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ASYMMETRIC EQUILIBRIA IN SECURE STRATEGIES

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We introduce a generalization of equilibrium in secure strategies, which takes into account inhomogeneous attitudes of players to security in noncooperative games. In the proposed Asymmetric Equilibrium in Secure Strategies players are divided into classes that unequally related to each other threats. Namely, a player in his behavior takes into account threats posed by players of his own or higher class, and ignores threats of lower-class players. We provide illustrative examples of these equilibria in matrix and continuous games, which have neither Nash-Cournot equilibrium no equilibrium in secure strategies. The proposed concept provides theoretical framework for studying games in which asymmetric and hierarchical logic plays an important role in the interaction of players.

Key words: Noncooperative games, Equilibrium in Secure Strategies, Asymmetric behavior, Blotto games, Product competition, Equilibrium existence

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Content

1. Introduction

A new model of cautious behavior in non-cooperative games was developed in (Iskakov 2005). A new formulation of this model was presented in (Iskakov and Iskakov 2012a). This model is based on the notion of ”threat”, which describes the situation when one player can increase his payoff and simultaneously decrease the payoff of some other player by unilateral deviation. An Equilibrium in Secure Strategies (EinSS) was defined by two conditions: (i) no player can threaten another and (ii) no player can increase payoff by a unilateral deviation without ending up worse off following a threat by some other player. This definition proved to be effective and allowed to find equilibrium positions in some well-known economic games that fail to have Nash-Cournot equilibrium: Hotelling price game with linear transportation costs; Tullock contest; the model of insurance market of Rothschild, Stiglitz and Wilson and Bertrand-Edgeworth price duopoly (Iskakov and Iskakov 2012b, 2014a). In contrast to the equilibrium in mixed strategies this approach provides an explicit solution, which can be easily interpreted in terms of cautious behavior. The proposed concept of EinSS may also be applied to the competitive stimulation systems (Novikov 2003) and may be considered as a promising area of game-theoretic modeling (Aleskerov 2013).

However, the definitions of EinSS assume that players avoid all potential threats. This assumption about cautious behavior is in fact quite strong. Under this assumption many games (for instance, product competition (Eaton and Lipsey 1975; and Shaked 1975) and Bertrand-Edgeworth price competition (1883, 1925)) posses neither Nash-Cournot equilibria nor EinSS. At the same time in these games some potential threats may prove to be unjustified or infeasible. It would be natural to assume that in these cases the players are also guided by considerations of security, but they take into account not all potential threats but only threats justified in some sense or another. In this paper we investigate an extension of EinSS through weakening the assumptions of security. We are looking for a generalization that would have a compelling interpretation and would retain the general logic of EinSS based on the concept of threat.

A natural weakening of the equilibrium concept can be obtained if we drop the first condition in the above definition of EinSS. For the first time this weakening was considered in (Iskakov and Iskakov 2012). Dropping the profile security condition in the definition of EinSS implies that there could be threats in the profile. However, in this case the condition of "profile security" is still retained in a weak form in a sense that players in the profile take into account mutual balance of "threats and counter-threats". The best known description of this behavioral pattern was proposed by Aumann and Maschler (1964) in the concept of bargaining set in cooperative games. The payoff configuration in cooperative game belongs to the bargaining set if any objection of one group of players against the other group can be answered by counter objection of the second group against the first one. Accordingly, for non-cooperative games one can introduce the notion of counter-threat in accordance with the EinSS concept. If a deviation of player creates a credible threat against him by some other player, such that following this threat the first player ends up worse off than he was before the deviation, then such threat is called *a counter-threat to the deviation* of the first player. Then a strategy profile can be called *an Equilibrium in Threats and Counter-Threats (ETCT)* if any profitable deviation of player is contained by a counter threat. This name is due to the fact that such a definition naturally satisfies the behavioral principle of "threats and counter-threats", i.e. each threat in the ETCT profile must be contained by a counter threat.

Despite the repeated attempts to formulate the behavioral principle of Aumann and Maschler in the framework of noncooperative games (for example, the strategies of threats and counter-threats of Vaisbord and Zhukovsky (1980) and the V-solutions of Vilkas (1990)) the concept of ETCT seems to be the first fully non-cooperative solution concept, which provides an analogue of the behavioral principle of "threats and counter threats" based purely on rational reflection of individual players. This concept brings a practical benefit since it actually reveals the quasi-equilibrium situations related to the players' strategic reflection, which can not be identified by Nash equilibrium and EinSS concepts. Namely, it reveals the quasi-equilibrium profiles in which all threats are effectively neutralized. Perhaps, the best known example of such quasi-equilibrium situation is the Prisoner's dilemma, when both prisoners choose a strategy of "silence". In this context the ETCT in (Iskakov and Iskakov 2012) has been called "*a threatening-proof profile*". The same equilibrium concept formalizes the principle of strategic containment of other players by counter-threats. Therefore, the ETCT has been investigated in (Iskakov and Iskakov 2014b) under the name of "*equilibrium contained by counter-threats*". The same equilibrium concept (ETCT) was independently proposed in (Sandomirskaia 2014) under the name "*Nash-2 Equilibrium*" using another conceptual interpretation from the standpoint of the theory of bounded rationality (with two steps of reasoning). It has been shown that this concept can explain the effect of tacit collusion in noncooperative games. The paper presents practical examples of such equilibria in the models of

Hotelling and Bertrand price competition. Thus, the question of a proper name for this concept of equilibrium is debatable. Each of the proposed names has its own interpretation and justification. In our opinion, the most appropriate name is "*an Equilibrium in Threats and Counter Threats (ETCT)*", since this concept defines the strategic reflection of the players according to the behavioral principle of "threats and counter-threats", first formulated by Aumann and Maschler (1964). Therefore this name expresses the intuitive essence of the concept (and we will use it in the following discussion).

The practical significance of ETCT concept is that it formalizes for non-cooperative games an intuitively obvious condition that in any quasi-equilibrium profile of the game there should not be unpunished profitable deviations. This condition must be met for any profile to be qualified as quasi-equilibrium in the generalized sense. Therefore any such generalized equilibria must be contained within the set of ETCT.

Several existence theorems are formulated and proved for ETCT. In (Sandomirskaia 2014, Theorems 3 and 4) it was proven that ETCT exists "in almost any" strictly competitive two-person game. In (Iskakov and Iskakov 2014b, Prpositions 2 and 6) it was proven that any two-person game with compact strategy sets and continuous payoff functions admits *a weak* ETCT. *A weak* ETCT here implies the weakening of ETCT concept if a strict inequality in the definition of the counter-threat is replaced by the non-strict inequality. An existence of an ϵ-analog of ETCT was proven for discontinuous two-person games, in which the payoff functions may be discontinuous (Propositions 2 and 8). Therefore, the undoubted advantage of ETCT is that these equilibria exist in many games. But the main disadvantage is the other side of this benefit. There are too many solutions of this kind and there is a problem of selection of the most stable profiles among ETCT.

Our aim is to propose in this paper such an equilibrium concept which is narrower than ETCT but wider than the EinSS. Since EinSS eliminates all threats at the profile and ETCT allows for all of them, a selective approach to threats implies an intermediate situation. The logic of this selection must be determined by the specifics of the model. One way to take into account this specifics is to assume that players are different in their attitudes to security: some of them are more cautious and some are more risky. In particular, we assume that players are divided into classes that unequally related to each other threats. Namely, a player in his behavior takes into account threats posed by players of his own or higher class, and ignores threats of lower-class players. This approach was suggested by Iskakov (2005, 2008) and applied to the Downs model of electoral competition. The proposed definition allowed to explore an asymmetric logic in the interaction of players. In practice, such asymmetry of players can either be determined externally (for example, by the amount of their resources, their market positions or unequal rules for different players) or arise spontaneously in the course of the game itself as a result of rational choice of roles by the players. Without going into the details which depend on the specifics of a particular model we simply assume that over the set of players the relation of desire for security is somehow introduced.

Let us illustrate how an equilibrium may occur as a result of unequal attitudes of players to security in the following matrix game, where Nash-Cournot equilibrium does not exist:

	t_1	t_2
s_1	(0,0)	(-3,2)
s_2	(-1,1)	(-2,-2)

In any profile of the game there is a threat, i.e. an unilateral deviation of a player which increases his payoff and reduces the payoff of opponent. Therefore an EinSS in the game does not exist either. Suppose, however, that s-player seek to play cautiously and t-player chooses the best response. Then they can choose as an equilibrium profile (s_2, t_1) with payoffs $(-1, 1)$. Although s-player can profitably deviate from this profile into the profile (s_1, t_1) with payoffs $(0, 0)$, he may lose more after the response deviation of the t-player from the profile (s_1, t_1) into the profile (s_1, t_2) with payoffs $(-3, 2)$. Therefore s-player may prefer to stay at (s_2, t_1) and keep his secure position. On the other hand, if t-player tries to find a secure position and s-player chooses the best response, for the same reasons they can choose profile (s_1, t_1) with payoffs $(0, 0)$ as an equilibrium point. Thus the profiles (s_1, t_1) and (s_2, t_1) can be regarded as "asymmetric" EinSS, which are realized under the assumption that players have different attitudes to their security. In this paper we provide a theoretical framework which allows us to formalize this logic.

In the next Section we give some preliminaries about EinSS from our earlier papers. In Section 3 we provide definitions of the chain EinSS, which is the most simple illustration of the asymmetric EinSS. In Section 4 we consider an important case of the two-person games and prove some interesting results. After several further examples of the chain EinSS in Section 5 we provide and discuss general definitions of the Asymmetric EinSS in Section 6. In Section 7 we consider general methods for finding such equilibria and summarize our results in the Conclusion.

2. Preliminaries

Here we briefly remind the definitions of (ordinary) EinSS from (Iskakov and Iskakov, 2012a). We consider n-person non-cooperative game in the normal form $G = \langle i \in N, s_i \in S_i, u_i \in R\rangle$. The concept of equilibria is based on the notion of threat and on the notion of secure strategy.

Definition 1. ***A threat*** *of player i against player j at strategy profile s is a deviation s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_j(s'_i, s_{-i}) < u_j(s)$.*

Definition 2. *A strategy s_i of player i is* ***a secure strategy*** *at strategy profile s if no player $j \neq i$ has a threat against player i at s. A strategy profile s is* ***a secure profile****, if all strategies are secure.*

Definition 3. ***A secure deviation*** *of player i at profile s is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ for any threat s'_j of player $j \neq i$ against player i at profile (s'_i, s_{-i}).*

Definition 4. *A secure strategy profile is* ***an Equilibrium in Secure Strategies (EinSS)*** *if no player has a secure (profitable) deviation.*

There are two conditions in the definition of EinSS. There are no threats in the profile and there are no profitable secure deviations.

A natural weakening of the EinSS concept may seem to be dropping the first condition in the definition. Therefore, we present here the definition of the corresponding concept from (Iskakov and Iskakov, 2012c)[1].

[1] In (Iskakov and Iskakov, 2012c) we called these equilibria "threatening-proof profiles".

Definition 5. *A strategy profile is* ***an Equilibrium in Threats and Counter Threats (ETCT)*** *if no player can make a secure (profitable) deviation.*

We choose this name due to the fact that there is an equivalent definition using the notion of counter threat.

Definition 6. *The deviation s'_i of player i at profile s such that $u_i(s'_i, s_{-i}) > u_i(s)$ is* ***contained by a counter threat*** *s'_j if there is a threat s'_j of player $j \neq i$ against player i at profile (s'_i, s_{-i}) such that $u_i(s'_i, s'_j, s_{-ij}) < u_i(s)$.*

A counter threat is defined only with respect to the particular profitable deviation s'_i. This implies that there can be threats s'_j against player i in the profile (s'_i, s_{-i}), which are not counter threats if $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$. Therefore the profile (s'_i, s_{-i}) is not necessarily a secure profile if there are no counter threats s'_j. Secondly, a counter threat is understood as an answer to a deviation which may or may not be a threat in the initial profile s. And finally note that a counter threat (like a threat) is directed from one particular player to another particular player rather than being directed against a strategy profile. Now we are ready to provide an equivalent formulation of ETCT:

Definition 5′. *A strategy profile is* ***an Equilibrium in Threats and Counter Threats (ETCT)*** *if any (profitable) deviation of player is contained by a counter threat.*

In particular, this definition implies that any threat in ETCT must be contained by a counter threat.

Note that the difference between EinSS and ETCT is only in the condition of profile security. However in many games the set of ETCT is very wide. It includes much narrower set of EinSS, which is sometimes empty. However, we assume, that any positions to be qualified as quasi-equilibrium in the generalized sense should be balanced in terms of mutual threats, i.e. they should belong to the set of ETCT. In this paper we shall employ the proximity of the two concepts and construct an intermediate set which is narrower than the ETCT but wider than the EinSS.

3. Chain equilibrium in secure strategies

In this section we show how to take into account an asymmetric attitude of players to mutual threats in the simplest case, when the players are strictly ordered by their relation to security. In this case we can easily specify the notion of security and secure deviation against threats of more risky players.

Definition 2′. *A strategy s_i of player i is* ***a secure strategy with respect to all players*** *$j > i$ at given strategies s_{-i} of all other players, if no player $j > i$ has a threat against player i.*

Definition 3′. *A deviation s'_i of player i at strategy profile s is* ***a [strict] secure deviation with respect to all players*** *$j > i$, if $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ $[u_i(s'_i, s'_j, s_{-ij}) > u_i(s)]$ for any threat s'_j of player $j > i$ against player i at profile (s'_i, s_{-i}).*

Now we can simply define a Chain EinSS in a way similar to EinSS.

Definition 4′. *A strategy profile is* ***a*** [***weak***] ***Chain Equilibrium in Secure Strategies (CEinSS)*** *if the players can be reindexed so that (i) each player i has a secure strategy w.r.t. all players $j > i$, and (ii) no player i has a [strict] secure deviation w.r.t. all players $j > i$.*

One can easily prove that a Chain EinSS is indeed an ETCT.

Proposition 1. *Any Chain Equilibrium in Secure Strategies is an Equilibrium in Threats and Counter Threats.*

Proof. In a Chain EinSS player i can not make a secure deviation w.r.t all players $j > i$ and therefore this player also can not make a secure deviation w.r.t. all players (since in this case the set of secure deviations w.r.t. all players is even narrower). □

Example 1. Let us reconsider the matrix game described in the Introduction in terms of these definitions and provide an intuitive interpretation. Let the first s-player be a developed (rich) country and the second t-player be a backward (poor) country. Each of these two country may choose either a peace-loving strategy or an aggressive one. The payoff matrix is:

	t_1	t_2
s_1	(0,0)	(-3,2)
s_2	(-1,1)	(-2,-2)

In the profile (s_1, t_1) mutually peaceful neutral policy provides basic zero payoffs $(0, 0)$. "Peaceful" occupation of a backward country in profile (s_2, t_1) gives her access to the capabilities of a developed country, and for a developed country it is inefficient waste of resources to the colony. Hence the payoffs are $(-1, 1)$. An aggression of a backward country against an advanced country not ready for this aggression in profile (s_1, t_2) implies big losses for an advanced country and a win for a backward country (with payoffs $(-3, 2)$ respectively). The mutual war in profile (s_2, t_2) brings great losses $(-2, -2)$ for both sides. The game admits neither Nash-Cournot equilibrium nor EinSS. But there are two Chain EinSS. Profile (s_1, t_1) is a chain EinSS with the order of players $\{t, s\}$, where a backward country behaves cautiously. Profile (s_2, t_1) is a chain EinSS with the order players $\{s, t\}$, in which developed country behaves cautiously. Obviously, there are no other configurations of Chain EinSS for two players.

4. Two-person games

A Chain EinSS has a particularly simple interpretation in the two-person games. In this case, the first player chooses secure profiles and improves its position only by secure deviations (i.e. behaves as he would behave in the EinSS). Second player chooses the best response (i.e. behaves as he would behave in the Nash equilibrium). The first player can be logically called *the leader*, and the second player can be called *the follower*. These names do not reflect, however, the order of moves as in Stackelberg model (1934), but rather behavioral attitudes of players. Nevertheless, chain EinSS in two-person games are closely linked with the concept of Stackelberg points. Let us consider their relationship. Let us consider matrix or continuous two-person games $G = \{(S_1, u_1), (S_2, u_2)\}$ with compact strategy spaces S_i and continuous payoff functions u_i. Following (D'Aspremont, Gerard-Varet 1980) for a game G we define *Stackelberg-point for player* $i \in \{1,2\}$ as a profile s^* such that

$$u_i(s^*) = \max_{s_i,\ s_{-i} \in BR_{-i}(s_i)} u_i(s_i, s_{-i}), \quad i \in \{1,2\} \tag{1}$$

where $BR_{-i}(s_i)$ is a best reply correspondence of player $(-i)$ in game G.

Proposition 2. *Stackelberg-points (1) in the two person game* G *are weak Chain Equilibria in Secure Strategies.*

Proof. Let us consider a Stackelberg point $s^* = (s_1^*, s_2^*)$ such that satisfy (1) with $i = 1$, i.e. $u_1(s^*) = \max_{s_1,\ s_2 \in BR_2(s_1)} u_1(s_1, s_2)$. For player 2 we have $s_2^* \in BR_2(s_1^*)$ and the definition of a weak CEinSS with the order of players $(\{1\},\{2\})$ is fulfilled for player 2. Player 1 has a secure strategy since player 2 has no possibility to increase his payoff by whatever deviation. Suppose player 1 can increase his payoff by deviating into s_1'. If $s_2^* \in BR_2(s_1')$ then according to (1) we obtain $u_1(s^*) = \max_{s_1,\ s_2 \in BR_2(s_1)} u_1(s_1, s_2) \geqslant u_1(s_1', BR_2(s_1')) = u_1(s_1', s_2^*)$, which contradicts our assumption. Therefore $s_2^* \notin BR_2(s_1')$. In this case there is a counter threat $(s_1', s_2^*) \to (s_1', BR_2(s_1'))$ of player 2 and in the result of its realization player 1 will gain $u_1(s_1', BR_2(s_1')) \leqslant u_1(s_1^*, BR_2(s_1^*))$, i.e. no more than in (s_1^*, s_2^*). Therefore the definition of a weak CEinSS with the order of players $(\{1\},\{2\})$ is fulfilled for player 1 as well. □

Corollary 1. *There is a weak Chain Equilibrium in Secure Strategies in the two-person game* G.

Proof. Follows from the fact that the best response and the maximum in (1) are always achieved on compact sets by continuous payoff functions. Therefore at least Stackelberg points (1) always exist. $\square$

However, the set of CEinSS is wider than the set of Stackelberg points. One can easily find examples of Chain EinSS, which is neither Stackelberg point nor (ordinary) EinSS.

Example 2. Let us consider the following matrix game:

	t_1	t_2	t_3
s_1	(0,0)	(1,2)	(4,1)
s_2	(3,2)	(2,0)	(-1,0)

One can easily check that both Stackelberg points coincide at profile (s_2, t_1) with payoffs $(3, 2)$. However, a profile (s_1, t_3) with payoffs $(4, 1)$ is also a Chain EinSS with the order of players $(\{t\}, \{s\})$ along with a profile (s_2, t_1). This CEinSS (s_1, t_3) is not an EinSS as there is a threat of $t-$player to deviate from (s_1, t_3) into (s_1, t_2). However, this deviation is not secure because of a counter threat of $s-$player to deviate from (s_1, t_2) into (s_2, t_2). Therefore if $t-$player as a "leader" in a Chain EinSS adopts a cautious behavior he would prefer to stay in profile (s_1, t_3) with payoffs $(4, 1)$.

An EinSS concept is of particular importance when considering games in which the payoff functions are discontinuous. In these games, the payoff functions may not attain a maximum. To obtain an existence conditions for these games it is necessary to weaken the notion of EinSS and use the concept of ϵ-equilibrium, where ϵ is understood as an arbitrarily small positive quantity. Let us now define a Chain ϵ-EinSS.

Definition 4*. *A strategy profile is* ***a Chain ϵ-Equilibrium in Secure Strategies (ϵ-CEinSS)*** *if the players can be reindexed so that (i) no player $j > i$ has a "more than ϵ" threat against player i (i.e. a threat s'_j, such that $u_i(s'_j, s_{-j}) < u_i(s) - \epsilon$), and (ii) no player i has a "more than ϵ" secure deviation w.r.t. all players $j > i$ (i.e. a deviation s'_i, such that $u_i(s'_i, s_{-i}) > u_i(s) + \epsilon$ and $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s) + \epsilon$ for any threat s'_j of player $j > i$ against player i at profile (s'_i, s_{-i})).*

Let us consider now the two-person discontinuous game $G = \{(S_1, u_1), (S_2, u_2)\}$ with discontinuous utility functions u_i. Let $BR_\epsilon(s_i, \epsilon)$ be an ϵ-best reply correspondence to s_i, i.e. the set $\{s_{-i} \mid u_{-i}(s_i, s_{-i}) > \sup\limits_{\tilde{s}_{-i}} u_{-i}(s_i, \tilde{s}_{-i}) - \epsilon\}$. Then a similar existence result for two-person discontinuous game can be formulated as follows:

Proposition 3. *Let in the two-person discontinuous game G for any $s_i \in S_i$, $i \in \{1, 2\}$ there exist a limit $\lim\limits_{\substack{\epsilon \to +0, \\ s_{-i} \in BR_\epsilon(s_i)}} u_i(s_i, s_{-i})$. Then, there is a Chain ϵ-Equilibrium in Secure Strategies in G.*

Proof. It becomes slightly more involved, but similar to the proof of Proposition 2. It is based on the concept of ϵ-Stackelberg points and is given in Appendix A. $\square$

Example 3. The Colonel Blotto two-person game (Borel 1953, Owen 1968).[2] Two players with limited resources A and B are tasked to distribute them over two battlefields. The player devoting the most resources to a battlefield wins that battlefield with the gain α or β. If both players devote equal resources they share the winning in half. Without loss of generality, we assume $A > B$ and $\alpha > \beta$. We also assume $A < 2B$, since otherwise the game solution is obvious. Let a and b be allocated resources at α-battlefield by A-player and B-player respectively. The resources allocated by players at β-battlefield are $A - a$ and $B - b$ respectively.

One can define the corresponding payoff functions of players:

$$U_A = \begin{cases} \beta, & a < b \\ \beta + \alpha/2, & a = b \\ \beta + \alpha, & b < a < b + (A - B) \\ \beta/2 + \alpha, & a = b + (A - B) \\ \alpha, & a > b + (A - B) \end{cases} ; \qquad (2)$$

$$U_B = \alpha + \beta - U_A$$

There are subsets in the strategy space (a, b) with different outcomes of the game, which are shown in Fig. 1.

There is no Nash equilibrium in the game. Moreover, there is also no EinSS, since there is a threat in any profile. Indeed, there is either a threat for B-player to get zero payoff, or (if B-player get zero payoff) there is a threat for A-player to decrease her payoff. However, a behavior logic peculiar to Chain EinSS may occur in the game, if one player behaves cautiously

[2] We thank V.Korepanov, who pointed out the existence of AEinSS in the Colonel Blotto games.

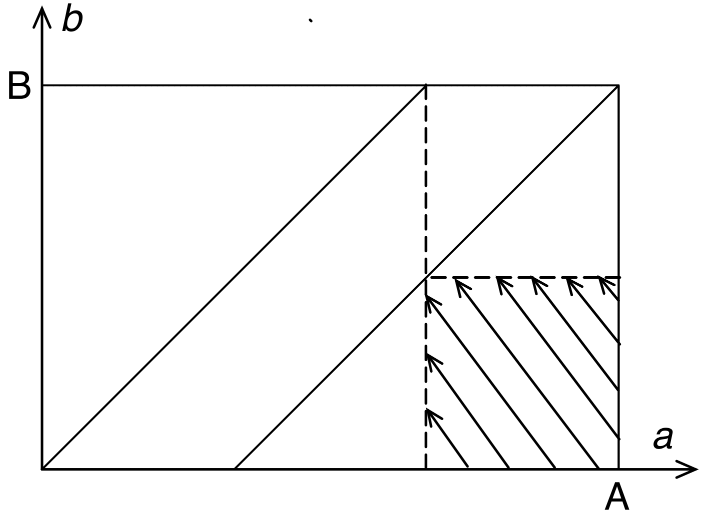

Fig. 1. Subsets in the strategy space (a, b) with different outcome in Blotto's game. The shaded area corresponds to a Chain EinSS with the order of players $(\{A\}, \{B\})$.

and the other player adjusts her strategy accordingly. For player B makes no sense to behave cautiously, since he can not guarantee any positive payoff. A stronger player A can get β, α or $\alpha + \beta$ (in ascending order). The maximum payoff $\alpha + \beta$ is reached in the "diagonal" zone $b < a < b + (A - B)$, but it is not guaranteed, since there is always a threat from player B to choose $b = B$ or $b = 0$ and reduce the payoff of player A. Nevertheless, player A can ensure a victory on the main battlefield (and thus ensure payoff α) by sending there more resources than player B has at his disposal (i.e. by choosing $a > B$). Otherwise, player A can get β at best response of player B. If a stronger player A behaves cautiously and chooses $a > B$

to guarantee victory on a major battlefield, and player B adjusts to it and chooses the best response $b < 2B - A$ to guarantee victory on a minor battlefield, such behavior corresponds to a Chain EinSS with the order of players $(\{A\}, \{B\})$. The corresponding set of Chain EinSS $\{(a, b)|\ a > B,\ b < 2B - A\}$ is indicated by hatching in Fig. 1. One can easily verify that the definition of CEinSS is fulfilled. Indeed, the strategy of player A is secure and he has no secure deviations. Player B can not increase his payoff by unilateral deviation. Obviously, in the area $\{(a, b)|\ a > B,\ 2B - A < b < a - (A - B)\}$ player A has a secure deviation and player B can not guarantee victory on a minor battlefield. Notice that the situation on the boundary of CEinSS can be different. In particular, if $\beta \geqslant \alpha/2$ then the boundary points at $a = B$ belong to the set of CEinSS. If $\beta < \alpha/2$ then the boundary points at $b = 2B - A$ belong to the set of CEinSS. One can also easily check that other strategy profiles in (a, b) can not correspond to the CEinSS.

5. Examples with more than two players

Let us consider now some examples of CEinSS in games with more than two players. One can expect that these CEinSS will also include some analogues of Stackelberg points. In fact, however, the behavior of players in CEinSS is different from the logic of players choosing the Stackelberg points. Players can assume in CEinSS a complex structure of threats existing between groups of players, but at the same time, unlike Stackelberg approach, they do not analyze the result of mutual deviations more than one move ahead. This removes certain difficulties associated with an ambiguous order of reflection.

Example 4. Let us consider the following three-person matrix game:

c_1 :		b_1	b_2
	a_1	(0,0,0)	(1,2,2)
	a_2	(2,2,1)	(2,1,2)

c_2 :		b_1	b_2
	a_1	(2,1,2)	(2,2,1)
	a_2	(1,2,2)	(0,0,0)

There are analogues of Stackelberg points (1) for three players taken in any order. However, there is no CEinSS in this game, since in any profile one of the players has a secure profitable deviation and the definition of CEinSS fails.

Example 5. Let us consider the cautious behavior of players A, B and C in the following matrix game:

c_1 :		b_1	b_2
	a_1	(0,0,0)	(-1,1,0)
	a_2	(-1,0,0)	(0,-1,0)

c_2 :		b_1	b_2
	a_1	(-1,0,1)	(-1,1,-1)
	a_2	(0,0,-1)	(0,-1,-2)

There is the following logic of behavior. Player A chooses strategy a_1 or a_2 and has only two options: either securely get -1, or get 0 with the threat from other players. Therefore, player A has nothing to lose and regardless of the threats of players B and C always chooses the best response with zero payoff. The payoff to player B does not depend on player C. On the other hand, given the predictable behavior of player A, player B secure himself against threats from player A. In particular, he chooses a strategy b_1 with zero payoff, since strategy b_2 taking into account the behavior of player A would give him -1. And finally, for player C threats from both players A and B are significant. Therefore, if player C behave cautiously, he chooses strategy c_1 with zero payoff, because otherwise taking into account predictable behavior of players A and B he gets negative payoff. It is natural to assume, that in the case of cautious behavior of players this game admits a CEinSS with the player's order $\{C, B, A\}$. The only such CEinSS in the game is a profile (a_1, b_1, c_1) with payoffs $(0, 0, 0)$.

Example 6. Product competition with a linear distribution of consumers (Eaton, Lipsey 1975, Shaked 1975). Let us consider a continuum of consumers, which have different preferences over a linear set $[0, 1]$ of product characteristics. Consumers are distributed with a non-atomic density $\rho(x)$, $x \in [0, 1]$. There are N firms, which produce a product at the same fixed price and choose the characteristic of the product x they will offer. Without loss of generality we assume their strategies to be $x_1 \leqslant x_2 \leqslant ... \leqslant x_N$. We also consider inelastic demand, i.e. each consumer purchases precisely one unit. Let $B_i(x) = \{x \in [0, 1] | \parallel x - x_i \parallel \leqslant \parallel x - x_j \parallel \quad \forall j \neq i\}$ be the firm i's market region. Then the profit function of firm i is:

$$U_i(x_i, x_{-i}) = (1 + n)^{-1} \int_{B_i(x)} \rho(x) dx \tag{3}$$

where n $(0 \leqslant n \leqslant N - 1)$ is the number of firms other than i, located at x_i. Eaton and Lipsey (1975) noted that for $N = 3$ and $\rho = 1$ the game admits no pure Nash equilibrium.

As an example, we consider the case when consumers are distributed with a linear density $\rho(x) = x$, $x \in [0, 1]$. For definiteness, consider the case of $N = 4$ (although further arguments can be easily generalized to an arbitrary $N > 2$). If three (or more) firms choose the same position, one of them, having displaced infinitely close, can always increase its payoff function at the expense of the other two firms. If two firms choose the same position and they have a neighbor on the right, one of these two firm, having displaced infinitely close, can always increase its utility either at the expense of another firm or at the expense of a neighboring firm on the right. If there is a firm, which does not coincide with other firms in its location and has a neighbor firm on its right, then it is always advantageous for this firm to move to the right (in the direction of increasing x) and thereby reduce the utility of the neighbor firm on its right. Therefore, at any locations chosen by firms there is no Nash-Cournot equilibrium in the game. Moreover, there is no EinSS, since all strategy profiles pose some threats. The logic of a Chain EinSS, however, reveals a rational decision, when players reflexively take into account mutual threats.

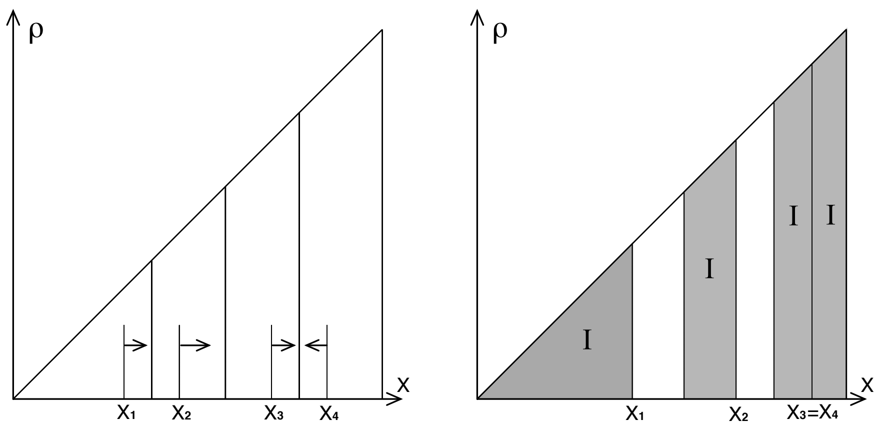

Fig. 2. Chain EinSS locations of players in a product competition with a linear distribution.

On the one hand (see Fig. 2), it is profitable for all players to move their location to the right (except for the rightmost player, for whom it is profitable to shift its location to the neighbor on the left). On the other hand, in the game there is a threat of

”undercutting” market share. If the market share to the right (or to the left) of some player exceeds the market share of any other player, it is profitable for this player to jump into the position immediately to the right (or to the left) of the first player. We assume, that the players behave cautiously and maximize their profit under the condition of security against losing market share to their right or to their left by undercutting.

We show, that this behavior corresponds to the CEinSS when consumers are distributed with a linear density $\rho(x) = x$. In this case the security condition can be satisfied only if all the players (except may be the players N and $N-1$) choose different locations $x_i \neq x_j$. Denote the payoff of firm i_{min} with a minimum market share as U_{min}. Consider the leftmost firm. It is always profitable for this firm to move its location to the right neighbor firm, until there is a threat of undercutting by player i_{min}, i.e. until the market area to the left of him does not exceed U_{min}. The next firm to the left must choose location $x_2 > x_1$, and hence $i_{min} \neq 1$. For this firm it is also always profitable to move its location to its right neighbor firm's location, until the market share of this firm to the left of it does not reach U_{min}. In moving its location to the right this firm takes into account the threat from firm i_{min}, but can ignore the threats from firms on its left. Indeed, their further shift to the right is already restricted by the condition of their security against the market share undercutting, and undercutting for these firms themselves is a priori not profitable, since their utility exceeds U_{min}. Proceeding recursively, one can see that each firm $i < N-1$ takes into account the threats only from the firms to the right of it $j > i$. For the last two rightmost firms it is profitable to move towards each other until they divide the remaining market area in half and there is the only possibility, that $i_{min} \in \{N, N-1\}$

(see Fig. 2). Let us write formally players payoff functions for $N = 4$:

$$\begin{aligned} U_1 &= \frac{1}{2}\left(\frac{x_1 + x_2}{2}\right)^2; \\ U_2 &= \frac{1}{2}\left[\left(\frac{x_2 + x_3}{2}\right)^2 - \left(\frac{x_1 + x_2}{2}\right)^2\right]; \\ U_3 &= \frac{1}{2}\left[\left(\frac{x_3 + x_4}{2}\right)^2 - \left(\frac{x_2 + x_3}{2}\right)^2\right]; \\ U_4 &= \frac{1}{2}\left[1 - \left(\frac{x_3 + x_4}{2}\right)^2\right] \end{aligned} \tag{4}$$

Appropriate security conditions against market share undercutting can be written as:

$$\begin{gathered} \frac{1}{2}\max\left\{x_1^2, \left(\frac{x_1 + x_2}{2}\right)^2 - x_1^2\right\} \leqslant \min\{U_2, U_3, U_4\}; \\ \frac{1}{2}\max\left\{x_2^2 - \left(\frac{x_1 + x_2}{2}\right)^2, \left(\frac{x_2 + x_3}{2}\right)^2 - x_2^2\right\} \leqslant \\ \leqslant \min\{U_3, U_4\}; \\ \frac{1}{2}\max\left\{x_3^2 - \left(\frac{x_2 + x_3}{2}\right)^2, \left(\frac{x_3 + x_4}{2}\right)^2 - x_3^2\right\} \leqslant U_4. \end{gathered} \tag{5}$$

Maximizing the payoffs (4) under conditions (5), we obtain the following solution:

$$x_1 = \sqrt{2U_{min}}, \quad x_2 = \frac{5}{3}\sqrt{2U_{min}}, \quad x_3 = x_4 = \sqrt{1 - 2U_{min}}, \tag{6}$$

where $2U_{min} \approx 0.17677$ is found from:

$$\frac{5}{3}\sqrt{2U_{min}} + \sqrt{1 - 2U_{min}} = 2\sqrt{1 - 4U_{min}} \qquad (7)$$

Generalizing the above argument to an arbitrary number of firms, we obtain the following proposition.

Proposition 4. *There is a unique (up to permutation of players) Chain EinSS in the product competition over a linear set* $[0, 1]$ *with the consumer distribution* $\rho(x) = x,\ x \in [0, 1]$*, which is given by*

$$x_1^2 = 1 - x_N^2 = 2U_{min}; \quad x_i^2 - \left(\frac{x_i + x_{i-1}}{2}\right)^2 = 2U_{min}, \qquad (8)$$
$$i = 2, ..., N - 1; \quad x_N = x_{N-1}$$

Proof. Formal proof follows the logic given above and is provided in Appendix B. □

6. General definitions

In this section we generalize the concept of Chain EinSS to the arbitrary relation of desire for security introduced over the set of players. This approach was suggested in (Iskakov 2005, 2008). However it was introduced in a very cumbersome form and included some inexactitudes. In this paper we provide a new intuitive formulation of this concept. Then we can specify the notion of security and secure deviation against threats of a particular group of players from the set $\tilde{N}$.

Definition 2″. *A strategy* s_i *of player* i *is* ***a secure strategy with respect to all players*** $j \in \tilde{N}$*, if no player* $j \in \tilde{N},\ j \neq i$ *has a threat against player* i*.*

Definition 3″. *A deviation* s'_i *of player* i *at strategy profile* s *is* ***a*** [***strict***] ***secure deviation with respect to all players*** $j \in \tilde{N}$*, if* $u_i(s'_i, s_{-i}) > u_i(s)$ *and* $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ $[u_i(s'_i, s'_j, s_{-ij}) > u_i(s)]$ *for any threat* s'_j *of player* $j \in \tilde{N},\ j \neq i$ *against player* i *at profile* (s'_i, s_{-i})*.*

Now we can define an Asymmetric EinSS in a way similar to a Chain EinSS.

Definition 4″. *A strategy profile is* ***a*** [***weak***] ***Asymmetric Equilibrium in Secure Strategies (AEinSS)*** *if there is a partition of players* $N = N_1 \cup ... \cup N_m$*, such that for all* $m' = 1, ..., m$*: (i) each player* $i \in N_{m'}$ *has a secure strategy w.r.t. all players* $j \in \bigcup\limits_{m' \leqslant k \leqslant m} N_k$*, and (ii) no player* $i \in N_{m'}$ *has a* [*strict*] *secure deviation w.r.t. all players* $j \in \bigcup\limits_{m' \leqslant k \leqslant m} N_k$*. The partition* $N = N_1 \cup ... \cup N_m$ *is* ***a structure of AEinSS****.*

Remark 1. An (ordinary) EinSS (Definition 4) and a Chain EinSS (Definition 4′) are the special cases of AEinSS with the trivial partition and the singleton partition of players respectively.

Remark 2. The same AEinSS may have multiple structures (or partitions). This means that the same AEinSS in a given game can be interpreted in several ways in terms of the behavioral attitudes of players. One can argue, that this ambiguity can be resolved by choosing the minimal in some way structure of AEinSS. However, we will not delve here into this question as it is hardly a principal one.

Remark 3. There can be multiple AEinSS. Moreover, the number of possible partitions of players in AEinSS grows very rapidly with the number of players. Choosing a particular partition includes an *ad hoc* element and brings a question of how the group formation occurs. Note, however, that in anonymous games, the number of variants of partitions can be significantly reduced (namely $n!$ times). In other games the structure of mutual threats can be defined by the rules of the game itself, for example by the size of the resources available to players, by their different positions in the market, by unequal rules of the game, etc. At the same time, it is important to note, that the definition does not impose any restrictions on the choice of partition. Equilibrium situation can occur spontaneously in the course of the game itself as a result of rational choice of roles by players. Here is an important advantage of the proposed formalism: it reveals the possibility of a hierarchical structure of mutual threats, which can not be identified by other approaches. We believe that the proposed theoretical framework will be useful primarily in the analysis of the problems in which there is a possibility of asymmetric behavior of players, such as are almost all practical problems. Below (in Examples 7 and 8) we shall present further game examples in which this logic works, despite the lack of Nash equilibrium and lack of EinSS.

Remark 4. In our definition players take into account the threats of players within their own subgroup. However, the attitudes towards mutual threats of players within their own subgroup could be defined differently. For example, one could assume that players ignore threats of other players in their subgroup or even differentiate these threats in a more complicated way. All such generalizations present no difficulties, but they do make the definitions more cumbersome.

Now we can prove that the AEinSS is indeed an intermediate set, which is narrower than the ETCT but wider than the EinSS. The following proposition is a generalization of Proposition 1.

Proposition 5. *Let M_{EinSS}, M_{AEinSS} and M_{ETCT} be the sets of Equilibria in Secure Strategies, Asymmetric Equilibria in Secure Strategies and Equilibria in Threats and Counter Threats respectively. Then $M_{EinSS} \subseteq M_{AEinSS} \subseteq M_{ETCT}$.*

Proof. The EinSS is also by definition a AEinSS with the trivial set partition with $N_1 = N$. Therefore $M_{EinSS} \subseteq M_{AEinSS}$. In AEinSS player from $N_{m'}$ can not make a secure deviation w.r.t. all players from $\bigcup\limits_{m' \leqslant k \leqslant m} N_k$. Therefore this player also can not make a secure deviation w.r.t. all players from N (since in this case the set of secure deviations is even narrower). Therefore $M_{AEinSS} \subseteq M_{ETCT}$. □

Let us consider an example of matrix game with asymmetric situations, that illustrate the intuitive meaning of Definition 4″.

Example 7. Equilibrium under "the Red Line".
Consider a team in some organization. Team members have an opportunity to break some formal rules for the benefit of themselves, because their implementation nobody really

watches. Each member of the team (players $B_1, B_2, ..., B_N$) can choose two strategies: to break or not to break the rules. If the amount of violations exceeds, however, a certain critical level (the Red Line), the administration will be forced to incur expenses and introduce some control mechanism. It will make trouble for all breakers. Thus, an administration (player A) also has two strategies: to conduct inspections or not (strategies a_1 and a_2 respectively). Let us denote as b_k the strategies of B-players, when k team members choose to break the rules, and accordingly $N-k$ team members prefer not to break them. Players' payoffs denote by three numbers: the payoff of player A, the payoff of B-players violating the rules, and the payoff of B-players complying with the rules. Let us then define the game using the following table:

	b_0	b_1	...	b_m	...	b_N
a_1	(0,#,0)	(-1,1,0)	...	(-m,1,0)	...	(-N,1,#)
a_2	(-m,#,0)	(-m,-1,0)	...	(-m,-1,0)	...	(-m,-1,#)

The situation without violations and without inspections corresponds to profile (a_1, b_0) where all players gets zero payoffs. It is profitable for B-players to break the rules if nobody watches. So the corresponding players in profiles $(a_1, b_k), k = 1, ..., N$ get the payoff $+1$. In the process the organization incurs losses in proportion to the number of violations. If the amount of violations exceeds m, it becomes more profitable for administration to start checking which also has the cost of m (i.e. to deviate into profiles $(a_2, b_k), k = m+1, ..., N$ respectively). In this case all breakers of the rules will suffer damage with payoff -1. If people of a particular culture keep sense of the red line and act carefully, then it is profitable for them to break the rules, but not to make the cup run over. In this case the level of violations will remain just below the "red line" in profile (a_1, b_m). This profile corresponds to the AEinSS with the partition of players $\{B_1, B_2, ..., B_N\} \cup \{A\}$, where cautious B-players take into account the threat of player A, and player A regulates their behavior by its reaction. One can easily see that this profile is not a Nash-Cournot equilibrium. It is also not an EinSS, as formally there is a threat for player A.

7. Calculation of AEinSS

First of all, notice a recursive property in the definition of AEinSS, which can be useful for its finding. Consider the game $G = \langle N, S_i, u_i, i \in N\rangle$ and the partition of the set of players $N = N_1 \cup N_2 \cup ... \cup N_m$. Denote by $\mathbf{s}_i$ the vector of strategies of players from N_i. At a given strategies $\mathbf{s}_1$ of players from N_1 let us define a restriction of the game G to $N \setminus N_1$ as $\tilde{G}_{\mathbf{s}_1} = \langle N \setminus N_1, S_i, \tilde{u}_i(\mathbf{s}_2, ..., \mathbf{s}_m) \equiv u_i(\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_m), i \in N \setminus N_1\rangle$. Then the following statement holds:

Proposition 6. *A profile* $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_m)$ *is an AEinSS in the game* G *with a structure* $N_1 \cup N_2 \cup ... \cup N_m$ *if and only if*

(i) all players from N_1 *have a secure strategies at profile* $\mathbf{s}$*,*

(ii) no player from N_1 *has a secure deviation at* $\mathbf{s}$*, and*

(iii) $(\mathbf{s}_2, ..., \mathbf{s}_m)$ *is an AEinSS in the game* $\tilde{G}_{\mathbf{s}_1}$ *with a structure* $N_2 \cup ... \cup N_m$.

Proof. The proof is in the direct examination of definitions of AEinSS. □

Thus, if we know AEinSS in the subgame $\tilde{G}_{\mathbf{s}_1}$ for any strategies $\mathbf{s}_1$, then all AEinSS in the game G can be found by checking for $\mathbf{s}_1$ the conditions (i) and (ii) of Proposition 6. In practical terms, this means that the task of finding AEinSS in the game G can be reduced to finding AEinSS in the game with fewer players.

Assume that the strategy sets of players are compact, and their payoff functions are continuous. For this case we present two concepts, which can be useful for finding AEinSS. The first concept is connected with the player's payoff after the realization of the worst possible for him in the profile threat. More specifically,

Definition 7. *A player's i* ***payoff function secured from threats*** *$v_i(s,\tilde{N})$ of all players $j \in \tilde{N}$ at strategy profile s equals either his payoff after realization of the worst threat of players $j \in \tilde{N}$ against player i (if any) or coincides with his payoff (if player i has a secure strategy w.r.t. all players $j \in \tilde{N}$ at strategy profile s), i.e.*

$$v_i(s,\tilde{N}) = \begin{cases} u_i(s), \ \ s_i \textit{ is a secure strategy} \\ \qquad\qquad\qquad\qquad \textit{w.r.t. all players } j \in \tilde{N} \\ \inf\limits_{\substack{j \in \tilde{N},\ j \neq i, \\ s'_j:\ u_j(s'_j, s_{-j}) > u_j(s)}} u_i(s'_j, s_{-j}), \ \ \textit{otherwise} \end{cases} \tag{9}$$

If the payoffs secured from threats $v_i(s,\tilde{N})$ can be easily calculated, then they can be used for finding AEinSS in accordance with the following statement.

Proposition 7. *A profile $\mathbf{s}^* = (\mathbf{s}_1^*, \mathbf{s}_2^*, ..., \mathbf{s}_m^*)$ is a weak AEinSS in the game G with a structure $N_1 \cup N_2 \cup ... \cup N_m$ if and only if* *the following system of equations are held:*

$$
\begin{aligned}
i \in N_1:\ & u_i(s^*) = v_i(s^*,\, N_1 \cup ... \cup N_m) = \\
& \qquad\qquad = \max_{s_i'} v_i(s_i', s_{-i}^*, N_1 \cup ... \cup N_m) \\
i \in N_2:\ & u_i(s^*) = v_i(s^*,\, N_2 \cup ... \cup N_m) = \\
& \qquad\qquad = \max_{s_i'} v_i(s_i', s_{-i}^*, N_2 \cup ... \cup N_m) \\
& \qquad ... \\
i \in N_m:\ & u_i(s^*) = v_i(s^*,\, N_m) = \max_{s_i'} v_i(s_i', s_{-i}^*, N_m)
\end{aligned} \tag{10}
$$

Proof. The condition of secure strategy of player i w.r.t. all players $j \in \tilde{N}$ (Definition $2''$), using the payoff function secured from threats (9), can be written as:

$$u_i(s^*) = v_i(s^*, \tilde{N}) \tag{11}$$

The condition of no strict secure deviation for player i w.r.t. all players $j \in \tilde{N}$ (Definition $3''$) implies that for the deviation $s_i' \neq s_i^*,\ s_i' \in S_i$ one of the following two situations occurs:

$$
\begin{gathered}
u_i(s_i^*) \geqslant u_i(s_i') \geqslant v_i(s_i', s_{-i}^*, \tilde{N}) \quad \text{or} \\
u_i(s_i^*) < u_i(s_i'), \quad v_i(s_i', s_{-i}^*, \tilde{N}) \leqslant u_i(s_i^*)
\end{gathered} \tag{12}
$$

Since S_i is a compact and the functions u_i and v_i are assumed to be continuous in s_i, then both of these conditions can be written as one:

$$u_i(s^*) \geqslant \max_{s_i' \neq s_i^*} v_i(s_i', s_{-i}^*, \tilde{N}) \tag{13}$$

Combining both conditions (11) and (13), we obtain one equivalent condition,

$$u_i(s^*) = v_i(s^*, \tilde{N}) = \max_{s_i'} v_i(s_i', s_{-i}^*, \tilde{N}),$$

which determines, in accordance with Definition 4″, the AEinSS conditions for players of different groups in the form of (10). □

Another concept, which can be useful for finding AEinSS, is a concept of the best secure response. The strategy of each player in the EinSS is the best secure response, i.e. the most profitable secure strategy of a given player at a given strategies of other players. Let us generalize this result to the case of AEinSS. First we define the best secure response to a group of players.

Definition 8. ***Best Secure Response of a player with respect to all players*** *$j \in \tilde{N}$ is the most profitable for him secure strategy with respect to all players $j \in \tilde{N}$.*

Proposition 8. *The strategy of each player from $N_{m'}$ in the AEinSS with structure $N = N_1 \cup \ldots \cup N_m$ is the Best Secure Response of this player w.r.t. all players $j \in \bigcup\limits_{m' \leqslant k \leqslant m} N_k$.*

Proof. Otherwise, a player could make a secure deviation into his Best Secure Response w.r.t. all players $j \in \bigcup\limits_{m' \leqslant k \leqslant m} N_k$. □

Denote by $BSR_i(s_{-i}, \tilde{N})$ the set of all best secure responses of player i to s_{-i} w.r.t. all players $j \in \tilde{N}$. Customarily, we will call $BS_i(\cdot)$ the best secure response function of player i. The Proposition 8 gives us only the necessary conditions of the AEinSS. They need to be supplemented by checking the insecurity of profitable deviations. To formalize this condition one can also use the payoff functions secured from threats v_i (9). Consider a matrix or continuous game G with continuous payoff functions.

Proposition 9. *A profile* $\mathbf{s}^* = (\mathbf{s}_1^*, \mathbf{s}_2^*, ..., \mathbf{s}_m^*)$ *is a weak AEinSS in the game* G *with a structure* $N_1 \cup N_2 \cup ... \cup N_m$ *if and only if the following system of equations is held for all* $1 \leqslant m' \leqslant m$*:*

$$\forall i \in N_{m'} : \ s_i^* = BSR_i(s_{-i}^*, \bigcup_{m' \leqslant k \leqslant m} N_k), \tag{14}$$

$$\forall i \in N_{m'} : \ u_i(s^*) \geqslant \max_{s_i' : u_i(s_i', s_{-i}^*) > u_i(s^*)} v_i(s_i', s_{-i}^*, \bigcup_{m' \leqslant k \leqslant m} N_k) \tag{15}$$

Proof. The necessity of (14) follows from Proposition 8. The necessity of (15) follows from the need of absence of secure deviations w.r.t. all players from a corresponding subgroup. The sufficiency follows from the fact, that the first condition implies the profile security of player i w.r.t. all players from a corresponding subgroup, and the second condition implies the absence of strict secure deviations of player i w.r.t. all players from a corresponding subgroup. □

For a given partition of players one can find a weak AEinSS according to Proposition 9 in four steps. First, analyzing mutual threats of players determine conditions for selectively secured profiles according to a given partition of players. Then find all BSR functions of players in (14) w.r.t. players from the corresponding groups. Then solve the corresponding maximization problem (14) in the set of selectively secure profiles. And finally, verify the definition of weak AEinSS for the found BSR-profiles, namely the conditions (15) of insecurity for all profitable deviations.

8. Conclusion

In this paper we introduce an extension of the concept of Equilibrium in Secure Strategies (Iskakov, 2005; Iskakov and Iskakov, 2012a), which takes into account non-uniform attitudes of players to security in non-cooperative games. We assume that the players are divided into classes, that unequally related to each other threats. Namely, the players in their behavior take into account threats posed by players of their own or higher class, and ignore threats of lower-class players. In practice, such asymmetry of players can either be determined by external parameters of the game, or arise spontaneously in the course of the game as a result of rational choice of roles by the players themselves.

First, we examined an asymmetric attitude of players to mutual threats in the simplest case, when all players are strictly ordered by their relation to security. A corresponding equilibrium we call a Chain Equilibrium in Secure Strategies (CEinSS). This concept has a particularly simple interpretation in the two-person games. In this case, the first player chooses secure profiles and improves its position only by secure deviations (i.e. behaves as he would behave in the EinSS). Second player chooses the best response (i.e. behaves as he would behave in the Nash equilibrium). In this case CEinSS are closely linked with the concept of Stackelberg-points (D'Aspremont, Gerard-Varet 1980). Specifically, Stackelberg-points in the matrix or continuous two-person games are weak CEinSS. Thus, in these games, there are always two weak CEinSS (which also may coincide). However, in the general case of two-person games the set of weak CEinSS is wider than the set of Stackelberg-points. For discontinuous games with two players, we prove a theorem on the existence of an ϵ-CEinSS.

As illustrative examples of CEinSS we considered a number of matrix games and two continuous games, which fails to have Nash-Cournot or (conventional) EinSS. The Colonel Blotto two-person game (Borel 1953, Owen 1968) for two battlefields with different price always admits a CEinSS with intuitive interpretation. The product competition of many players (Eaton, Lipsey 1975, Shaked 1975) with a linear distribution of consumers on a segment always admits a unique (up to permutation of players) CEinSS solution.

We introduced an Asymmetric Equilibrium in Secure Strategies (AEinSS), which generalizes the concept of Chain EinSS to the arbitrary relation of desire for security introduced over the set of players. Each AEinSS corresponds to a certain partition of the players into security classes, which we call *a structure* of AEinSS. The same AEinSS in the game may have multiple structures (or partitions). There can be also multiple AEinSS with different partitions. Moreover, the number of possible partitions of players in AEinSS grows very rapidly with the number of players. All these inconveniences, however, can be partly mitigated when considering specific games. In anonymous games for example the number of variants of partitions can be significantly reduced. In other games the structure of mutual threats can be defined by the rules of the game itself, for example by the size of the resources available to players, by their different positions in the market, by unequal rules of the game, etc. At the same time, it is important to note, that the definition does not impose any restrictions on the choice of partition. Equilibrium situation can occur spontaneously in the course of the game itself as a result of rational choice of roles by players. Here is an important advantage of the proposed formalism: it reveals the possibility of a hierarchical structure of mutual threats, which can not be identified by other approaches. We believe that the proposed theoretical framework will be useful primarily in the analysis of the problems, in which there is a possibility of asymmetric behavior of players, such as are almost all practical problems. As an illustration of the intuitive meaning of AEinSS we considered a matrix game with a lot of cautious players (team members) adapting to the rules dictated by one player (the organization). An intuitively expected equilibrium position is an AEinSS, which is neither Nash-Cournot equilibrium nor EinSS.

Finally, we proposed some methods for finding AEinSS in Propositions 7, 8 and 9. Unfortunately, they do not allow to pre-select one or another partition of players. Therefore, to find all AEinSS, we have to enumerate different possible structures, and for each of them to build the appropriate secured payoff functions $v_i(s, \tilde{N})$ or best secure responses $BSR_i(s_{-i}, \tilde{N})$. The corresponding AEinSS can be found then as a solution of (10) or (14-15) respectively. Nevertheless, it is hoped, that the anonymity of games, their symmetry by players, the monotonic dependence of desired points on strategies of individual players, will allow to decompose many games by players and to search for the AEinSS recursively.

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APPENDIX A: Proof of Proposition 3

Proposition 3. *Let $G = \{(S_1, u_1), (S_2, u_2)\}$ be a two-person game with discontinuous utility functions u_i. Let $BR_\epsilon(s_i, \epsilon)$ be an ϵ-best reply correspondence to s_i, i.e. the set $\{s_{-i} \mid u(s_i, s_{-i}) > \sup_{\tilde{s}_{-i}} u_{-i}(s_i, \tilde{s}_{-i}) - \epsilon\}$. Let for any $s_i \in S_i,\ i \in \{1, 2\}$ there exist a $\lim_{\substack{\epsilon \to +0, \\ s_{-i} \in BR_\epsilon(s_i)}} u_i(s_i, s_{-i})$. Then, there is a Chain ϵ-Equilibrium in Secure Strategies in G.*

Proof. From an assumption, that for any $s_i \in S_i,\ i \in \{1, 2\}$ there exist a $\lim_{\substack{\epsilon \to +0, \\ s_{-i} \in BR_\epsilon(s_i)}} u_i(s_i, s_{-i})$ it also follows, that for any $s_i \in S_i,\ i \in \{1, 2\}$ there exist a $\hat{u}_i(s_i) \equiv \lim_{\epsilon \to +0} \sup_{s_{-i} \in BR_\epsilon(s_i)} u_i(s_i, s_{-i})$. For arbitrarily small $\epsilon > 0$ choose s_i^* such that $\hat{u}_i(s_i^*) > \sup_{s_i \in S_i} \hat{u}_i(s_i) - \epsilon/5$. According to the bassc assumption, $\forall \epsilon > 0\ \ \exists \delta > 0, \delta < \epsilon$ such that $|\sup_{s_{-i} \in BR_\delta(s_i^*)} u_i(s_i^*, s_{-i}) - \hat{u}_i(s_i^*)| < \epsilon/5$. Choose $s_{-i}^* \in BR_\delta(s_i^*)$ such that $|u_i(s_i^*, s_{-i}^*) - \sup_{s_{-i} \in BR_\delta(s_i^*)} u_i(s_i^*, s_{-i})| < \epsilon/5$. Thus, for an arbitrarily small positive ϵ, there exists a point (s_i^*, s_{-i}^*) such that $u_i(s_i^*, s_{-i}^*) > \hat{u}_i(s_i^*) - 2\epsilon/5 > \sup_{s_i \in S_i} \hat{u}_i(s_i) - 3\epsilon/5$ and $u_{-i}(s_i^*, s_{-i}^*) > \sup_{\tilde{s}_{-i}} u_{-i}(s_i^*, \tilde{s}_{-i}) - \delta > \sup_{\tilde{s}_{-i}} u_{-i}(s_i^*, \tilde{s}_{-i}) - \epsilon$. This point can be called ϵ-Stackelberg point with a leader player i and follower player $(-i)$.

Let us prove that it is an ϵ-CEinSS. For definiteness we consider the case $i = 1$, $-i = 2$. First of all, note that $u_2(s_1^*, s_2^*) > \sup_{s_2} u_2(s_1^*, s_2) - \epsilon$, i.e. player 2 does not have whatever "more than ϵ" deviations, and therefore the definition of an ϵ-CEinSS with the order of players $(\{1\}, \{2\})$ is fulfilled for player 2.

Any profitable deviation of player 2 will be within the set $BR_\delta(s_1^*)$. Since we assume, that there exist a $\lim_{\substack{\delta \to +0, \\ s_2 \in BR_\delta(s_1)}} u_1(s_1, s_2)$, then by choosing in our previous arguments δ sufficiently small, we can ensure that the payoff function of player 1 is not reduced by more than ϵ. So there are no "more than ϵ" threats to player 1 in the profile (s_1^*, s_2^*).

Let us now suppose that player 1 can increase her payoff by deviating into s_1'.
If $u_1(s_1', s_2^*) \leqslant \hat{u}_1(s_1') + 2\epsilon/5 \leqslant \sup_{s_1 \in S_1} \hat{u}_1(s_1) + 2\epsilon/5 < u_1(s_1^*, s_2^*) + \epsilon$ and the deviation of player 1 into s_1' is "no more that ϵ" deviation.
If $u_1(s_1', s_2^*) > \hat{u}_1(s_1') + 2\epsilon/5$ then in accordance with the assumption of Proposition 3 for selected $\epsilon > 0$ $\exists \delta' > 0, \delta' < \epsilon$ such that for $\forall s_2' \in BR_{\delta'}(s_1')$ holds $u_1(s_1', s_2') < \hat{u}_1(s_1') + 2\epsilon/5$. Thus, $s_2^* \notin BR_{\delta'}(s_1')$ and the deviation of player 2 from s_2^* into $\forall s_2' \in BR_{\delta'}(s_1')$ is profitable for him. At the same time the payoff of player 1 strictly decreases and after the counter reply of player 2 it becomes $u_1(s_1', s_2') < \hat{u}_1(s_1') + 2\epsilon/5 \leqslant \sup_{s_1 \in S_1} \hat{u}_1(s_1) + 2\epsilon/5 <$

$u_1(s_1^*, s_2^*) + \epsilon$, i.e. it is no more than by ϵ exceeds her payoff at (s_1^*, s_2^*), i.e the deviation of player 1 into s_1' is "no more than ϵ" secure deviation. Therefore the definition of an ϵ-CEinSS with the order of players $(\{1\}, \{2\})$ is fulfilled for player 1 as well. □

APPENDIX B: Proof of Proposition 4

Proposition 4. *There is a unique (up to permutation of players) Chain EinSS in the product competition over a linear set* $[0, 1]$ *with the consumer distribution* $\rho(x) = x, \; x \in [0, 1]$*, which is given by*

$$x_1^2 = 1 - x_N^2 = 2U_{min}; \quad x_i^2 - \left(\frac{x_i + x_{i-1}}{2}\right)^2 = 2U_{min}, \qquad (\Pi.1)$$
$$i = 2, ..., N-1; \quad x_N = x_{N-1}$$

Proof. It is easy to verify that the positions defined by the system (П.1) are a Chain EinSS with the players indexed from left to right. Let us prove that this CEinSS is unique (up to permutation of players). Consider an arbitrary CEinSS in the product competition with a linear density of consumers $\rho(x) = x$.

1. Note that there can not be in the CEinSS three (or more) firms choosing the same position. Indeed, in this case any of these firms having displaced infinitely close can always increase its payoff function at the expense of the other two (or more) firms. However there can not be mutual threats in the CEinSS. For the same reason, if two firms in CEinSS choose the same position then the market share to the left and to the right of them must be equal.

2. Any firm always has a threat against its immediate neighbor on the right (if any) by a small shift to the right (in the direction of increasing $\rho(x)$). Therefore, in any CEinSS firms can only be indexed from left to right (by their position). Thus, we can consider the firms locations: $x_1 \leqslant x_2 \leqslant ... \leqslant x_N$ to be indexed exactly in accordance with their order in CEinSS.

3. By definition of CEinSS the position of the rightmost player must be his best response. If the rightmost player does not share his position with some other player, then his position can not be his best response, because it is always profitable for him to move a little to the left. Therefore, the two rightmost players in CEinSS always choose the same location: $x_{N-1} = x_N$. As it was already mentioned, the market share to the left and to the right of them must be equal.

4. Let firms $i-1$ and i having a neighbor to the right choose the same location in the CEinSS. Then firm i can increase the payoff by a small shift to the right. In this case "the undercutable market share" to the right of the firm i would not exceed "the undercutable market share" to the left of the firm $i+1$. Therefore there are no counter threats of undercutting firm i (after its small shift to the right) by firms $j > i+1$ (located to the right) provided that there were not threats of undercutting firm i by these players in the initial position of CEinSS. There is also no counter threat against firm i by firm $(i+1)$ to move to the left, since it is not profitable for firm $(i+1)$. Even if the firm $(i+1)$ is the rightmost player, according to point 3, there will be some other firm to the right of him after his move to the left. We obtained that a small move of firm i to the right is a secure deviation w.r.t. firms $j > i$, which contradicts the definition of CEinSS. Therefore, we conclude that all firms (except for the two rightmost firms N and $N-1$) choose different locations in CEinSS.

5. Given the above considerations, further construction of the system of equations (П.1) to determine the position of players in arbitrary CEinSS can be carried out in exactly the same way as in the preceding example. The system (П.1) can be easily reduced to two equations for x_N and U_{min}: $x_N^2 = 1 - 2U_{min}$; $x_N = f(U_{min})$, where $f(U_{min})$ – is a strictly increasing function. Hence it follows that this system always has a unique solution. Therefore, the CEinSS in the product competition with a linear density of consumers is unique (up to permutation of players) by construction. □

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Представлено обобщение концепции равновесий в безопасных стратегиях, которое учитывает неоднородное отношение игроков к безопасности в бескоалиционных играх. В предложенном *асимметричном* равновесии в безопасных стратегиях игроки делятся на классы, неодинаково относящиеся к угрозам друг друга, а именно, игрок в своем поведении учитывает угрозы со стороны игроков своего или высшего класса и игнорирует угрозы со стороны игроков низших по отношению к нему классов. Представлены иллюстративные примеры таких равновесий в матричных и непрерывных играх, в которых не существует равновесия Нэша – Курно и обычного равновесия в безопасных стратегиях. Излагаемая концепция является новой формулировкой *сложного* равновесия в безопасных стратегиях М. Искакова (2005) и обеспечивает теоретическую основу для изучения игр, в которых важную роль играет асимметричная и иерархическая логика взаимодействия игроков.

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