If \mathcal{V} is empty, then V is preserved by the action of a certain non-trivial subtorus of $(\mathbb{C} \setminus 0)^n$.

2) Let $(\mathbb{C} \setminus 0)^k$ be the maximal subtorus of $(\mathbb{C} \setminus 0)^n$, preserving V , so that $V = V' \times (\mathbb{C} \setminus 0)^k$, then there exists a polytope $S \subset \mathbb{Q}^{n-k}$, such that we have $e(V, B) = -(-1)^{\dim V+k} P_{\text{Trop } \mathcal{V}'}(B)$ for any n -dimensional $B \supset S$.

The proof is the same as for Proposition 2.30: let x_1, \dots, x_n be monomial coordinates on $(\mathbb{C} \setminus 0)^n$, such that $(\mathbb{C} \setminus 0)^k = \{x_{k+1} = \dots = x_n = 1\}$, then the number of critical points of h can be represented as the intersection of $\mathcal{V}' \times (\mathbb{C} \setminus 0)^k$ and the set $\{(x, y) \mid y_i = x_i \frac{\partial f}{\partial x_i} - a_i f \text{ for } i \leq k \text{ and } \frac{\partial f}{\partial x_i} = 0 \text{ for } i > k\}$.

PROPOSITION 2.32. There exists a polytope $S \subset \mathbb{Q}^n$, such that, for polytopes $B_0, \dots, B_m \supset S$, we have $e(V, B_0, \dots, B_m) =$

$$e(V) - (-1)^{\dim V} \left(P_{\text{Trop } \mathcal{V}, n} + (-1)^m \sum_{j < n-m} P_{\text{Trop } \mathcal{V}, j} \sum_{\substack{b_0 + \dots + b_m = n-j \\ b_0 \in \mathbb{N}, \dots, b_m \in \mathbb{N}}} [B_0]^{b_0} \dots [B_m]^{b_m} \right).$$

PROOF. In the notation of Lemma 2.14, let f_I be the polynomial $\sum_{i \in I} \lambda_i f_i$ on $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^I$, and let B_I be its Newton polytope. Consider the action of the complex circle $(\mathbb{C} \setminus 0)$ on $(\mathbb{C} \setminus 0)^I$ by coordinatewise multiplication and the projection $pi_I : (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^I \rightarrow (\mathbb{C} \setminus 0)^n$. By the additivity of the Euler characteristic, the desired $e(V \cap \{f_0 = \dots = f_m = 0\})$ equals $\sum_I e(\{f_I = 0\} \cap (V \times (\mathbb{C} \setminus 0)^n) / (\mathbb{C} \setminus 0)) - me(V)$, because the restriction of the projection $\mathbb{C}\mathbb{P}^m \times V \rightarrow V$ to $\{\sum_i \lambda_i f_i\} / (\mathbb{C} \setminus 0)$ has fibers $\mathbb{C}\mathbb{P}^{m-1}$ over the points of $\{f_0 = \dots = f_m = 0\}$ and $\mathbb{C}\mathbb{P}^m$ over the other points of V .

We can now compute $e(\{f_I = 0\} \cap (V \times (\mathbb{C} \setminus 0)^n) / (\mathbb{C} \setminus 0))$ for every I by Proposition 2.30 or 2.31 in terms of $P_{\text{Trop } \mathcal{V}}(B_I)$, and simplify the answer by Lemma 2.14. \square

PROPOSITION 2.33. If \mathcal{V} is non-empty (i. e. $V \subset (\mathbb{C} \setminus 0)^n$ is not preserved by non-trivial tori), then

$$P_{\text{Trop } \mathcal{V}}(B) = \text{Trop}(V)[B]^{\dim V} + \dots + (-1)^{\dim V} e(V).$$

PROOF. The equalities $P_{\text{Trop } \mathcal{V}, j} = 0$, $j < n - \dim V$, $P_{\text{Trop } \mathcal{V}, n - \dim V} = \text{Trop } V$ and $P_{\text{Trop } \mathcal{V}, n} = (-1)^{\dim V} e(V)$ follow from Proposition 2.32 for $m = \dim V - 1$, because $e(V, B_0, \dots, B_m)$ equals $[B_0] \dots [B_m] \text{Trop } V$ for all B_0, \dots, B_m . \square

PROPOSITION 2.34. The equality of Proposition 2.32 is valid without the assumptions $\dim B = n$ and $B \supset S$.

PROOF. For $m = \dim V - 1$, the statement follows from Proposition 2.33. Assume that we have proved the statement for all m greater than a given value m_0 . Then we can prove it for $m = m_0$ by induction on the number p of the polytopes B_i that do not contain S or have dimension less than n . For $p = 0$, the desired statement coincides with the one of Proposition 2.32. For $p > 0$, apply Lemma 2.29 with $k = m$, B_0, \dots, B_{p-1} containing S and n -dimensional, and B'_0, B_p, \dots, B_m arbitrary. Note that $B_0 + B'_0$ is also n -dimensional and containing S , so all the terms in the equality of Lemma 2.29, except for $e(M, B'_0, B_1, \dots, B_m)$, can be computed by the inductive assumption. This can be used to compute $e(M, B'_0, B_1, \dots, B_m)$ and leads to the desired formula for it. \square

DEFINITION 2.35. Let $V : (\mathbb{C} \setminus 0)^n \rightarrow \mathbb{R}$ be a constructible function (that is, a linear combination of characteristic functions of algebraic subsets). Define its characteristic class $[V] = \sum_i [V]_i$, $[V]_i \in \mathcal{K}_i^0(\mathbb{Z}^k)$, as follows.

1) If V is (the characteristic function of) a smooth subvariety $W \subset (\mathbb{C} \setminus 0)^n$, then define $[W]_i$ as $(-1)^{i+n+\dim W} P_{\text{Trop } \mathcal{W}, i}$.

2) If V is (the characteristic function of) a smooth subset of $\{f \neq 0\}$, $f : (\mathbb{C} \setminus 0)^n \rightarrow \mathbb{C}$, then define $[V]_i$ as the image of $[V \times (\mathbb{C} \setminus 0) \cap (\text{the graph of } f)]_i$ under the projection $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0) \rightarrow (\mathbb{C} \setminus 0)^n$.

3) For arbitrary V , represent it as a linear combination $\sum_i \alpha_i V_i$ of functions V_i of the form (2), and define $[V]_i$ as $\sum_j \alpha_j [V_i]_j$.

THEOREM 2.36. 1) For all lattice polytopes B_1, \dots, B_k in \mathbb{Q}^n and generic $f_i \in \mathbb{C}^{B_i}$, the Euler characteristic integral of V over $\{f_1 = \dots = f_k = 0\}$ (see e.g. [M74] and [V88]) equals $(-1)^{\dim V+k} \sum_{j \leq n-k} [V]_j \sum_{\substack{b_1+\dots+b_k=n-j \\ b_1 \in \mathbb{N}, \dots, b_k \in \mathbb{N}}} [B_1]^{b_1} \dots [B_k]^{b_k}$.

2) We have $[U \times V] = [U] \times [V]$.

3) $[V]_n$ equals the Euler characteristic integral of V .

4) $[V]$ does not depend on the decomposition of V into a linear combination of characteristic functions, chosen in Definition 2.35.

5) For generic $c \in (\mathbb{C} \setminus 0)^n$, we have $[U \cap (cV)] = [U][V]$.

6) For a generic hypersurface V with the Newton polytope Δ , we have $[V] = \frac{[\Delta]}{1+[\Delta]}$.

PROOF. For characteristic functions of smooth subvarieties of tori, (1) is Proposition 2.34, (3) is Proposition 2.33, and (2) follows from multiplicativity of $P_A(B)$ (see Proposition 2.17). For arbitrary characteristic functions, (1,2,3) follow by the additivity of Euler characteristic. They imply (4), because $[V]_i$, $i < n$, are uniquely determined by (1) for $k = 1$, and $[V]_n$ is uniquely determined by (3). To prove (5), represent $U \cap (cV)$ as $U \times V \cap \{(x, y) \mid x_i - c_i y_i = 0\}$, then the desired equality follows from (1,2). Part (6) follows from Proposition 2.30, because, computing $e(V, B)$ in its statement by the Kouchnirenko-Bernstein-Khovanskii formula [Kh77b], we have $P_{\text{Trop } \mathcal{V}, j} = [\Delta]^j$. \square

We now prove the relation of tropical characteristic classes to Schwarz-MacPherson classes (Part 6 of Definition 2.2 and Proposition 2.12). For this, recall the description of the cohomology ring of a smooth toric variety in terms of tropical fans. Let Σ be a \mathbb{Z} -simple fan, and let X_Σ be the corresponding smooth toric compactification of the torus $(\mathbb{C} \setminus 0)^n$. We say that a k -dimensional tropical fan (P, φ) is **compatible** with Σ , if its support set P is contained in the union of the k -dimensional cones of Σ .

PROPOSITION 2.37 ([B96], [FS94]). *The cohomology ring of X_Σ is naturally isomorphic to the ring of tropical fans compatible with Σ . The isomorphism sends a codimension k cohomology class α to the tropical fan (P, φ) , where P is the union of the k -dimensional cones of Σ , and the value of φ on every such cone C equals the restriction of α to the closure of the C -orbit in X_Σ .*

We identify the cohomology and homology of X_Σ by the Poincaré duality. For a subvariety $V \subset (\mathbb{C} \setminus 0)^n$, we consider the Schwarz-MacPherson class $c_{i, \Sigma}^{SM}(V) \subset H^i(X_\Sigma)$ of a non-closed set $V \subset X_\Sigma$ as a tropical fan, compatible with Σ .

THEOREM 2.38. 1) *If the closure of V in X_Σ is smooth and transversal to the orbits, then the multiplicity of every cone $C \in \Sigma$ in the tropical fan $c_{i, \Sigma}^{SM}(V)$ equals the Euler characteristic of $(\text{closure of } V) \cap (C\text{-orbit})$.*

2) If $U \subset (\mathbb{C} \setminus 0)^n$ is arbitrary, $V \subset (\mathbb{C} \setminus 0)^n$ is a generic complete intersection, defined by the equations with the Newton polytopes N_1, \dots, N_k , and the dual fans $[N_1], \dots, [N_k]$ are compatible with Σ , then $c_\Sigma^{SM}(U \cap V) = c_\Sigma^{SM}(U)c_\Sigma^{SM}(V)$.

3) For arbitrary $U \subset (\mathbb{C} \setminus 0)^n$, choose Σ compatible with the tropical characteristic class $[U]$, constructed above. Then $[U]$ equals $c_\Sigma^{SM}(U)$.

LEMMA 2.39. (see Theorem 3.1 in [A11]) Let M be a smooth compact variety, $V \subset M$ be an arbitrary subvariety, and D be a smooth hypersurface, transversal to V . Then 1) $c^{SM}(V \cap D) = c^{SM}(V) \cdot c^{SM}(D)$ in the cohomology ring $H^\bullet(M)$,

2) $c^{SM}(V \cap D) \in H^\bullet(D)$ is equal to the restriction of $c^{SM}(V \setminus D) \in H^\bullet(M)$.

Proof of Theorem. Part 1 follows by Lemma 2.39(2) and Proposition 2.37, because the highest Schwarz-MacPherson class equals the Euler characteristic. Part 2 follows by Lemma 2.39(1) for a generic hypersurface V and by induction for a generic complete intersection V .

Part 3 for a generic complete intersection follows from Part 1 and Theorem 2.36(6). For an arbitrary U , choose a generic complete intersection V , such that the dual fans of the Newton polytopes of its equations are compatible with Σ . Then we have $c_\Sigma^{SM}(V) = [V]$, and the highest components of $c_\Sigma^{SM}(U) \cdot c_\Sigma^{SM}(V)$ and $[U] \cdot [V]$ are both equal to the Euler characteristic of $U \cap V$ (by Part 2 and Theorem 2.36(5) respectively). This equality for arbitrary V implies that $[U] = c_\Sigma^{SM}(U)$, because the cohomology ring of X_Σ is generated by complete intersections, and the multiplication of cohomologies of complementary dimensions is a nondegenerate bilinear form. \square

2.7 Characteristic classes of critical complete intersections

Non-degenerate complete intersections are the simplest example of schön varieties. We shall need the next simplest example: the set of critical points of a projection $\{f = 0\} \subset (\mathbb{C} \setminus 0)^n \xrightarrow{p} (\mathbb{C} \setminus 0)^k$, where f is a Laurent polynomial with generic coefficients, and p is a morphism of complex tori. This complete intersection is degenerate, but still schön, as we prove below.

Denote the $(n+1)$ -dimensional space of affine linear functions on \mathbb{Z}^n by $(\mathbb{Z}^n)^\star$, to be distinguished from the dual space $(\mathbb{Z}^n)^\circ$. For every subspace $L \subset \mathbb{Z}^n$, let $L^\perp \subset (\mathbb{Z}^n)^\star$ be the set of affine functions on \mathbb{Z}^n that vanish on L . For a Laurent polynomial $f(x) = \sum_{b \in \mathbb{Z}^n} c_b x^b$ on $(\mathbb{C} \setminus 0)^n$ with the standard coordinates $x = (x_1, \dots, x_n)$ and an element $\alpha \in (\mathbb{Z}^n)^\star$, denote the polynomial $\sum_{b \in \mathbb{Z}^n} \alpha(b) c_b x^b$ by $\partial_\alpha f$.

EXAMPLE 2.40. If α is the i -th coordinate function, then $\partial_\alpha f = x_i \frac{\partial f}{\partial x_i}$.

DEFINITION 2.41. The set $\{\partial_D f = 0\} \subset (\mathbb{C} \setminus 0)^n$, given by the equaltions $\partial_\alpha f = 0$, $\alpha \in D$, is called a **critical complete intersection**.

PROPOSITION 2.42. 1) For every finite $B \subset \mathbb{Z}^n$, every vector subspace $D \subset (\mathbb{Z}^n)^\star$ and a generic Laurent polynomial $f(x) = \sum_{b \in B} c_b x^b$, the critical complete intersection $\{\partial_D f = 0\}$ is indeed a regular complete intersection.

2) Moreover, it is schön.

Note that the schön complete intersection $\{\partial_D f = 0\}$ is not in general non-degenerate with respect to its Newton polytopes ([Kh77a]) even for generic f .

Part 1 is a standard corollary of the Bertini—Sard theorem, see the subsequent paragraph. The proof of Part 2, including an explicit construction of a schön compactification, occupies the rest of this subsection.

PROOF OF PART 1. With no loss in generality we can assume that B is not contained in an affine hyperplane. Then we can choose a basis $\alpha_0, \dots, \alpha_k$ in D and points b_0, \dots, b_k in B such that the matrix $\alpha_i(b_j)$ is triangular and non-degenerate, i.e. $\partial_{\alpha_i} f$ does not depend on $c_{b_{i+1}}, \dots, c_{b_k}$. Then, by induction on i , we can prove that $\partial_{\alpha_0} f = \dots = \partial_{\alpha_i} f = 0$ is a regular complete intersection for almost all f : for every i , take c_{b_i} to be a regular value of the restriction of the function $-\sum_{b \in B \setminus \{b_i\}} \frac{\alpha_i(b)}{\alpha_i(b_i)} c_b x^{b-b_i}$ to the regular set $\partial_{\alpha_0} f = \dots = \partial_{\alpha_{i-1}} f = 0$. \square

The proof of Part 2 is based on an explicit construction of a schön compactification for $\{\partial_D f = 0\}$. We shall use the following notation to describe it. For every $\gamma \in (\mathbb{Z}^n)^*$ and $B \subset \mathbb{Z}^n$, let B^γ be the set of points at which the restriction of γ to B attains its maximum. For every sequence of sets $B_k \subset \mathbb{Z}^n$, denote the affine span of $\bigcup_{i < k} B_k$ by $B_{<k}$.

DEFINITION 2.43. 1) For every $\gamma \in (\mathbb{Z}^n)^*$, define a finite sequence of subsets $B_k^\gamma \subset B$ inductively by $k \geq 0$ as follows: the set B_k^γ consists of all points at which the restriction of the function γ to $B \setminus B_{<k}^\gamma$ attains its maximal value, unless $(B_{<k}^\gamma)^\perp \cap D = 0$, and B_k is not defined in the latter case.

2) For a finite sequence of non-empty non-intersecting subsets $B_k \subset B$, the cone C_{B_\bullet} is the set of all $\gamma \in (\mathbb{Z}^n)^*$ such that the sequence B_k^γ equals the sequence B_k . The fan that consists of the cones C_{B_\bullet} , as B_\bullet runs over all sequences of subsets of B , will be denoted by C_B .

REMARK 2.44. The codimension of a non-empty cone C_{B_\bullet} equals the dimension of $\sum_i B_i$.

EXAMPLE 2.45. If $B = \{0, e_1, e_2, e_3\}$ is the standard simplex in \mathbb{R}^3 , and D is generated by the functions 1 and $x + y$, then C_B is the minimal subdivision of the dual fan of B by the line generated by $x + y$. For example, the 1-dimensional cone generated by $-x - y$ corresponds to the sequence $B_\bullet = (B_0, B_1) = (\{0, e_1\}, \{e_2, e_3\})$.

Now Proposition 2.42(2) can be restated in more detail as follows. For every sequence of non-intersecting sets $B_k \subset \mathbb{Z}^n$, choose a direct sum decomposition $D = \bigoplus_i B^i$ such that $\bigoplus_{i \geq k} B^i$ equals $B_{<k}^\perp \cap D$ for every k . For every subset $B' \subset B$ and a polynomial $f(x) = \sum_{b \in B} c_b x^b$, denote the **restriction of f to B'** by $f^{B'}(x) = \sum_{b \in B'} c_b x^b$.

Choose any subdivision \tilde{C}_B of the fan C_B such that the \tilde{C}_B -toric variety is smooth. For every cone C_{B_\bullet} , the orbits of the \tilde{C}_B -toric variety, corresponding to subcones of C_{B_\bullet} , will be called the C_{B_\bullet} -orbits.

PROPOSITION 2.46. 1) For almost all f of the form $\sum_{b \in B} c_b x^b$, the closure of the critical complete intersection $\{\partial_D f = 0\}$ in the \tilde{C}_B -toric variety is smooth and intersects its orbits transversally.

2) The closure of $\{\partial_D f = 0\}$ intersects a C_{B_\bullet} -orbit by the same set as the closure of the set given by the equations $\partial_{B^i} f^{B^i} = 0$ for all i .

PROOF. Choose a basis $\alpha_{0i}, \dots, \alpha_{k_i}$ in B_i . The union of these bases is a basis in D . For every linear function $l \in C_{B_\bullet}$, the maximum of l on the Newton polytope of the polynomial $\partial_{\alpha_{ji}} f$ is attained on its face B_i . Thus, the intersection of a C_{B_\bullet} -orbit and the

closure of $\{\partial_D f = 0\}$ is contained in the intersection of the C_{B_\bullet} -orbit and the closure of $\cap_{i,j}\{\partial_{\alpha_{ji}} f = 0\}$, and contains the regular part of the latter intersection. However, the latter intersection is regular for generic f , because, for every i , the equations $\partial_{B^i} f^{B^i} = 0$ define a regular set S_i by Proposition 2.42(1), and, for generic $c_i \in (\mathbb{C} \setminus 0)^n$, the sets $c_i S_i$ intersect transversally. Thus we have proved that the intersection of the closure of $\{\partial_D f = 0\}$ with every C_{B_\bullet} -orbit is regular for every cone C_{B_\bullet} , and is given by the equations $\partial_{B^i} f^{B^i} = 0$. This concludes the proof of Propositions 2.46 and 2.42. \square

Proposition 2.12 reduces computation of tropical characteristic classes of critical complete intersections to computation of the Euler characteristics of critical complete intersections. For every sequence B_\bullet , let Z_{B_\bullet} be (the set given by the equations $\partial_{B^i} f^{B^i} = 0$ for all i) / (the subtorus of $(\mathbb{C} \setminus 0)^n$ generated by t^γ over all $\gamma \in C_{B_\bullet}$).

COROLLARY 2.47. *The j -th tropical characteristic class of $\{\partial_D f = 0\}$ consists of the cones C_{B_\bullet} over all sequences B_\bullet such that $j = \dim \sum_i B_i$, with the multiplicities equal to the Euler characteristics of Z_{B_\bullet} .*

REMARK 2.48. For most of sequences B_\bullet , the cone C_{B_\bullet} is either empty or has multiplicity 0. For example, C_{B_\bullet} is empty unless the sets B_i are non-empty and non-overlapping, and B_0 is contained in the boundary of the convex hull of B . The multiplicity of C_{B_\bullet} also equals 0 if $\dim D \cap B_{<k}^\perp > \dim \sum_{i < k} B_i$ for some k .

The sets of the form Z_{B_\bullet} that we face in the framework of this paper are very simple (points and lines). However we notice that such Euler characteristics can be computed in the general case by methods of [E10]. For example, we give the answer for 0-dimensional critical complete intersections.

NOTATION. Assume that $D \subset (\mathbb{Z}^n)^*$ is a hyperplane. It is uniquely determined by the common zero $z \in \mathbb{R}\mathbb{P}^n$ of all affine linear functions $\alpha \in D$. Let \mathfrak{A} be the set $\{\Gamma \mid \Gamma \text{ is a face of the convex hull of } A, \text{ and the affine span of } \Gamma \text{ contains } z\}$. Let $N(A, D)$ be the sum $\sum_{\Gamma \in \mathfrak{A}} e_A^\Gamma \text{Vol } \Gamma$, where e_A^Γ is the Euler obstruction of A at Γ , see [E10] for the definition.

LEMMA 2.49. *If A generates \mathbb{Z}^n and D is a hyperplane, then, for a generic Laurent polynomial $f(x) = \sum_{a \in A} c_a x^a$, the 0-dimensional critical complete intersection $\{\partial_D f = 0\}$ consists of $N(A, D)$ points.*

This is the Bernstein formula [B76] with some roots hidden at infinity (more precisely, at the orbits of the A -toric variety, corresponding to the faces from \mathfrak{A}), see [E10] for details.

3 Affine Plucker formulas

The problematics of affine multisingularity theory is introduced in Section 3.1. Its relation to tropical enumerative geometry is outlined in Section 3.2. The affine version of the Plucker formulas is obtained in Sections 3.3–3.5: the tropical fan of the universal singularity stratum of codimension 1 (i.e. the Newton polytope of the A -discriminant for a non-dual-defective set $A \subset \mathbb{Z}^n$) is computed in Section 3.3; additional assumptions on A (k -versality, Definition 3.19), playing the role of dual defectiveness for the codimension 2 universal multisingularity strata, are introduced in Section 3.4; the tropical

fans of these strata are described in Section 3.5 (see Section 3.8 for a proof of this description). Section 3.6 explains how to specialize these results to various non-universal settings such as generic polynomial maps of generic affine hypersurfaces, and an example of the computation for a simple special case is given in Section 3.9. Application of affine Plucker formulas requires somewhat tedious computations with secondary fans that can be simplified with a formula for the volume of a fiber body, presented in Section 3.7.

3.1 Affine multisingularity theory

A singularity theory is an equivalence relation on the set of germs of algebraic varieties. An equivalence class is called a singularity.

EXAMPLE 3.1. We shall study \mathcal{A}_1 and \mathcal{A}_2 singularities. The \mathcal{A}_i singularity is the class of hypersurface germs in \mathbb{C}^n that can be represented as $x_1^{i+1} + x_2^2 + \dots + x_n^2 = 0$ in suitable local coordinates. In particular, for $n = 2$, the \mathcal{A}_1 and \mathcal{A}_2 singularities are nodes and cusps of plane curves, and we shall denote them by \times and \prec respectively.

A finite tuple of isolated singularities $S = (S_1, \dots, S_k)$ is called a multisingularity. The S -multisingularity stratum of a proper morphism of varieties $N \rightarrow M$ is the set of all $x \in M$ whose preimages have exactly k singularities S_1, \dots, S_k . One of the first goals of singularity theory is to describe the topology (e. g. the homology class) of a multisingularity stratum in terms of the topology (e. g. the characteristic classes) of M and N , provided that the morphism is “generic enough”.

However, this theory does not suit the study of multisingularity strata of polynomial mappings $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ or $(\mathbb{C} \setminus 0)^n \rightarrow (\mathbb{C} \setminus 0)^m$, as explained in the introduction. A possible way to understand singularity theory in this affine setting is as follows: a generic mapping $f = (f_1, \dots, f_m)$ is a mapping whose components f_i are generic linear combinations of given monomials, and the goal of our singularity theory is to describe the cohomology classes of multisingularity strata of the map f in the ring of conditions of $(\mathbb{C} \setminus 0)^n$, that is, in the ring of tropical fans $\mathcal{K}^0(\mathbb{Z}^n)$.

Instead of studying individual maps, we first study the **universal setting**: identify elements of \mathbb{Z}^n with monomials in the variables x_1, \dots, x_n , let \mathbb{C}^A be the space of linear combinations of monomials from a finite set $A \subset \mathbb{Z}^n$, choose finite sets A_1, \dots, A_m in \mathbb{Z}^n . The universal S -multisingularity stratum is the set of all tuples of polynomials $f \in \mathbb{C}^{A_1} \oplus \dots \oplus \mathbb{C}^{A_m}$, such that the variety $f = 0$ has exactly k singularities S_1, \dots, S_k , and our goal is to describe tropical fans of universal multisingularity strata in terms of the sets A_1, \dots, A_m . Then we explain how to reduce individual problems regarding generic mappings $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ or $(\mathbb{C} \setminus 0)^n \rightarrow (\mathbb{C} \setminus 0)^m$ to the universal case.

A lot of recent research can be considered from this point of view: Newton polytopes of A -discriminants, sparse resultants and mixed discriminants [GKZ94], [S94], [G00], [EKh08], [STY07], [DFS07], [ST08], [E10], [E13], [CCdRDS13], [DEK14], [AT12] are tropical fans of the (\times) -singularity stratum. The tropical fan of higher dual toric varieties [DdRP14] is the tropical fan of the stratum of hypersurfaces with a single singularity of a given order.

If the sets A_1, \dots, A_m are “small enough”, then collections of isolated singularities do not appear at all in the fibers $f = 0$, $f \in \mathbb{C}^{A_1} \oplus \dots \oplus \mathbb{C}^{A_m}$, and one can be interested instead in the strata Σ_k of all f such that the singular locus of f has a given dimension k . For $m = 1$, the lowest codimension non-empty stratum Σ_k is the A -discriminant

variety, and its tropical fan is computed in [DFS07] (for any k). If $A_1 = \dots = A_m$ is the standard simplex, then the strata Σ_k are the determinantal varieties. In [S13], all triples (m, n, k) are classified, for which the minors of the matrix form a tropical basis of the ideal of the determinantal variety. This, in particular, gives the tropical fan of Σ_k . For other triples (m, n, k) , the tropical fan is unknown.

Tropical enumeration of plane curves with nodes and cusps [M05], [BBM14], [S06], [S12] can also be seen as a partial computation of tropical fans of corresponding Severi varieties, that is, the universal $(\times, \dots, \times, \prec, \dots, \prec)$ -multisingularity strata, see Section 3.2 and [Y13] for details.

Summarizing this, the only singularity strata, whose complete tropical fans are known, are

- the lowest codimension singularity strata,
- the single singularity strata for singularities of a given order,
- the strata with known tropical bases of their defining ideals.

We shall compute the tropical fan of the simplest possible strata outside of this list, namely, the codimension 2 multisingularity strata: the $(\times \times)$ -stratum of all $f \in \mathbb{C}^A$ with two nondegenerate singular points and the (\prec) -stratum of all $f \in \mathbb{C}^A$ with a single minimally degenerate singular point.

Many relations for tropical characteristic classes of multisingularity strata follow directly from additivity of the Euler characteristic. For example, let X be a smooth toric variety, and let $V \subset X \times (\mathbb{C} \setminus 0)^N$ be a subvariety. For every multisingularity $S = (S_1, \dots, S_k)$, let $e(S)$ be the sum of the reduced Euler characteristics of the Milnor fibers of the isolated singularities S_1, \dots, S_k , let $[S]$ be the tropical characteristic class of the S -multisingularity stratum of the projection $\pi : V \rightarrow (\mathbb{C} \setminus 0)^N$, and let p be the codimension of the set of all points $x \in (\mathbb{C} \setminus 0)^N$, such that the fiber $\pi^{-1}(x)$ has a non-isolated singularity. If a generic fiber of π is smooth, then, denoting its Euler characteristic by e_0 , we have for $i < p$:

$$e_0 \cdot [\pi(V)]_i - \sum_S e(S) \cdot [S]_i = \sum_{T \subset X} \pi_* [V \cap T]_{i + \dim T}, \quad (*)$$

where T runs over all orbits of X , and S runs over all (isolated) multisingularities. This follows from the additivity of the Euler characteristics and the functoriality of tropical characteristic classes (Part 5 of Definition 2.2).

3.2 Relation to tropical correspondence theorems

In this subsection, we restate tropical correspondence theorems in terms of intersection numbers of tropical fans of multisingularity strata. This allows us to consider tropical enumeration as the classical approach to enumerative geometry, up to substitution of conventional cohomology rings, Chern classes and Thom polynomials with the ring of tropical fans, tropical characteristic classes and tropical fans of multisingularity strata.

Recall that we write tropical objects in Fraktur. Every codimension 1 tropical fan with positive multiplicities equals the hypersurface $\mathfrak{f} = \mathfrak{o}$ for some tropical polynomial \mathfrak{f} with unit coefficients. Conversely, for every tropical hypersurface $\{\mathfrak{x} \mid \mathfrak{f}(\mathfrak{x}) = \mathfrak{o}\}$ and every point \mathfrak{x}_0 in $(\mathbb{T} \setminus \mathfrak{o})^m$, the shifted hypersurface $\{\mathfrak{x} \mid \mathfrak{f}(\mathfrak{x}/\mathfrak{x}_0) = \mathfrak{o}\}$ coincides in a small neighborhood of $(\mathbf{1}, \dots, \mathbf{1})$ with a certain codimension 1 tropical fan that we denote by $\{\mathfrak{f} = \mathfrak{o}\}_{\mathfrak{x}_0} \in \mathcal{K}_1^0(\mathbb{Z}^m)$.

DEFINITION 3.2. For any collection of tropical fans and hypersurfaces $\mathfrak{H}_1, \dots, \mathfrak{H}_k$ in $(\mathbb{T} \setminus \mathfrak{o})^m$, whose codimensions sum up to m , the number $(\mathfrak{H}_1)_{\mathfrak{x}} \cdots (\mathfrak{H}_k)_{\mathfrak{x}} \in \mathcal{K}_m^0(\mathbb{Z}^m) = \mathbb{Q}$ is called the **tropical intersection number** of $\mathfrak{H}_1, \dots, \mathfrak{H}_k$ at the point $\mathfrak{x} \in (\mathbb{T} \setminus \mathfrak{o})^m$.

Recall that the following **displacement rule** allows us to count the tropical intersection number and can be taken as its definition. For a rational affine subspace $L \subset \mathbb{R}^m$, by a slight abuse of notation, denote the lattice $\{l \in (\mathbb{Z}^m)^* \mid l(L) = \text{const}\}$ by L^\perp .

PROPOSITION 3.3. 1) Assume that $\mathfrak{H}_1, \dots, \mathfrak{H}_k$ intersect transversally at \mathfrak{x} , i.e. $\mathfrak{H}_i \subset (\mathbb{T} \setminus \mathfrak{o})^m = \mathbb{R}^m$ coincides with a rational affine space $L_i \ni \mathfrak{x}$ endowed with a multiplicity m_i in a small neighborhood of \mathfrak{x} , and $\sum_i L_i^\perp$ is an m -dimensional lattice \mathcal{L} . Then the tropical intersection number of $\mathfrak{H}_1, \dots, \mathfrak{H}_k$ at \mathfrak{x} equals $m_1 \dots m_k \cdot |(\mathbb{Z}^m)^* / \mathcal{L}|$.

2) In the general case, let $\tilde{\mathfrak{H}}_i$ be the copy of \mathfrak{H}_i shifted by a small generic vector. Then $\tilde{\mathfrak{H}}_1, \dots, \tilde{\mathfrak{H}}_k$ intersect at finitely many points $\mathfrak{x}_1, \dots, \mathfrak{x}_N$ near \mathfrak{x} (if any), the intersection is transversal at each of these points, and the intersection number of $\mathfrak{H}_1, \dots, \mathfrak{H}_k$ at \mathfrak{x} equals the sum of the intersection numbers of $\tilde{\mathfrak{H}}_1, \dots, \tilde{\mathfrak{H}}_k$ at $\mathfrak{x}_1, \dots, \mathfrak{x}_N$.

From now on, let \mathbb{K} denote either \mathbb{C} or \mathbb{T} .

Let $A \subset \mathbb{Z}^n$ be a finite set containing 0. Let \mathbb{K}^A be the set of Laurent polynomials of the form $\sum_{a \in A} c_a x^a$, $c_a \in \mathbb{K}$, and let its subset $(\mathbb{K} \setminus 0)_1^A$ consist of all polynomials, whose constant term c_0 equals 1, and whose other coefficients are non-zero. Let $S \subset (\mathbb{K} \setminus 0)_1^A$ be the (S_1, \dots, S_k) -multisingularity stratum, and let $B_{g_i} \subset (\mathbb{K} \setminus 0)_1^A$, $g_i \in G_i$, be a family of conditions of incidence parameterized by the set G_i . For instance, $g_i \in G_i = (\mathbb{K} \setminus 0)^n$, and B_{g_i} is the set of all $f \in (\mathbb{K} \setminus 0)_1^A$, such that $f(g_i) = 0$; or $g_i \in G_i = \mathbb{C}^\Delta$, $\Delta \subset \mathbb{Z}^n$, and B_{g_i} is the closure of the set of all f such that $f = 0$ is tangent to $g_i = 0$ at some point.

We shall be interested in counting hypersurfaces of the form $f = 0$, $f \in (\mathbb{C} \setminus 0)_1^A$, with prescribed singularities S_1, \dots, S_k , satisfying prescribed conditions of incidence B_{g_1}, \dots, B_{g_l} for generic parameters $g_i \in G_i$. More accurately, we shall be interested in the number of *isolated* intersection points of the stratum S and the pencils B_{g_i} for generic g_i . Most of classical problems of enumerative geometry can be restated in this form.

REMARK 3.4. However, note that the desired number is not in general equal to the global intersection number of S and B_{g_i} in the ring of conditions, because the intersection $S \cap \bigcap_i B_{g_i}$ may have non-isolated components even for generic g_i . A classical example is the count of conics that are tangent to five generic lines: the intersection of the corresponding pencils $\bigcap_{i=1}^5 B_{g_i}$ in the 5-dimensional space of conics contains a two-dimensional component that consists of all two-fold lines.

We now describe the tropical version of counting hypersurfaces with prescribed singularities and conditions of incidence. Let \mathfrak{S} be the tropical fan of the stratum S . It is a tropical fan in the real part of the Lie algebra of the complex torus $(\mathbb{C} \setminus 0)_1^A$, this real part is naturally identified with the tropical torus $(\mathbb{T} \setminus \mathfrak{o})_1^A$.

DEFINITION 3.5. A tropical hypersurface $f = 0$ is said to have the (S_1, \dots, S_k) -multisingularity, if f is contained in \mathfrak{S} .

Assume that each parameter space G_i is a complex torus, denote the real part of its Lie algebra by \mathfrak{G}_i , and consider the tropical fan \mathfrak{B}_i of the set $\{(f, g) \mid f \in B_g, g \in G_i\}$.

DEFINITION 3.6. For every $\mathbf{g}_i \in \mathfrak{G}_i$, define the **tropical condition of incidence** $\mathfrak{B}_{\mathbf{g}_i}$ as the tropical intersection of \mathfrak{B}_i and the plane $(\mathbb{T} \setminus \mathfrak{o})_1^A \times \{\mathbf{g}_i\}$.

It is convenient to give tropical conditions of incidence a more geometric interpretation in important special cases. For instance, if $g_i \in G_i = (\mathbb{C} \setminus 0)^n$, and B_{g_i} is the set of all $f \in (\mathbb{C} \setminus 0)_1^A$, such that $f(g_i) = 0$, then $\mathbf{g}_i \in \mathfrak{G}_i = (\mathbb{T} \setminus \mathfrak{o})^n$, and \mathfrak{B}_{g_i} is the set of all $\mathbf{f} \in (\mathbb{T} \setminus \mathfrak{o})_1^A$, such that $\mathbf{f}(\mathbf{g}_i) = \mathfrak{o}$. If $g_i \in G_i = \mathbb{C}^\Delta$, $\Delta \in \mathbb{Z}^2$, and B_{g_i} is the closure of the set of all $f \in (\mathbb{C} \setminus 0)_1^A$, such that the curves $f = 0$ and $g = 0$ are tangent, then $\mathbf{g}_i \in \mathfrak{G}_i = (\mathbb{T} \setminus \mathfrak{o})_1^A$, and \mathfrak{B}_{g_i} is the the set of all $\mathbf{f} \in (\mathbb{T} \setminus \mathfrak{o})_1^A$, such that the tropical curves $\mathbf{f} = 0$ and $\mathbf{g}_i = 0$ are tangent in the sense of [BBM14].

DEFINITION 3.7. The problem of counting hypersurfaces with prescribed singularities S_1, \dots, S_k , satisfying prescribed conditions of incidence B_{g_1}, \dots, B_{g_I} , is said to be **tropicalizable**, if the answer equals the number of isolated points in the intersection $\mathfrak{S} \cap \bigcap_i \mathfrak{B}_{\mathbf{g}_i}$ for generic $\mathbf{g}_i \in L_i^*$ (counted with intersection multiplicities, see Definition 3.2).

In these terms, the correspondence theorem [M05] reduces to computation of the tropical fan of the Severi variety, because the problem of counting plane curves with a prescribed number of self-intersections through prescribed generic points $\mathbf{g}_1, \dots, \mathbf{g}_I$, is tropicalizable (see Proposition 3.8 below). Recall that the Severi variety S is the closure of the set of all $f \in (\mathbb{C} \setminus 0)^A$, $A \subset \mathbb{Z}^2$, such that the curve $f = 0$ is nodal with d nodes.

Similarly, the correspondence theorem [BBM14] reduces to computation of the tropical fan \mathfrak{S} of the Severi variety, because the problem of counting plane rational curves, containing prescribed generic points and tangent to prescribed generic lines $\mathbf{g}_1, \dots, \mathbf{g}_I$, is tropicalizable (this is difficult, but can be extracted from [BBM14]). More specifically, the tropical curves counted in the correspondence theorems are given by the equations $\mathbf{f} = 0$, such that \mathbf{f} is an isolated point of the intersection $\mathfrak{S} \cap \bigcap_i \mathfrak{B}_{\mathbf{g}_i}$, and the multiplicity, assigned to such a curve by the correspondence theorem, equals the tropical intersection multiplicity of \mathfrak{S} and $\mathfrak{B}_{\mathbf{g}_i}$, $i = 1 \dots, I$, at \mathbf{f} .

However, the class of tropicalizable enumerative problems is much wider than counting nodal curves. E.g. the problem of counting hypersurfaces with an arbitrary given multisingularity, passing through a given collection of generic points, is tropicalizable:

PROPOSITION 3.8. *Let S be any codimension p subset of $(\mathbb{C} \setminus 0)_1^A$ (for example, a multisingularity stratum). Then, for generic points $g_1, \dots, g_p \in (\mathbb{C} \setminus 0)^n$ and $\mathbf{g}_1, \dots, \mathbf{g}_p \in (\mathbb{T} \setminus \mathfrak{o})^n$, there are finitely many $f \in S$ such that $f(g_1) = \dots = f(g_p) = 0$, and this number coincides with the tropical intersection number of the tropical fan \mathfrak{S} and the conditions of incidence $\{\mathbf{f} \in (\mathbb{T} \setminus \mathfrak{o})_1^A \mid \mathbf{f}(\mathbf{g}_i) = \mathfrak{o}\}$, $i = 1, \dots, p$.*

This follows from Proposition 2.21(2), because the condition of passing through a generic point obviously has no base points at infinity (see Definition 2.20).

3.3 Tropical fan of the discriminant

For a finite set A in the character lattice L of the complex torus $(\mathbb{C} \setminus 0)^n$, let us define the **tautological polynomial** s on $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^A$ by the equality $s(x, f) = f(x)$. We shall describe the tropical characteristic classes of the critical complete intersection $s = \partial s / \partial x_1 = \dots = \partial s / \partial x_n = 0$, where (x_1, \dots, x_n) are the standard coordinates on

$(\mathbb{C} \setminus 0)^n$. As a corollary, we shall obtain a new description of the Newton polytope of the A -discriminant, provided that it is a hypersurface, and a new characterisation of dual defect toric varieties (projective toric varieties such that their projectively dual variety is not a hypersurface).

The tropical fan of $s = \partial s / \partial x = 0$ lives in the dual character lattice of $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^A$, i.e. in $L^* \oplus (\mathbb{T} \setminus \mathfrak{o})^A$. We shall use the following notation to describe it. For every $\gamma \in L^* \oplus (\mathbb{T} \setminus \mathfrak{o})^A$, consider its components $\gamma' \in L^*$ and $\gamma'' \in (\mathbb{T} \setminus \mathfrak{o})^A$, $\gamma = \gamma' + \gamma''$, as functions on A : the function γ' is the restriction of the function $\gamma' : L \rightarrow \mathbb{Z}$ to $A \subset L$, and the value of γ'' at $a \in A$ is the coefficient of the degree a monomial in the tropical polynomial $\gamma'' \in (\mathbb{T} \setminus \mathfrak{o})^A$. For every sequence of sets $A_k \subset L$, denote the affine span of $\bigcup_{i < k} A_k$ by $A_{<k}$. Assume that A is not contained in an affine hyperplane.

DEFINITION 3.9. 1) For every $\gamma \in L^* \oplus (\mathbb{T} \setminus \mathfrak{o})^A$, define a finite sequence of subsets $A_k^\gamma \subset A$ inductively by $k \geq 0$ as follows: the set A_k^γ consists of all points at which the restriction of the function $\gamma'' - \gamma' : A \rightarrow \mathbb{Z}$ to $A \setminus A_{<k}^\gamma$ attains its maximal value, provided that $A_{<k}^\gamma \subsetneq L$, and A_k is not defined otherwise.

2) For a finite sequence of non-empty non-intersecting subsets $A_k \subset A$, the cone C_{A_\bullet} is the set of all $\gamma \in L^* \oplus (\mathbb{T} \setminus \mathfrak{o})^A$ such that the sequence A_k^γ equals the sequence A_k . The fan that consists of the cones C_{A_\bullet} , as A_\bullet runs over all sequences of non-empty non-intersecting subsets of A , will be denoted by C_A .

REMARK 3.10. The codimension of the cone C_{A_\bullet} equals $\sum_i (|A_i| - 1)$.

REMARK 3.11. It would be interesting to prove that the fan C_A is regular (i.e. can be represented as the set of domains of linearity of a convex piecewise-linear function).

For every sequence of sets $A_k \subset L$, decompose the space L^* (see Section 2.7) into a direct sum $\bigoplus_i A^k$ such that $\bigoplus_{i \geq k} A^i$ is the orthogonal complement to $A_{<k}$ (that is, $\{l \mid l(A_{<k}) = 0\}$) for every k . For every subset $B \subset A$ and a polynomial $f(x) = \sum_{a \in A} c_a x^a$, denote the **restriction of f to B** by $f^B(x) = \sum_{a \in B} c_a x^a$, and define the corresponding tautological polynomial s^B on $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^A$ by the equality $s^B(f, x) = f^B(x)$. Applying Proposition 2.46(2) to the critical complete intersection $s = \partial s / \partial x = 0$, we obtain the following.

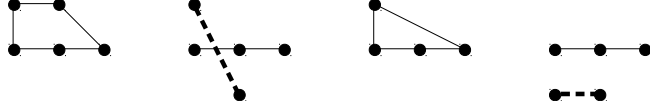
PROPOSITION 3.12. 1) *The closure of $s = \partial s / \partial x = 0$ in the C_A -toric variety intersects its orbits transversally.*

2) *The closure of $s = \partial s / \partial x = 0$ intersects the C_{A_\bullet} -orbit by the same set as the closure of the complete intersection given by the equations $\partial_{A_i} s^{A_i} = 0$ for all i .*

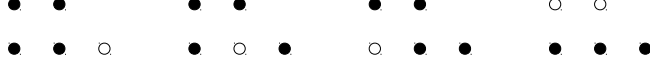
This fact leads to the following description of the characteristic classes of $s = \partial s / \partial x = 0$.

DEFINITION 3.13. The **rank** $\text{rk}_k B$ of a subset $B \subset A_k$ is $|B| - \dim(\text{affine span of } B \cup A_{<k}) + \dim A_{<k} - 1$. A sequence A_k is said to be **essential**, if $\text{rk}_k B < \text{rk}_k A_k$ for every k and every $B \subsetneq A_k$. The number of minimal subsets $B \subset A_k$, such that $\text{rk}_k B = \text{rk}_k A_k - 1$, is denoted by $m(A_k)$. The number $m(A_k)$ for the maximal k , such that $\text{rk}_k A_k > 0$, is denoted by $m(A_\bullet)$. If the number $|L / \bigcup_i \{a - b \mid a \text{ and } b \in A_i\}|$ is finite, then denote it by $i(A_\bullet)$, otherwise set $i(A_\bullet) = 0$.

EXAMPLE 3.14. We shall work out all the notation for the following examples (in what follows, A_0 is always depicted in solid lines and A_1 in dashed lines).



In the first example, $i(A_\bullet) = 1$, $D_0 = L^*$, $\text{codim } C_{A_\bullet} = |A_0| - 1 = 4$, $\text{rk}_0 A_0 = 2$, and the four minimal sets $B \subset A_0$ with rank 1 are as follows (so that $m(A_\bullet) = 4$):



In the second example, $i(A_\bullet) = 2$, $\text{codim } C_{A_\bullet} = |A_0| - 1 + |A_1| - 1 = 3$, D_1 is generated by the function $l(x, y) = x$, and D_0 is any plane such that $D_0 + D_1 = L^*$. The third example is not essential, and the last one is essential with $i(A_\bullet) = 0$.

COROLLARY 3.15. 1) *The support set of the characteristic class $[s = \partial s / \partial x. = 0]_j$ is contained in the union of the cones C_{A_\bullet} , where A_\bullet runs over all essential sequences such that $\sum_i (|A_i| - 1) = j$.*

2) *For $j \leq n$, we have $[s = \partial s / \partial x. = 0]_j = 0$.*

3) *The multiplicity of the cone C_{A_\bullet} in the tropical fan $[s = \partial s / \partial x. = 0]_{n+1}$ equals 1 for every essential A_\bullet such that $\sum_i (|A_i| - 1) = n + 1$.*

4) *The multiplicity of the cone C_{A_\bullet} in the tropical fan $[s = \partial s / \partial x. = 0]_{n+2}$ equals $2 - m(A_\bullet)$ for every essential A_\bullet such that $\sum_i (|A_i| - 1) = n + 2$.*

PROOF. Choose any sequence A_\bullet and denote $\sum_i (|A_i| - 1)$ by d . Proposition 3.12.2 implies that the intersection of the closure of $s = \partial s / \partial x. = 0$ and the C_{A_\bullet} -orbit in the C_A -toric variety is a $(d - n - 1)$ -dimensional variety V in a d -dimensional complex torus. The following facts immediately follow from the definitions:

1) If A_\bullet is not essential, then V is empty.

2) If A_\bullet is essential, and $d = n + 1$, then V consists of one point.

3) If A_\bullet is essential, and $d = n + 2$, then V is a projective line minus $m(A_\bullet)$ points.

The statement follows from these facts and Propositions 3.12 and 2.12. \square

One can compute multiplicities of all cones for all the characteristic classes of $s = \partial s / \partial x. = 0$ in the same way as we do in the proof of Parts 3 and 4 for the first two non-zero classes. The complete answer will be given elsewhere, because we do not need it to study multisingularities in codimension 1 and 2.

EXAMPLE 3.16. It may well happen that the multiplicity $2 - m(A_\bullet)$ is 0 in the circumstances of Part 4, i.e. the cone C_{A_\bullet} for some essential A_\bullet with $\sum_i (|A_i| - 1) = n + 2$ does not contribute to $[s = \partial s / \partial x. = 0]_{n+2}$. For example, this happens for $A_0 = \{(0, 0, 0), (0, 0, \pm 1)\}$ and $A_1 = \{(0, \pm 1, 0), (\pm 1, 0, 0)\}$. In particular, Part 1 gives a non-sharp estimate on the support sets of the characteristic classes.

The aim of our computation is the image $\pi_*[s = \partial s / \partial x. = 0]_j$ under the projection $\pi : (\mathbb{C} \setminus 0)^n \oplus (\mathbb{C} \setminus 0)^A \rightarrow (\mathbb{C} \setminus 0)^A$ rather than the class $[s = \partial s / \partial x. = 0]_j$ itself. Let w_{A_\bullet} be the weight of the cone C_{A_\bullet} in the characteristic class $[s = \partial s / \partial x. = 0]_j$, $\sum_i (|A_i| - 1) = j$. Then, by Proposition 2.6 and Corollary 3.15.1, we have the following.

PROPOSITION 3.17. *The tropical fan $\pi_*[s = \partial s / \partial x. = 0]_j$ consists of the cones $\pi(C_{A_\bullet})$ with weights $w_{A_\bullet} \cdot i(A_\bullet)$, where A_\bullet runs over all essential sequences such that $\sum_i (|A_i| - 1) = j$ and $i(A_\bullet) > 0$.*

Together with Corollary 3.15.2–4, this completely describes $\pi_*[s = \partial s/\partial x. = 0]_j$ for $j \leq n+2$. In particular, since the closure of the image of the complete intersection $s = \partial s/\partial x. = 0$ under the projection $\pi : (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^A \rightarrow (\mathbb{C} \setminus 0)^A$ is the A -discriminant, this gives new information about tropical fans of A -discriminants. Recall that a finite set in \mathbb{Z}^n is called an **iterated circuit**, if it can be decomposed into the union of an essential sequence of non-empty non-intersecting subsets A_k , such that $\sum_i (|A_i| - 1) = n + 1$ and $\dim \sum_i A_i = n$. A more explicit version of this definition is given in the introduction and in [E10], where Part 2 of the following statement was conjectured.

COROLLARY 3.18. 1) *The dual fan of the Newton polytope of the A -discriminant equals $\pi_*[s = \partial s/\partial x. = 0]_{n+1}$.*

2) *The A -toric variety is not dual defect if and only if A contains an iterated circuit.*

PROOF. Part 1 follows from Remark 2.9. The “if” direction of Part 2 is proved in [E10], so it remains to prove the “only if” direction: if the A -toric variety is not dual defect, then the A -discriminant is a non-empty hypersurface, then the dual fan of its Newton polytope is not empty, then $[s = \partial s/\partial x. = 0]_{n+1} \neq 0$ according to Part 1, then, by Corollary 3.15(1), there exists an essential sequence of subsets $A_k \subset A$ such that $\sum_i (|A_i| - 1) = n + 1$. Its union is an iterated circuit. \square

Actually the “if” direction of Part 2 also follows from Corollary 3.15(3) and Proposition 3.17, because, for every sequence A_\bullet , such that $\dim \sum_i A_i = n$, we have $i(A_\bullet) > 0$.

3.4 Tropical fan of the $(\times \times)$ and (\prec) strata: the assumptions

We now compute tropical fans of the codimension 2 multisingularity strata. Let A be a finite set in the character lattice L of the complex torus $(\mathbb{C} \setminus 0)^n$.

Let $\{\times\}$, $\{\times \times\}$ and $\{\prec\}$ be the sets of all $f \in \mathbb{C}^A$, such that the closure of the hypersurface $\{f = 0\}$ in the smooth toric variety X_A has one \mathcal{A}_1 singularity, two \mathcal{A}_1 singularities, and one \mathcal{A}_2 singularity respectively. Denote the tropical characteristic classes of these strata by $[\times]$, $[\times \times]$ and $[\prec]$. Note that, unlike in the classical multisingularity theory, we do not switch to the closures of the sets $\{\times\}$, $\{\times \times\}$ and $\{\prec\}$: these sets are smooth (which is not important for us), but not closed! Passing to the closure would not change their tropical fans (that is, their highest characteristic class), but would affect the other characteristic classes.

Theorems 3.23, 3.24 and 3.28 below provide three independent linear equations on the tropical fans $[\times]_2$, $[\times, \times]_2$ and $[\prec]_2$. We can solve this linear system and compute $[\times \times]_2$ and $[\prec]_2$, the desired tropical fans of the $\{\times \times\}$ and $\{\prec\}$ multisingularity strata. In Section 3.9, we give an example of the explicit computation of $[\times]_2$, $[\times \times]_2$ and $[\prec]_2$ for a particular set A (i.e. we list all cones and their weights in these tropical fans).

We shall obtain these results under certain assumptions on A , generalizing non-dual-defectiveness.

DEFINITION 3.19. For a Laurent polynomial $g \in \mathbb{C}[L] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, let $I_g \subset \mathbb{C}[L]$ be the ideal generated by g and its partial derivatives $\partial g/\partial x_i$. The set A is said to be versal in codimension k , if the set of all points $f \in \mathbb{C}^A$, such that the natural map $\mathbb{C}^A \rightarrow \mathbb{C}[L]/I_f$ is not of the full rank at f , has codimension $> k$.

Informally, this means that, for each tuple of singularities S_1, \dots, S_m , whose Tyurina numbers sum up to $M \leq k$, the (S_1, \dots, S_m) -multisingularity stratum has codimension

M and the same adjacencies as in the product of the versal deformations of S_1, \dots, S_m (c.f. the notions of T-smoothness and deformation completeness of families of curves, see e.g. [GLSh]). For codimension 2, we can restate this definition more explicitly:

DEFINITION 3.20. The set A is versal in codimension 2, if there exists a codimension 3 set $\Sigma \subset \mathbb{C}^A$ such that every $f \in \mathbb{C}^A \setminus \Sigma$ is among the following three types:

1) f is not in the image of the projection $\{s = \partial s / \partial x. = 0\} \rightarrow (\mathbb{C} \setminus 0)^A$. In this case, the hypersurface $f = 0$ has no singularities.

2) f is a regular value of the projection $\{s = \partial s / \partial x. = 0\} \rightarrow (\mathbb{C} \setminus 0)^A$ with one or two preimages $(z_{(i)}, f)$. In this case, $z_{(i)}$ are the only singular points of the hypersurface $f = 0$, and both of them are \mathcal{A}_1 . Moreover, the tautological hypersurface $s = 0$ is given by the equation $z_{(i)1}^2 + \dots + z_{(i)n}^2 = y_i$ for suitable local coordinates $(y_1, \dots, y_{|A|})$ near $f \in \mathbb{C}^A$ and $(z_{(i)1}, \dots, z_{(i)n}, y_1, \dots, y_{|A|})$ near $(z_{(i)}, f)$.

3) f is a critical value of the projection $\{s = \partial s / \partial x. = 0\} \rightarrow (\mathbb{C} \setminus 0)^A$ with one preimage (z, f) . In this case, z is the only singular point of the hypersurface $f = 0$, and it is \mathcal{A}_2 . Moreover, the tautological hypersurface $s = 0$ is given by the equation $z_1^3 + y_1 z_1 + z_2^2 \dots + z_n^2 = y_2$ for suitable local coordinates $(y_1, \dots, y_{|A|})$ near $f \in \mathbb{C}^A$ and $(z_1, \dots, z_n, y_1, \dots, y_{|A|})$ near (z, f) .

In what follows, we always assume A to be versal in codimension 2. For instance, every non-dual defect set in \mathbb{Z}^2 and every set containing $4 \cdot$ (simplex of volume 1) is versal.

REMARK 3.21. It would be interesting to obtain more accurate sufficient conditions of versality in a small codimension and prove its monotonicity similarly to the monotonicity of non-dual defectiveness (if a subset of $A \subset L$ is not dual defect and is not contained in a proper affine sublattice of L , then A is not dual defect as well, see [E10]). One obvious but weak version of monotonicity is as follows: $A \subset L$ will be called *k-strongly versal*, if it is versal, and every polynomial $g \in \mathbb{C}[L]$, such that $\dim_{\mathbb{C}} \mathbb{C}[L]/I_g \leq k$, admits $f \in \mathbb{C}^A$, such that $I_g = I_f$. If a *k-strongly versal* A is not contained in a proper affine sublattice of L and is contained in B , then B is $(k - 1)$ -strongly versal.

In what follows, we also always assume that $A \subset L$ is simple in edges ([T01]), i.e. A is the set of lattice points in a lattice polytope, whose every edge is contained in $n - 1$ facets, and the $n - 1$ external normal covectors to these facets form a part of a basis in $(\mathbb{Z}^n)^*$. Geometrically, this means that the A -toric variety X_A has at most isolated singularities. This assumption is similar to versality of A in codimension 2 with respect to singularities at infinity, see the proposition below.

For every $B \subset \mathbb{Z}^n$ and every $f = \sum_{a \in A} c_a x^a \in (\mathbb{C} \setminus 0)^A$, let f^B be the polynomial $\sum_{a \in B} c_a x^a$, and denote $s^B(x, f) = s(x, f^B)$. If B is a face of the convex hull of A , then let $T_B \subset (\mathbb{C} \setminus 0)^n$ be the subtorus given by the equations $x^a = x^b$ for all a and $b \in B$, so that the B -orbit of the toric variety X_A is naturally identified with $(\mathbb{C} \setminus 0)^n / T_B$. Similarly, the intersection of $(B\text{-orbit}) \times (\mathbb{C} \setminus 0)^A$ with the closure of $\{s^B = 0\}$ is naturally identified with $\{s^B = 0\} / T_B$. For every facet Γ of the convex hull of A , let Γ' be the maximal face of the convex hull of $A \setminus \Gamma$, parallel to Γ and “looking in the same direction” (so that $\Gamma' + \Gamma$ is a face of $(A \setminus \Gamma) + A$).

PROPOSITION 3.22. *Assume that A is simple in edges. Then one can choose a codimension 3 set $\Sigma \subset \mathbb{C}^A$ such that every $f \in \mathbb{C}^A \setminus \Sigma$ is among the following two types:*

1) f is in the image of the projection $\{s^\Gamma = \partial s^\Gamma / \partial x = s^{\Gamma'} = 0\} / T_\Gamma \rightarrow (\mathbb{C} \setminus 0)^A$ for some facet Γ of the convex hull of A . In this case, the unique preimage of f is a point (z, f) , where $z \in (\Gamma\text{-orbit of } X_A)$ is the unique singular point of the closure of $f = 0$, and this singularity is \mathcal{A}_1 . Moreover, $(\Gamma\text{-orbit}) \times (\mathbb{C} \setminus 0)^A$ and the closure of the hypersurface $s = 0$ are given by the equations $z_1 = 0$ and $(z_1 - y_1)^2 + z_2^2 + \dots + z_n^2 = y_2$ respectively for suitable local coordinates $(y_1, \dots, y_{|A|})$ near $f \in \mathbb{C}^A$ and $(z_1, \dots, z_n, y_1, \dots, y_{|A|})$ near (z, f) .

2) Otherwise, the closure of $f = 0$ in the toric variety X_A is smooth at every point outside $(\mathbb{C} \setminus 0)^n$.

The proof is based on the same arguments as Proposition 2.46 (note that the lattice distance between the parallel hyperplanes, containing Γ and Γ' , equals 1, because A is simple in edges).

3.5 Tropical fan of the $(\times \times)$ and (\prec) strata: the answer

Recall that we assume that $A \subset \mathbb{Z}^n$ is simple in edges and versal in codimension 2 (see Section 3.4).

THE FIRST EQUATION. Consider the equality $(*)$ of Section 3.1 in codimension 2 for the projection $\pi : X_A \times (\mathbb{C} \setminus 0)^A \rightarrow (\mathbb{C} \setminus 0)^A$, restricted to the closure of the tautological hypersurface $\{s = 0\} \subset X_A \times (\mathbb{C} \setminus 0)^A$, $s(x, f) = f(x)$. If A is versal in codimension 2 and simple in edges, then, by Proposition 3.22, it will reduce to the following.

THEOREM 3.23. *If A is versal in codimension 2 and simple in edges, then*

$$[\times]_2 + 2[\times \times]_2 + 2[\prec]_2 = \sum_{\Gamma} \pi_*[\{s^\Gamma = 0\} / T_\Gamma]_{\dim \Gamma + 2},$$

where Γ runs over all positive-dimensional faces of the convex hull of A .

Here $[\{s^\Gamma = 0\} / T_\Gamma]_j$ for all j can be expressed in terms of the Newton polytope of s^Γ , see Example 2.8. So the only unknown terms are on the left hand side.

THE SECOND EQUATION. Consider the equality $(*)$ of Section 3.1 in codimension 2 for the projection $\pi : X_A \times (\mathbb{C} \setminus 0)^A \rightarrow (\mathbb{C} \setminus 0)^A$, restricted to the first Boardman stratum $\{s = \frac{\partial s}{\partial x} = 0\}$. If A is versal in codimension 2 and is simple in edges, then, by Proposition 3.22, it will reduce to the following.

THEOREM 3.24. *If A is versal in codimension 2 and simple in edges, then*

$$[\times]_2 + 2[\times \times]_2 + [\prec]_2 = \pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+2} + \sum_{\Gamma} \pi_*[\{s^\Gamma = s^{\Gamma'} = \frac{\partial s^\Gamma}{\partial x} = 0\} / T_\Gamma]_{n+1},$$

where Γ runs over all facets of the convex hull of A .

Here $[\{s^\Gamma = s^{\Gamma'} = \frac{\partial s^\Gamma}{\partial x} = 0\}]_{n+1} = [\{s^\Gamma = \frac{\partial s^\Gamma}{\partial x} = 0\}]_n \cdot [s^{\Gamma'} = 0]_1$ and $[s = \frac{\partial s}{\partial x} = 0]_{n+2}$ are computed by Corollary 3.15. So the only unknown terms are on the left hand side.

REMARK 3.25. The equalities from Theorems 3.23 and 3.24 are already enough to compute the tropical fan of the singularity stratum $\{\prec\}$. In the same way as we deduced Theorems 3.23 and 3.24, we can also compute the left hand side of the equality (*) of Section 3.1 for all higher Boardman strata of the projection $\{s = 0\} \rightarrow \mathbb{C}^A$. However, for the (1,1)-Boardman stratum we shall obtain an equation on $[\prec]_2$ that is linearly dependent with Theorems 3.23 and 3.24, and for higher Boardman strata the equations will be trivial in codimension 2. So the equality (*) alone is not enough to find the tropical fan even for the simplest multisingularity stratum $\{\times\times\}$.

THE THIRD EQUATION. To formulate it, we define a certain continuous piecewise-linear function \mathbf{l}_A on the support set of the tropical fan $[s = \frac{\partial s}{\partial x} = 0]_{n+1}$. Its definition relies upon the notions and notation from Section 3.3.

Consider $\gamma \in (\mathbb{T} \setminus \mathfrak{o})^n \oplus (\mathbb{T} \setminus \mathfrak{o})^A$ such that $i(A_\bullet^\gamma) > 0$. Recall that we consider its components $\gamma' \in (\mathbb{T} \setminus \mathfrak{o})^n$ and $\gamma'' \in (\mathbb{T} \setminus \mathfrak{o})^A$, $\gamma = \gamma' + \gamma''$, as functions on A : the function γ' is the restriction of the function $\gamma' : \mathbb{R}^n \rightarrow \mathbb{Z}$ to $A \subset \mathbb{Z}^n$, and the value of γ'' at $a \in A$ is the coefficient of the degree a monomial in the tropical polynomial $\gamma'' \in (\mathbb{T} \setminus \mathfrak{o})^A$. For every k , denote the value of the function $\tilde{\gamma} = \gamma'' - \gamma' : A \rightarrow \mathbb{Z}$ at A_k^γ by $\tilde{\gamma}_k$, and define $A_k^\gamma(r)$ as the set of all $a \in A$ such that $\tilde{\gamma}(a) \in [\tilde{\gamma}_k, \tilde{\gamma}_k + r]$. Define the function

$$i_\gamma(r) = |\mathbb{Z}^n / \bigcup_i \{a - b \mid a \text{ and } b \in A_k^\gamma(r)\}|$$

on \mathbb{R} and the function

$$\mathbf{l}_A(\gamma) = \int_0^{+\infty} (i_\gamma(r) - 1) dr$$

on the set $\{\gamma \mid i(A_\bullet^\gamma) > 0\}$.

The function $\mathbf{l}_A : \{\gamma \mid i(A_\bullet^\gamma) > 0\} \rightarrow \mathbb{R}$ is well-defined, because $i(A_\bullet^\gamma) > 0$ implies $i_\gamma(r) \leq i_\gamma(0) < \infty$, and the assumption that A affinely generates \mathbb{Z}^n implies $i_\gamma(r) = 1$ for r large enough. Moreover, the function $\mathbf{l}_A : \{\gamma \mid i(A_\bullet^\gamma) > 0\} \rightarrow \mathbb{R}$ is continuous, because, for γ_1 in a small neighborhood of γ , the functions i_γ and i_{γ_1} coincide outside a small neighborhood of the discontinuities of i_γ .

Let $\tilde{\mathbf{l}}_A$ be any continuous extension of the function \mathbf{l}_A to the support set of the fan $[s = \frac{\partial s}{\partial x} = 0]_{n+1}$, then the product $\tilde{\mathbf{l}}_A \cdot [s = \frac{\partial s}{\partial x} = 0]_{n+1}$ is an element of the component $\mathcal{K}_{n+1}^1(\mathbb{Z}^n \oplus \mathbb{Z}^A)$ of the differential ring of tropical fans (see Section 2.2). Since its image $\pi_*(\tilde{\mathbf{l}}_A \cdot [s = \frac{\partial s}{\partial x} = 0]_{n+1})$ under the projection $\pi : (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^A \rightarrow (\mathbb{C} \setminus 0)^A$ does not depend on the choice of an extension $\tilde{\mathbf{l}}_A$ by Proposition 3.17, we shall denote it by

$$\pi_*(\mathbf{l}_A \cdot [s = \frac{\partial s}{\partial x} = 0]_{n+1}) \in \mathcal{K}_1^1(\mathbb{Z}^A).$$

DEFINITION 3.26. The corner locus

$$\delta\pi_*(\mathbf{l}_A \cdot [s = \frac{\partial s}{\partial x} = 0]_{n+1}) \in \mathcal{K}_2^0(\mathbb{Z}^A)$$

is a codimension 2 tropical fan in $(\mathbb{T} \setminus \mathfrak{o})^A$ and will be called the **ternary fan** of A (by analogy with the codimension 1 **secondary fan** introduced in [GKZ94] and containing $\pi_*([s = \frac{\partial s}{\partial x} = 0]_{n+1}) \in \mathcal{K}_1^0(\mathbb{Z}^A)$ as a summand).

REMARK 3.27. Note that the corner locus of the product $\mathbf{f}F \in \mathcal{K}_1^1$ of a continuous piecewise-linear function $\mathbf{f} \in \mathcal{K}_0^1$ and a tropical fan $F \in \mathcal{K}_1^0$ is known as the Cartier divisor of \mathbf{f} on F (see e. g. [M06]), and the geometric meaning of $\delta\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1})$ is similar. However, the element $\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1}) \in \mathcal{K}_1^1(\mathbb{Z}^A)$ that we have to differentiate cannot be decomposed into the product of a continuous piecewise linear function from $\mathcal{K}_0^1(\mathbb{Z}^A)$ and a tropical fan with constant weights from $\mathcal{K}_1^0(\mathbb{Z}^A)$.

THEOREM 3.28. *If A is versal in codimension 2 and simple in edges, then*

$$[\times]_2 - [\prec]_2 = \delta\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1}) - (\pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+1})^2.$$

See Section 3.8 for a proof. The only unknown classes in this theorem are on the left hand side, and, together with Theorems 3.23 and 3.24, we obtain three independent equations on $[\times]_2$, $[\times, \times]_2$ and $[\prec]_2$.

REMARK 3.29. The geometric meaning of the piecewise linear function $\pi_*(\mathbf{1}_A \cdot [s = \frac{\partial s}{\partial x} = 0]_{n+1})$ is clear: it is the tropicalization of the function on the A -discriminant $D = \pi_*\{s = \frac{\partial s}{\partial x} = 0\}$, assigning the determinant of the Jacobian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(z)\right)_{i,j=1,\dots,n}$ to every $f \in D$ with a unique point $z \in (\mathbb{C} \setminus 0)^n$ such that $f(z) = 0$ and $df(z) = 0$. However, I do not know how to prove Theorem 3.28 using this interpretation.

REMARK 3.30. The cones of the fan $(\pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+1})^2$ correspond to the two-dimensional faces of the polytope, dual to the fan $\pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+1}$. Multiplicities of the cones equal the areas of the corresponding faces. Every such face is either a parallelogram (whose area is easy to compute) or the secondary polygon of a certain subset of A (whose area can be conveniently computed by the formula from Example 3.39 below).

We now conjecture how the tropical fan of the $\underbrace{(\times, \dots, \times)}_m$ -stratum might look like for arbitrary m .

DEFINITION 3.31. The tropical fan $\delta^k\pi_*(\mathbf{1}^k[s = \frac{\partial s}{\partial x} = 0]_{n+1})$ is called the k -ary fan of A and is denoted by A^k .

For every linear $\gamma : \mathbb{Z}^n \rightarrow \mathbb{Z}$, define $A^\gamma = A_1^\gamma$ as the set of all point of A where γ attains its maximum, and then inductively $A_{k+1}^\gamma = (A \setminus \bigcup_{i=1}^k A_i^\gamma)^\gamma$.

CONJECTURE 3.32. *There exists a universal function P_m of a collection of disjoint sets $B_1, \dots, B_m \subset \mathbb{Z}^n$, taking values in $\mathcal{K}_m^0(\mathbb{Z}^{\bigcup_i B_i})$, such that*

1) *For given $\dim B_1, \dots, \dim B_m$, the function $P_m(B_1, \dots, B_m)$ is a polynomial of the k -ary fans $(B_i)^k$ and the characteristic classes $\pi_*[s_i = \frac{\partial s_i}{\partial x} = 0]_j$ for the tautological polynomials $s_i = \sum_{b \in B_i} c_b x^b$ on $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^{B_i}$;*

2) *the tropical fan of the $\underbrace{(\times, \dots, \times)}_m$ -stratum in $(\mathbb{C} \setminus 0)^A$ is the sum of $P_m(A_1^\gamma, \dots, A_m^\gamma)$ over all primitive γ (including $\gamma = 0$).*

3.6 Reduction to the universal case

We have described the tropical fans of the universal multisingularity strata $\{\times\times\}$ and $\{\prec\}$ in $(\mathbb{C}\setminus 0)^A$. We now outline how to apply this result to more down-to-earth settings, such as the multisingularity strata $\{\times\times\}$ and $\{\prec\}$ of a generic polynomial map $(\mathbb{C}\setminus 0)^m \rightarrow \mathbb{C}^n$. The details will be given in a separate paper.

NON-MORSE POLYNOMIALS. Let $A \not\cong 0$ be a finite set, generating \mathbb{Z}^n . The set of polynomials $f \in (\mathbb{C}\setminus 0)^A$ such that $f : (\mathbb{C}\setminus 0)^n \rightarrow \mathbb{C}$ has a degenerate critical point is called the *caustics*; the set of polynomials $f \in (\mathbb{C}\setminus 0)^A$ such that $f : (\mathbb{C}\setminus 0)^n \rightarrow \mathbb{C}$ takes the same value at two critical points is called the *Maxwell stratum*, see [LZ04] for the study of the case $n = 1$.

Denote $A \cup \{0\}$ by A' and the natural projection $(\mathbb{C}\setminus 0)^{A'} \rightarrow (\mathbb{C}\setminus 0)^A$ by p , then the tropical fan of the caustics equals $p_*([\prec]_2)$, and the tropical fan of the Maxwell stratum equals $p_*([\times\times]_2)$, where $\{\times\times\}$ and $\{\prec\}$ are the codimension 2 multisingularity strata in $(\mathbb{C}\setminus 0)^{A'}$.

$\{\times\times\}$ AND $\{\prec\}$ FOR COMPLETE INTERSECTIONS. For a collection of finite sets $A_0, \dots, A_k \subset \mathbb{Z}^m$, let $\{\times\}$ be the set of all $(f_1, \dots, f_k) \in \mathbb{C}^{A_0} \oplus \dots \oplus \mathbb{C}^{A_k}$ such that $f_1 = \dots = f_k = 0$ is a complete intersection with a unique singular point, and its type is \mathcal{A}_1 . Similarly, define $\{\prec\}$ and $\{\times\times\} \subset \mathbb{C}^{A_0} \oplus \dots \oplus \mathbb{C}^{A_k}$ as the sets of all complete intersections with one \mathcal{A}_2 singularity and two \mathcal{A}_1 singularities respectively. Define the Cayley configuration $A_0 * \dots * A_k \subset \mathbb{Z}^n \times \mathbb{Z}^k$ as the union of $A_0 \times \{0\}$ and $A_i \times \{i\text{-th vector of the standard basis}\}$ over $i = 1, \dots, k$, and consider the natural isomorphism $p : (\mathbb{C}\setminus 0)^{A_0 * \dots * A_k} \rightarrow (\mathbb{C}\setminus 0)^{A_0} \times \dots \times (\mathbb{C}\setminus 0)^{A_k}$.

PROPOSITION 3.33. *Assume that none of the A_i is contained in an affine hyperplane, and $A_0 * \dots * A_k$ is 2-versal. Then we have*

$$[\times]_1 = p_*[\times]_1, [\prec]_2 = p_*[\prec]_2 \text{ and } [\times\times]_2 = p_*[\times\times]_2.$$

Proof. Denoting the closure of the highest dimension component of a constructible set S by S_0 , the statement of the proposition follows from the equalities

$$\{\times\}_0 = p(\{\times\}_0), \{\prec\}_0 = p(\{\prec\}_0), \{\times\times\}_0 = p(\{\times\times\}_0).$$

For the discriminant $\{\times\}_0$, this equality is proved in [E10] and [CCdRDS13], and for the codimension 2 strata the proof follows the same lines. \square

Note that however the inclusion $p(\{\times\}) \subset \{\times\}$ is proper, and thus $[\times]_2 \neq p_*[\times]_2$ in general.

$\{\times\times\}$ AND $\{\prec\}$ FOR PROJECTIONS OF COMPLETE INTERSECTIONS. Let B_0, \dots, B_k be finite sets in $\mathbb{Z}^n \oplus \mathbb{Z}^m$, $g_i \in (\mathbb{C}\setminus 0)^{B_i}$ be generic polynomials, and π be the restriction of the projection $(\mathbb{C}\setminus 0)^n \times (\mathbb{C}\setminus 0)^m \rightarrow (\mathbb{C}\setminus 0)^m$ to the complete intersection $g_0 = \dots = g_k = 0$. We shall compute the tropical fans of the multisingularity strata $\{\times\times\}_\pi$ and $\{\prec\}_\pi \subset (\mathbb{C}\setminus 0)^m$ of π .

For this, denote the image of B_i under the projection $\mathbb{Z}^n \oplus \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ by A_i , and let x and y be the standard coordinates on $(\mathbb{C}\setminus 0)^n$ and $(\mathbb{C}\setminus 0)^m$ respectively. In the product $(\mathbb{C}\setminus 0)^{A_0} \times \dots \times (\mathbb{C}\setminus 0)^{A_k} \times (\mathbb{C}\setminus 0)^m$, consider the sets $G_i = \{(f_0, \dots, f_k, y) \mid \forall x f_i(x) = g_i(x, y)\}$ and the projections p to $(\mathbb{C}\setminus 0)^m$ and q to $(\mathbb{C}\setminus 0)^{A_0} \times \dots \times (\mathbb{C}\setminus 0)^{A_k}$. The tropical fans of $\{\times\times\}_\pi$ and $\{\prec\}_\pi$ equal

$$p_*([G_0]_{|A_0|} \cdot \dots \cdot [G_k]_{|A_k|} \cdot q^*[\times\times]_2) \text{ and } p_*([G_0]_{|A_0|} \cdot \dots \cdot [G_k]_{|A_k|} \cdot q^*[\prec]_2). \quad (*)$$

In this formula, the tropical fan $[G_i]_{|A_i|}$ of the set G_i is known for generic $g_i \in (\mathbb{C} \setminus 0)^{B_i}$: the set G_i is a non-degenerate complete intersection, so its tropical fan is the product of the dual fans of the Newton polytopes of its equations. Computation of the tropical fans $[\times \times]_2$ and $[\prec]_2$ on the right hand side is reduced to Theorems 3.23, 3.24 and 3.28 by Proposition 3.33.

REMARK 3.34. This result covers the setting of Example 1.1, but reducing the general answer (*) to the elementary one given in Example 1.1 is a non-trivial combinatorial problem, which will be treated in a subsequent paper. On the other hand, the answer (*) remains valid for generic hypersurfaces $M \subset (\mathbb{C} \setminus 0)^3$ with arbitrary Newton polytopes.

$\{\times \times\}$ AND $\{\prec\}$ FOR MAPS OF COMPLETE INTERSECTIONS. Let $A_0, \dots, A_p, D_1, \dots, D_m$ be finite sets in \mathbb{Z}^n , $f_i \in (\mathbb{C} \setminus 0)^{A_i}$ and $h_i \in (\mathbb{C} \setminus 0)^{D_i}$ be generic polynomials, and H be the restriction of the map $(h_1, \dots, h_m) : (\mathbb{C} \setminus 0)^n \rightarrow \mathbb{C}^m$ to the complete intersection $f_0 = \dots = f_k = 0$. We shall compute the tropical fans of the multisingularity strata $\{\times \times\}_H$ and $\{\prec\}_H \subset (\mathbb{C} \setminus 0)^m$ of H . For this, note that $\{\times \times\}_H$ and $\{\prec\}_H$ equal $\{\times \times\}_\pi$ and $\{\prec\}_\pi$, where π is the restriction of the projection $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^m \rightarrow (\mathbb{C} \setminus 0)^m$ to the complete intersection $f_0 = \dots = f_p = h_1 - y_1 = \dots = h_m - y_m = 0$, and (y_1, \dots, y_m) are the standard coordinates on $(\mathbb{C} \setminus 0)^m$. The tropical fans of $\{\times \times\}_\pi$ and $\{\prec\}_\pi$ can now be computed as in (*).

REMARK 3.35. If we try to apply Proposition 3.33 to computation of the tropical fans $[\times \times]_2$ and $[\prec]_2$ on the right hand side of (*) in this setting, we observe that the assumption of Proposition 3.33 is not satisfied. Nevertheless one can easily verify that the conclusion of Proposition 3.33 remains valid in this particular setting.

3.7 Volume of a fiber body

Choose a codimension k subspace $L \subset \mathbb{R}^n$. There exists a unique additive symmetric function M_L of $k+1$ convex bodies in \mathbb{R}^n , taking values in convex bodies in L , such that $M_L(\underbrace{A, \dots, A}_{k+1})$ is the fiber body of A for every convex body $A \subset \mathbb{R}^n$. See Example 1.3 or [BS92] for the definition of the fiber body, and [M04] or [E08] for a proof of the fact.

DEFINITION 3.36. The convex body $M_L(A_0, \dots, A_k)$ is called the **mixed fiber body** of A_0, \dots, A_k .

See [EKh08] for a relation of mixed fiber polytopes to algebraic geometry and the following characterization.

PROPOSITION 3.37. *The mixed fiber body of A_0, \dots, A_k is the unique convex body X such that the Euclidean mixed volume of $A_0, \dots, A_k, B_1, \dots, B_{n-k-1}$ in \mathbb{R}^n equals the Euclidean mixed volume of $M_L(A_0, \dots, A_k), B_1, \dots, B_{n-k-1}$ in L .*

PROPOSITION 3.38. *Let L be a codimension k subspace in \mathbb{R}^n , and $A_i^j \subset \mathbb{R}^n$, $i = 0, \dots, k$, $j = 1, \dots, n-k$, be convex bodies. Denote the space $\{(v_1, \dots, v_{n-k} \mid v_1 + \dots + v_{n-k} = 0\} \subset L^{n-k}$ by K and the image of*

$$\underbrace{\{0\} \times \dots \times \{0\}}_{i-1} \times A_i^j \times \{0\} \times \dots \times \{0\} \subset (\mathbb{R}^n)^{n-k}$$

under the projection $(\mathbb{R}^n)^{n-k} \rightarrow (\mathbb{R}^n)^{n-k}/K$ by \tilde{A}_j^i . Then the mixed volume of the mixed fiber bodies $M_L(A_0^i, \dots, A_k^i)$, $i = 1, \dots, n-k$ equals the mixed volume of \tilde{A}_i^j , $i = 0, \dots, k$, $j = 1, \dots, n-k$.

The proof is by $n-k-1$ applications of the equality from Proposition 3.37.

An important special case of a fiber body is a secondary polytope ([GKZ94]), and Proposition 3.38 gives a formula for its volume (see e.g. [Or99] for one motivation to study volumes of secondary polytopes). We shall need the following special case.

EXAMPLE 3.39. Let $A \subset \mathbb{R}^n$ be a lattice polytope. For every its vertex v , let c_A^v be the volume of the difference of A and the convex hull of $A \cap \mathbb{Z}^n \setminus \{v\}$. If the secondary polytope of A is 2-dimensional, then its lattice area equals $(n+1)! \text{Vol } A - n! \sum_v c_A^v$.

Another important special case of a mixed fiber body is the Newton polytope of a sparse resultant ([EKh08]), and Proposition 3.38 gives a positive formula for its volume. This may be already of interest in the case of the Newton polytope of the determinant, that is, the Birkhoff polytope (see e.g. [LLY09] for the study of its volume).

3.8 Proof of Third equation

Let $i = (i_1, i_2, \dots)$ be a sequence of integer numbers stabilizing at 1, such that $i_r | i_{r-1}$ for every r .

DEFINITION 3.40. An ***i -forking paths singularity*** is a plane singularity with i_1 distinct regular branches $\varphi_{q_1, q_2, \dots} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$, $q_r = 1, \dots, i_r / i_{r+1}$, such that the intersection number of $\varphi_{p_1, p_2, \dots}$ and $\varphi_{q_1, q_2, \dots}$ at 0 equals the minimal number r such that $p_r \neq q_r$.

LEMMA 3.41. *The Milnor number of an i -forking paths singularity equals $\mu(i) = i_1 \sum_r (i_r - 1) - (i_1 - 1)$.*

PROOF. Perturb the branches of the singularity independently, then the union U of the perturbations has exactly $N = \sum_r \frac{i_1 i_r (i_r - 1)}{i_r}$ nodes. The Euler characteristic of the normalization of U equals i_1 and differs by $2N$ from the desired Euler characteristic of the Milnor fiber. \square

PROOF OF THEOREM 3.28. We can now prove the following equality, equivalent to the statement of Theorem 3.28. Note that the tropical fan $\pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+1}$ is the dual fan of the Newton polytope \mathcal{N}_A of the Gelfand-Kapranov-Zelevinsky A -discriminant D_A .

PROPOSITION 3.42. *For a generic lattice polytope $B \subset \mathbb{R}^A$ (that is, every polytope outside finitely many hypersurfaces in the cone of polytopes), we have*

$$\begin{aligned} & -[\mathcal{N}_A]^2[B]^{|A|-2} - (|A| - 2)[\mathcal{N}_A][B]^{|A|-1} = \\ & = [\times]_2[B]^{|A|-2} - (|A| - 2)[\times]_1[B]^{|A|-1} - [\prec]_2[B]^{|A|-2} - \delta\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1})[B]^{|A|-2}. \end{aligned}$$

This is equivalent to Theorem 3.28, because the second terms on both sides mutually eliminate: $[\mathcal{N}_A] = [\times]_1$, and then the multiplier $[B]^{|A|-2}$ can be cancelled by the nondegeneracy of the intersection number pairing in the ring of tropical fans $\mathcal{K}^0(\mathbb{Z}^A)$.

PROOF. For $M \in \mathbb{Z}$, consider the map $\mathcal{M} : \mathbb{C}^A \rightarrow \mathbb{C}^A$, $\mathcal{M}(\sum_a c_a x^a) = \sum_a c_a^{M!} x^a$, rising the standard coordinates to a large power. Choose generic polynomials $h_1, \dots, h_{|A|-2} \in \mathbb{C}^{\mathbb{Z}^A \cap B}$, generic $g \in \mathbb{C}^{\mathbb{Z}^A \cap M\mathcal{N}_A}$ close to $D_A \circ \mathcal{M} \in \mathbb{C}^{\mathbb{Z}^A \cap M\mathcal{N}_A}$, a smooth toric compactification \mathcal{X} of \mathbb{C}^A , compatible with the polytopes B and \mathcal{N}_A , and denote its big torus by T . Denote the closure of $h_1 = \dots = h_{|A|-2} = 0$, $h_1 = \dots = h_{|A|-2} = g = 0$ and $h_1 = \dots = h_{|A|-2} = D_A \circ \mathcal{M} = 0$ in \mathcal{X} by V , C and \tilde{C} . The smooth curve \tilde{C} is a smoothening of the curve C in the smooth surface V , so we have

$$e(T \cap \tilde{C}) = e(\text{smooth part of } C) + \sum_{p \in \text{sing } C} e(T \cap (\text{Milnor fiber of } C \text{ at } p))$$

where all points of C outside the big torus T are considered as singular.

The curve C has the following singularities:

1) \mathcal{A}_2 -singularities appear at the points of the intersection of V and the singularity stratum $\mathcal{M}^{-1}\{\prec\}$ (because the discriminant of the versal deformation of an \mathcal{A}_2 -singularity is itself a \mathcal{A}_2 -singularity).

2) \mathcal{A}_1 -singularities appear at the points of the intersection of V and the multisingularity stratum $\mathcal{M}^{-1}\{\times \times\}$.

3) For large M , all singularities of C outside the big torus of \mathcal{X} are forking paths singularities (this is not always true for $M = 1$).

Thus, the previous formula can be rewritten more precisely as follows:

$$\begin{aligned} e(T \cap \tilde{C}) &= e(\text{smooth part of } C) - (\text{the number of } \mathcal{A}_2\text{-singularities of } C) + \\ &+ \sum_{p \in T \cap \text{sing } C} e(T \cap (\text{Milnor fiber of } C \text{ at } p)). \end{aligned} \quad (1)$$

As we evaluate the terms of this equality, they turn out to be equal to the respective terms of the desired equality in the statement.

By the Khovanskii formula [Kh77b], the left hand side of (1) equals

$$-M!^2 \cdot [\mathcal{N}_A]^2 [B]^{|A|-2} - M! \cdot (|A| - 2) [\mathcal{N}_A] [B]^{|A|-1}.$$

For the first term on the right hand side of (1), we have

$$\begin{aligned} e(\text{smooth part of } C) &= [\text{smooth part of } C]_0 = ([h_1 = 0] \cdot \dots \cdot [h_{|A|-2} = 0] \cdot [\times])_0 = \\ &= M!^2 \cdot [\times]_2 [B]^{|A|-2} - M! \cdot (|A| - 2) [\times]_1 [B]^{|A|-1}. \end{aligned}$$

For the second term on the right hand side of (1), the \mathcal{A}_2 -singularities appear at the points of the intersection of V and the singularity stratum $\mathcal{M}^{-1}\{\prec\}$, so there are $M!^2 \cdot [\prec]_2 [B]^{|A|-2}$ of them.

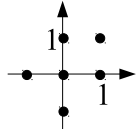
For the last term of (1), let us describe the forking paths singularities of C in more detail. For a generic lattice polytope $B \subset \mathbb{Z}^n$, the tropical fans $[B]^{|A|-2}$ and $[\mathcal{N}_A]$ intersect transversally (outside the origin), and the 1-dimensional fan $[B]^{|A|-2} [\mathcal{N}_A]$ consists of finitely many rays with weights $m_\alpha \in \mathbb{Z}$ and primitive generators $\lambda_\alpha \in (\mathbb{Z}^A)^*$. Considering $\lambda_\alpha \in (\mathbb{Z}^A)^* \cong \mathbb{Z}^A$ as a function $A \rightarrow \mathbb{Z}$, we can find a linear (but not necessarily integer!) function $\gamma'_\alpha : \mathbb{Z}^n \rightarrow \mathbb{Q}$ such that the function $\lambda_\alpha - \gamma'_\alpha : A \rightarrow \mathbb{Z}$ attains its maximum on a certain set $A_\alpha \subset A$ that is not the set of vertices of a simplex.

Moreover, for a generic B , such A_α is unique and is a circuit. Define $\gamma_\alpha = (\lambda_\alpha, \gamma'_\alpha) \in (\mathbb{Q}^A \oplus \mathbb{Q}^n)^*$. For $r \in \mathbb{Z}$, define i_r^α as $i(\frac{r-1}{M!})$, where the function $i(\cdot)$ is introduced as in Definition 3.26 for $\gamma = \gamma_\alpha$.

In this notation and under these genericity assumptions for the lattice polytope $B \subset \mathbb{R}^A$, the codimension 1 orbit of \mathcal{X} , corresponding to the covector λ_α , contains $m_\alpha M! / i_1^\alpha$ singularities of the curve C , and each of them is an $(i_1^\alpha, i_2^\alpha, \dots)$ -forking paths singularity, all of whose branches are transversal to the orbit. Other orbits of \mathcal{X} do not contain singularities of C . Counting the Euler characteristic of the Milnor fibers of the forking paths singularities by Lemma 3.41, we conclude that the last term in (1) equals $-M!^2 \cdot \delta\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1})[B]^{|A|-2}$. \square

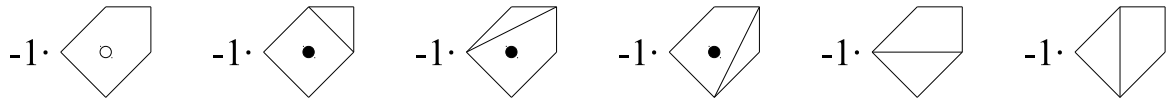
3.9 Example

We compute the tropical fans of the $\{\times\times\}$ and $\{\prec\}$ -strata in \mathbb{C}^A , with A as follows:

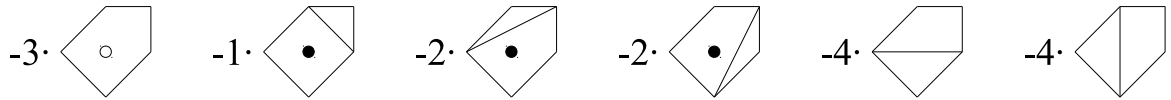


We shall describe cones in \mathbb{Q}^A that we encounter in the process of computation by drawing coherent subdivisions of the convex hull of A : a coherent subdivision will stand for the cone that consists of all $\gamma \in \mathbb{Q}^A$, such that the upper faces F of the convex hull of the graph of $\gamma : A \rightarrow \mathbb{Q}$ form the given subdivision. A point a is shown in black or white depending on whether $(\gamma(a), a)$ is in F or not, as in [GKZ94].

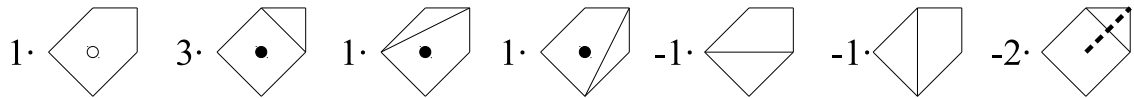
The right-hand side in Theorem 3.23 consists of the cones



The right-hand side in Theorem 3.24 consists of the cones

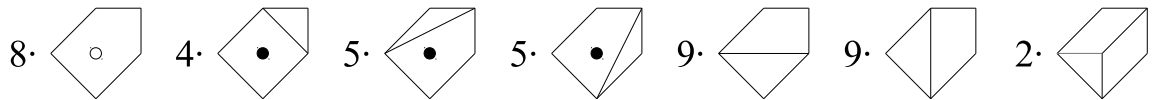


The fan $d\pi_*(\mathbf{1}[s = \frac{\partial s}{\partial x} = 0]_{n+1})$ consists of the cones

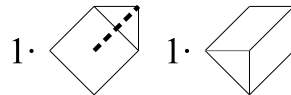


Here the last picture denotes the cone that consists of all γ such that A_0^γ is the square and A_1^γ is the dotted segment. This is the simplest example of a cone of a multisingularity stratum that cannot be represented as the set of all $\gamma \in \mathbb{Z}^A$ that induce a given regular subdivision of A . Although there is only one such cone in our example, most of cones are like this for large A . They are not among the cones of Severi varieties, whose weights are described in [Y13].

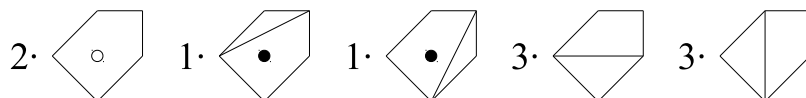
The fan $(\pi_*[s = \frac{\partial s}{\partial x} = 0]_{n+1})^2$ consists of the cones



See Remark 3.30 on how to compute it. As a result, the tropical fan of the $\{\times\times\}$ -stratum consists of the cones



and the tropical fan of the $\{\prec\}$ -stratum consists of the cones



which can be easily verified manually for this example.

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