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A COMPARISON OF POLARIZATION AND BI-POLARIZATION INDICES IN SOME SPECIAL CASES

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An estimation of a degree of a polarization is very common in different areas of sociology, political science and economics. I study the properties of two main polarization and two main bi-polarization indices – those by Esteban-Ray, Aleskerov-Golubenko, Wolfson-Foster and Wang-Tsui. Two new properties for the indices are proposed and some key features and differences are described.

Keywords: Indices of polarization, polarization, bi-polarization, Esteban-Ray index, Aleskerov-Golubenko index, Wolfson-Foster index, Wang-Tsui index

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1 Introduction

Causes of social conflicts always attract attention of researchers in social sciences and economics. In the early studies much attention was paid to the inequality in income distribution: it was assumed to be a one of the reasons of conflicts between groups in society [1–3]. Indeed, inequality of wealth can cause the antagonism between rich and poor segments of society; however, noneconomic characteristics of individuals may also be reasons of division of society into separate groups.

There are two basic approaches in the analysis of social diversity: fractionalization [4–6] and polarization [7–9] measures. The fractionalization is a characteristic of social dissociation into groups according to some attribute. It depends only on the number of groups: the greater the number of groups, the higher the level of fractionalization. Polarization takes into account the degree of similarity between groups: less the similarity, greater polarization and degree of antagonism in society. Polarization, like fractionalization, can be measured in different aspects of invididual life : income, ethnic, religion, linguistic group, political views and so on [10].

The fundamental work in the theory of measure of polarization is [7]. In this paper the following definition of polarization is suggested:

> Suppose that a population of individuals may be grouped according to some vector of characteristics into clusters, such that each cluster is very similar in terms of the attributes of its members, but different clusters have members with very dissimilar attributes. In that case, we say that the society is polarized.

Therefore, the degree of social polarization depends on how much the values of attributes differ for different groups, and on the number and sizes of groups.

Polarization of society is directly related to existence of social tension and hence with a probability of social conflict. Thus, development of methods for polarization measure allows quantifying social tension and predicting social conflicts.

There are several polarization indices. The Esteban-Ray index [7] is a generalization of the Gini index . In [9] the Aleskerov-Golubenko index was introduced, which is based on ideas from mechanics.

Bi-polarization measures are very common as a subset of polarization measures. While a polarization measure reflects the division of society into groups (regardless of number of groups), bi-polarization measure is closely related with concept of "middle class" - the lower the size of middle class, the higher the degree of bi-polarization of the system and vice versa. The Wolfson-Foster index [11] is well-known bi-polarization index. Similarly to the Gini index, Wolfson-Foster index is calculated as twice the area under polarization curve. The generalization of the Wolfson-Foster index was proposed in [12].

In this work, polarization indices in some special cases are investigated and compared. In Section 2 formulation of indices is given. In Section 3 some new properties for indices are formulated. In Section 4 polarization indices are compared with formulated properties. Results and conclusions are presented in Section 5.

2 Formal Definition

Let us consider a society divided into n groups, where coordinate of i-th group is $y_i \in [0,1]$ and the size is p_i, where $\sum\limits_{i=1}^{n} p_i = 1$.

In [7] the model of "identification-alienation" was suggested, according to which any individual, on the one hand, identified with his group (and a sense of identification enhanced by the size of group), and on the other hand alienated from individuals from the other groups (a sense of alienation enhanced by the distance between groups). Based on this model the index was obtained

$$ER = 2^{\alpha+1} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i^{1+\alpha} p_j \left| y_i - y_j \right| \tag{1}$$

where $\alpha \in (0, 1.6]$ is a parameter of the model, that describes the degree of "polarization sensitivity" of the system (if $\alpha = 0$ ER is equal to Gini inequality index).

In [9] an alternative approach to measure polarization was proposed, it was used to estimate the degree of polarization in Finland parliamentary election in 1999 and 2003. The approach is based on the idea of physical characteristics of mass distribution in the system - the static moment of forces [13]. The Aleskerov-Golubenko index was introduced as

$$AG_0 = 2 \sum_{i=1}^{n} p_i \left| y_i - c \right| \tag{2}$$

where $c = \frac{\sum\limits_{i=1}^{n} p_i y_i}{\sum\limits_{i=1}^{n} p_i}$ is the barycenter of the system.

Further, the modification of (2) will be used

$$AG = \frac{4}{n} \sum_{i=1}^{n} p_i \left| y_i - c \right| \tag{3}$$

The main difference between (3) and (2) is that if $n \to \infty$ $AG \to 0$, while $AG_0 \to \frac{1}{2}$. Since in many practical cases the number of groups is large, it is more convenient to use an index with zero limit value: that index takes a low value not only in the case of a single group, but in the case of sufficiently large number of equal groups too.

As it was mentioned in Introduction the class of bi-polarization measures can be distinguished among polarization measures. By analogy with the Lorentz curve, in [11] a notion of polarization curve was proposed, and Wolfson-Foster index was determined as twice area under second-order polarization curve. In [11] the following formula for Wolfson-Foster index was obtained

$$WF = (T - G)\frac{\mu}{m} \tag{4}$$

where μ is the mean and m is the median values of attribute, G is Gini index,
$T = 1 - 2L(0.5)$, $L(0.5)$ is the value of Lorenz curve in point 0.5 [11].

The other bi-polarization measure - Wang-Tsui index - is the generalization of Wolfson-Foster index [12].

$$WT = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{y_i - m}{m} \right|^r = \sum_{i=1}^{n} p_i \left| \frac{y_i - m}{m} \right|^r, \tag{5}$$

where m is the median value of attribute, N is the number of individuals, n is the number of groups, $r \in (0, 1)$.

Note that extreme values of (1), (3) and (4), (5) are different. The Esteban-Ray and Aleskerov-Golubenko indices take maximal values in the case of two equal groups and this value is equal to one [7, 9]; the Wolfson-Foster and Wang-Tsui indices tend to infinity in the case of complete inequality.

Further we compare the indices (1), (3), (4) and (5) in some special cases and show important differences between them.

3 Properties for Polarization Indices

In this section, I propose two properties, which in my opinion polarization indices should satisfy. Verification of such properties allows us to detect differences between indices, what may be helpful for choosing the right index in specific case.

The first property states that polarization should decrease when the number of groups grow up and groups have equal sizes.

Property 1. *Let us consider aciety divided on n groups of equal size, which are placed on $[0,1]$, so that any two neighboring groups are equally distanced. Then increasing of the number of groups leads to polarization decreasing, and if $n \to \infty$ polarization tends to zero.*

In the next property only cases of odd n are considered. For example, let us consider the case of three equal groups and let the middle group move toward one of the groups on the boundary. When the middle group merges with one of the extreme, there will be only two groups. Such a system seems to be more polarized than the initial system with three groups.

Property 2. *Let us consider a society divided on n groups of equal size, which are placed on $[0,1]$, so that any two neighboring groups are equally distanced and n is odd. Let the middle group move from $y_{\frac{n+1}{2}}$ by ϵ, such that $|\epsilon| < \dfrac{1}{2(n-1)}$. Then polarization should in socrease with $|\epsilon|$ increasing.*

We use these properties to observe important differences between polarization indices.

4 A Comparison of Polarization Indices

In this Section the properties formulated above are verified and important differences between indices are demonstrated.

4.1 A Comparison of Polarization Indices in a Case of Equal Groups

Consider the situation from Property 1: society is divided on n equal groups which are placed on $[0,1]$, so that any two neighboring groups are equally distanced (Fig.1). We call this distribution **"equal"**. Although this case is not likely to be met in real systems, it is interesting for consideration: as it will be shown, indices are not sensitive to small perturbations of groups sizes and positions.

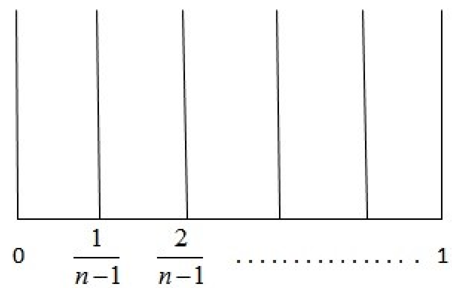

Figure 1. Equal distribution of n groups

Theorem 1. *Let ER_n and AG_n be the values of Esteban-Ray and Aleskerov-Golubenko indices in the case of equal distribution of n groups. Then*

- $ER_{n+1} < ER_n$;
- $AG_{n+1} < AG_n$.

Proof of this and all other theorems is given in the Appendix.

Theorem 1 means that Esteban-Ray and Aleskerov-Golubenko indices decrease monotonically with n increasing. This theorem is consistent with the results obtained in [7, 9], that the indices (1), (2) takes maximum values in the case of two equal groups.

In contrast to Aleskerov-Golubenko and Esteban-Ray indices bi-polarization indices do not decrease monotonically with number of groups increasing.

Theorem 2. *Let WT_n and WF_n be values of Wang-Tsui and Wolfson-Foster indices in the case of equal distribution of n groups. Then*

- $\forall n > 1$, $\forall r \in (0, 1)$ $WT_{2n} > WT_{2n+2}$, $WT_{2n-1} < WT_{2n+1}$, $WT_{2n} > WT_{2n+1}$;
- $\forall n > 1$ $WF_{2n} > WF_{2n-1} > WF_{2n+1}$, $WF_{2n} > WF_{2n+2}$.

Theorem 2 demonstrates some interesting features of bi-polarization indices. Firstly, values of bi-polarization indices depend on whether the number of groups odd or even, and in the case of even number of groups the indices values are greater. Secondly, while the Wolfson-Foster index non-monotonically decreases with n growing, the Wang-Tsui index is increasing with growing odd n and decreasing with even n.

Behavior of indices in this case is demonstrated on Fig.2. The graph shows that while Aleskerov-Golubenko and Esteban-Ray indices tends to zero for large values of n, bi-polarization indices vary around some positive values. This fact is formulated in Theorem 3.

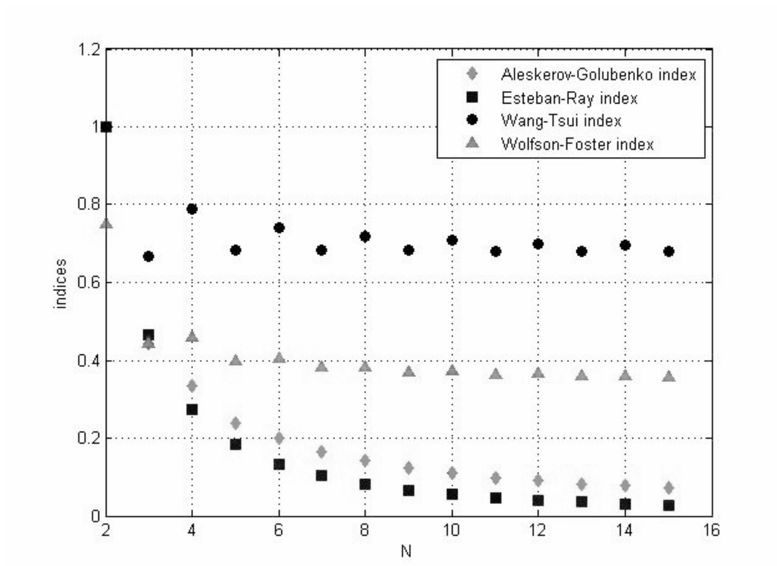

Figure 2. Behavior of indices in the case of equal distribution, $\alpha = 1.6$, $r = 0.5$. The graph is obtained via computer simulations.

Theorem 3. *Let us consider the equal distribution of* n *groups. Then if* $n \to \infty$

- $\forall \alpha \ ER_n \to 0,\ AG_n \to 0$;
- $WT_n \to f(r) > 2^{r-2}$;
- $WF_n \to \dfrac{1}{6}$

Thus if $n \to \infty$ the values of Wang-Tsui and Wolfson-Foster indices stay positive. Note that in the case of two equal groups the Wang-Tsui index value is equal to one for all $r \in (0,1)$. But if r is close to zero the limit value of index (when $n \to \infty$) is close to one, although polarization seems to be very low in the limit case. Similarly the Wolfson-Foster index changes its value very slowly since $n > 3$.

Theorems 1-3 can be formulated as the next proposition

Proposition 1. *Property 1 holds for Aleskerov-Golubenko and Esteban-Ray indices, and does not hold for Wang-Tsui and Wolfson-Foster index.*

Results of the comparison of polarization indices are presented in Theorems 4-5.

Theorem 4. *Let us consider the equal distribution of* n *groups. Then there is* $n^*(\alpha)$ *such that*

- *if* $n \le n^*$ $AG_n < ER_n$;
- *if* $n > n^*$ $AG_n > ER_n$;

Theorem 5. *Let us consider the equal distribution of* n *groups. Then* $\forall n > 1$, $r \in (0,1)$ $WT_n > WF_n$.

4.2 A Comparison of Polarization Indices in the Case of Odd Number of Equal Groups

Let us consider situation from Property 2: society is divided on n equal groups which are placed on $[0,1]$, so that any two neighboring groups are equally distanced, n is odd and the middle group is placed on
$x \in \left[\frac{n-3}{2(n-1)}, \frac{n+1}{2(n-1)}\right]$ (Fig.3).

Theorem 6. *Let us consider the equal distribution of* n *groups, when* n *is odd and the middle group is placed on* $x \in \left[\frac{n-3}{2(n-1)}, \frac{n+1}{2(n-1)}\right]$. *Then regardless of* x, $ER = \left(\frac{2}{n}\right)^{1+\alpha} \frac{n+1}{3}$.

Theorem 7. *Let us consider the equal distribution of* n *groups, when* n *is odd and the middle group is placed on* $x \in \left[\frac{n-3}{2(n-1)}, \frac{n+1}{2(n-1)}\right]$. *Then*

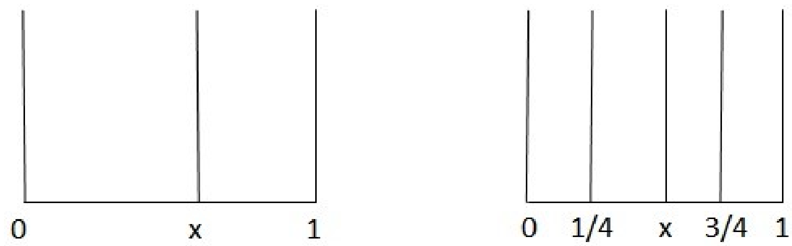

Figure 3. Middle group placed on x: cases of three and five groups)

Aleskerov-Golubenko index takes the minimum value on $x = 0.5$ *and the maximum value on* $x = \dfrac{n-3}{2(n-1)}$, $x = \dfrac{n+1}{2(n-1)}$.

Study of behavior of bi-polarization indices gives interesting results: it turns out that Wang-Tsui and Wolfson-Foster indices are not symmetric with respect to $x = 0.5$. This fact presented in Theorem 8.

Theorem 8. *Let us consider the equal distribution of* n *groups, when* n *is odd and the middle group is placed on* $x \in \left[\dfrac{n-3}{2(n-1)}, \dfrac{n+1}{2(n-1)}\right]$. *Then* WT *and* WF *decrease monotonically with* x *increasing.*

Theorems 6-8 are illustrated on Fig.4 and can be formulated as the next proposition.

Proposition 2. *Property 2 holds for Aleskerov-Golubenko index and does not hold for Esteban-Ray, Wolfson-Foster and Wang-Tsui indices.*

4.3 A Response to A Random Perturbation (Computational Experiment)

The case of equal groups is very easy for analytical investigation, but it is unlikely to be common case in reality. Therefore, a question about indices sensitivity to small perturbation of equal distribution is appropriate. We estimate this sensitivity using computer modeling.

Equal distribution system was uniformly perturbed in several ways: only sizes or only positions of groups or both were changed. Computer modeling

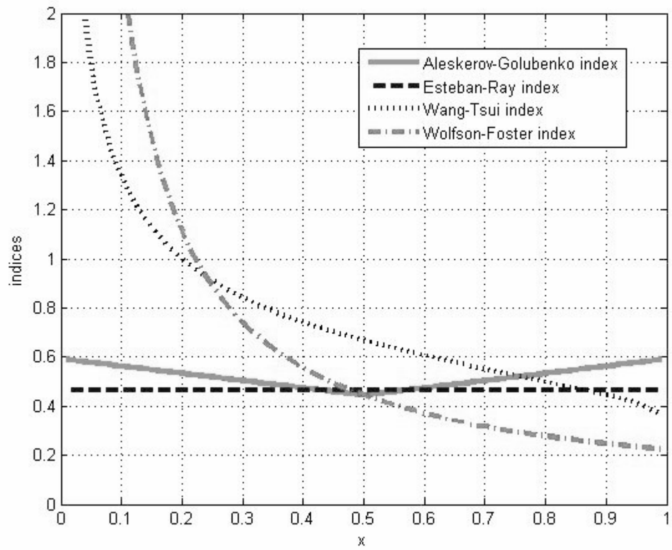

Figure 4. Behavior of indices depending on coordinate of the middle group x in the case of three equal groups ($\alpha = 1.6$, $r = 0.5$). The graph is obtained by computer simulations.

was carried out in Matlab, for number of groups from 2 to 15. The magnitude of perturbation was always $\delta = 10\%$ from initial value. There were 10,000 iterations for each n and value of $\Delta I = |I - I_0| \cdot \frac{100\%}{I_0}$ was calculated on each iteration (here I_0 is initial index value, I - perturbed index value). Final value of ΔI calculated as an average over all iterations.

Results of modeling are presented in Tables 1-3.

Tables 1-3 shows that sensitivity of Aleskerov-Golubenko and Esteban-Ray indices is less than two percent in most cases. In addition, sensitivity of both indices is decreasing monotonically with n increasing except the cases of two and three equal groups. Comparison of Tables 1 and 2 suggests that both indices are more sensitive to perturbation of sizes of groups than its positions.

The bi-polarization indices behavior is radically different. Firstly, an important feature of Wang-Tsui and Wolfson-Foster indices is that their

Table 1. ΔER, ΔAG, ΔWT and ΔWF in the case of equal distribution with perturbed sizes of groups. $\alpha = 1.6$, $r = 0.5$

Number of groups	ΔAG,%	ΔER,%	ΔWT,%	ΔWF,%
2	0.33	0.17	26.19	73.11
3	2.49	2.03	2.25	2.62
4	1.27	1.85	15.73	30.08
5	1.47	1.72	1.24	5.32
6	1.09	1.60	9.51	14.79
7	1.13	1.50	0.89	7.47
8	0.95	1.42	6.95	9.42
9	0.96	1.35	0.72	8.99
10	0.86	1.29	5.41	7.06
11	0.85	1.25	0.61	10.01
12	0.78	1.19	4.42	9.93
13	0.77	1.67	0.54	10.55
14	0.73	1.13	3.76	8.31
15	0.71	1.09	0.49	10.74

sensitivity depends on whether the number of groups is odd or even. This is consistent with Theorem 2. Secondly, bi-polarization indices are extremely sensitive to a small perturbation: sensitivity can reach 70-90% if the number of groups is small. This feature of bi-polarization indices should be taken into account in real life problems.

5 Conclusions

In our attempt to identify differences and key features of main polarization indices we introduced two new properties for polarization measures and then verified them for four known indices of polarization.

Property 1 deals with a simple case of n equal groups. This property helps to understand important differences between polarization and bi-polarization indices. In the case of equal distribution of n groups all indices decrease with n increasing, but while Esteban-Ray and Aleskerov-Golubenko indices tend to zero as n tends to infinity, Wolfson-Foster and Wang-Tsui indices tend to some positive value. Difference between indices

Table 2. ΔER, ΔAG, ΔWT and ΔWF in the case of equal distribution with perturbed positions of groups. $\alpha = 1.6$, $r = 0.5$

Number of groups	ΔAG,%	ΔER,%	ΔWT,%	ΔWF,%
3	1.10	0	86.63	3.31
4	1.17	0.47	8.30	2.83
5	0.90	0.46	30.21	3.34
6	0.73	0.42	4.39	1.72
7	0.60	0.37	17.53	2.76
8	0.50	0.33	3.24	1.18
9	0.43	0.29	12.08	2.32
10	0.37	0.26	2.66	0.87
11	0.32	0.23	9.13	1.96
12	0.29	0.21	2.29	0.57
13	0.26	0.19	7.27	1.74
14	0.23	0.17	2.02	0.48
15	0.21	0.16	6.02	1.54

is not only in their limiting values, but also in the type of decreasing. The bi-polarization indices decrease non-monotonically and depend on whether the number of groups is odd or even.

In Property 2 the same case is considered, but the number of groups is odd, and the coordinate of the middle group $x \neq 0.5$. In this case, the following features of indices are found

- the Esteban-Ray index value is constant regardless of x;
- the Aleskerov-Golubenko index value is minimal if $x = 0.5$ and increasing with the middle group coordinate moving towards one of the extreme groups;
- the Wolfson-Foster and Wang-Tsui indices monotonically decrease with x increasing.

A sensitivity of the indices to small random perturbations was studied. It was shown that Esteban-Ray and Aleskerov-Golubenko indices have low sensitivity (about 1% for 10% perturbation), and Wang-Tsui and Wolfson-

Table 3. ΔER, ΔAG, ΔWT and ΔWF in the case of equal distribution with perturbed sizes and positions of groups. $\alpha = 1.6$, $r = 0.5$

Number of groups	ΔAG,%	ΔER,%	ΔWT,%	ΔWF,%
2	0.33	0.17	26.19	73.11
3	2.68	2.00	86.64	4.17
4	1.70	1.86	8.28	30.01
5	1.70	1.76	30.21	6.18
6	1.31	1.63	4.44	14.69
7	1.28	1.54	17.50	7.91
8	1.09	1.45	3.25	9.54
9	1.04	1.38	12.08	9.30
10	0.94	1.31	2.66	7.19
11	0.91	1.27	9.12	9.99
12	0.83	1.21	2.29	8.97
13	0.82	1.18	7.27	1.60
14	0.76	1.15	2.04	8.40
15	0.75	1.13	6.02	10.79

Foster indices are extremely sensitive (sensitivity reaches 80% for 10% perturbation in the case of small number of groups).

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7 Appendix. Proof of Theorems

Proof of Theorem 1

1. Let us prove the first statement of the Theorem 1: $\forall n\ ER_{n+1} < ER_n$. In the case of equal distribution of n groups $p_i = \dfrac{1}{n}$ and $y_i = \dfrac{i-1}{n-1}$ $\forall i = 1, 2, ...n$. Then according to (1)

$$ER_n = \frac{2^{1+\alpha}}{n^{2+\alpha}(n-1)} \sum_{i,j=1}^{n} |i-j| = \frac{2^{1+\alpha}}{3} \frac{n+1}{n^{1+\alpha}} \tag{6}$$

Hence

$$\frac{ER_{n+1}}{ER_n} = \left(\frac{n}{n+1}\right)^{\alpha} \frac{n(n+2)}{(n+1)^2} < \frac{n(n+2)}{(n+1)^2} = 1 - \frac{1}{(n+1)^2} < 1$$

2. Let us prove the second statement of the Theorem 1: $\forall n\ AG_{n+1} < AG_n$. In the case of equal distribution $c = \dfrac{1}{2}$, therefore, according to (3)

$$AG_n = \frac{4}{n^2} \sum_{i=1}^{n} \left| \frac{i-1}{n-1} - \frac{1}{2} \right| = \begin{cases} \dfrac{1}{n-1}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{n^2}, & \text{if } n \text{ is odd} \end{cases} \tag{7}$$

Let us show $AG_{n+1} < AG_n$. If n is even and $n+1$ is odd, then

$$AG_n - AG_{n+1} = \frac{1}{n-1} - \frac{n+2}{(n+1)^2} = \frac{n+3}{(n-1)(n+1)^2} > 0$$

If n is odd and $n+1$ is even, then

$$AG_n - AG_{n+1} = \frac{n+1}{n^2} - \frac{1}{n} = \frac{1}{n^2} > 0$$

Hence $\forall n\ AG_n > AG_{n+1}$. $\square$

Proof of Theorem 2

1. Let us prove the first statement of the Theorem 2: $\forall n > 1$, $\forall r \in (0, 1)$ $WT_{2n} > WT_{2n+2}$, $WT_{2n-1} < WT_{2n+1}$, $WT_{2n} > WT_{2n+1}$.

(a) Let us prove that $WT_{2n} > WT_{2n+2}$. In the case of equal distribution of $2n$ groups $p_i = \frac{1}{2n}$, $y_i = \frac{i-1}{2n-1}$, $m = \frac{1}{2}$, hence according to (5)

$$WT_{2n} = \frac{2^r}{2n} \sum_{i=1}^{2n} \left| \frac{i-1}{2n-1} - \frac{1}{2} \right|^r \tag{8}$$

Because of the symmetry of considered distribution

$$WT_{2n} = \frac{1}{n(2n-1)^r} \sum_{i=1}^{n} (2n-2i+1)^r \tag{9}$$

Note that WT_{2n} is non-increasing function of $r \in (0,1)$:

$$\frac{dWT_{2n}}{dr} = \frac{1}{n(2n-1)^r} \sum_{i=1}^{n} (2n-2i+1)^r \ln \frac{2n-2i+1}{2n-1} \leq 0,$$

and also

$$WT_{2n}\Big|_{r=0} = 1 \tag{10}$$

Let us compare the values of WT_{2n} and WT_{2n+2} for $r = 1$. According to (9)

$$WT_{2n+2} = WT_{2(n+1)} = \frac{1}{(n+1)(2n+1)^r} \sum_{i=1}^{n+1} (2n-2i+3)^r \tag{11}$$

Hence

$$WT_{2n}\Big|_{r=1} = \frac{1}{n(2n-1)} \sum_{i=1}^{n} (2n-2i+1) = \frac{n}{2n-1} \tag{12}$$

$$WT_{2n+2}\Big|_{r=1} = \frac{1}{(n+1)(2n-1)} \sum_{i=1}^{n+1} (2n-2i+3) = \frac{n+1}{2n+1} \tag{13}$$

So, $\forall n \; WT_{2n}\Big|_{r=1} > WT_{2n+2}\Big|_{r=1}$. Therefore, according to (10) and monotonic non-increasing of WT_{2n} on $r \in (0,1)$, $WT_{2n} > WT_{2n+2}$ holds for $r \in (0,1)$ for every n.

(b) Let us prove $\forall n \; WT_{2n-1} < WT_{2n+1}$. According to (5)

$$WT_{2n-1} = \frac{2^r}{2n-1} \sum_{i=1}^{2n-1} \left| \frac{i-1}{2n-2} - \frac{1}{2} \right|^r \tag{14}$$

Because of the symmetry of considered distribution

$$WT_{2n-1} = \frac{2}{(2n-1)(n-1)^r} \sum_{i=1}^{n} (n-i)^r - \frac{1}{2n-1} \left(\frac{n}{n-1} \right)^r \tag{15}$$

Note that WT_{2n-1} is decreasing function of $r \in (0,1)$. Indeed,

$$\frac{dWT_{2n-1}}{dr} = \frac{2}{(2n-1)(n-1)^r} \sum_{i=1}^{n-1} (n-i)^r \ln \frac{n-i}{n-1} -$$

$$- \frac{1}{2n-1} \left(\frac{n}{n-1} \right)^r \ln \frac{n}{n-1} < 0,$$

and also

$$WT_{2n-1} \Big|_{r=0} = 1 \tag{16}$$

Let us compare the values of WT_{2n-1} and WT_{2n+1} for $r = 1$. According to (15)

$$WT_{2n-1} \Big|_{r=1} = \frac{n(n-2)}{(2n-1)(n-1)}, \tag{17}$$

$$WT_{2n+1} \Big|_{r=1} = \frac{n^2-1}{n(2n+1)}, \tag{18}$$

From (17), (18) $WT_{2n+1}\Big|_{r=1} > WT_{2n-1}\Big|_{r=1}$ follows. Hence, according to (16) and monotonic decreasing of WT_{2n-1} on $r \in (0,1)$, $WT_{2n+1} > WT_{2n-1}$ holds for $r \in (0,1)$ and every n.

(c) Let us prove $WT_{2n} > WT_{2n+1}$. According to (12) and (18) $WT_{2n}\Big|_{r=1} > WT_{2n+1}\Big|_{r=1}$. Hence, because of (A5), (A11) and monotonic non-increasing of WT_{2n} and WT_{2n+1} on $r \in (0,1)$, the statement holds.

2. Let us prove the second statement of Theorem 2: $\forall n > 1$ $WF_{2n} >$ $> WF_{2n-1} > WF_{2n+1}$, and $WF_{2n} > WF_{2n+2}$. In the case under consideration $\mu = m = 0.5$, so according to (4) $\forall n > 1$ and well-known equation for Gini index

$$WF_n = T_n - G_n = \begin{cases} \dfrac{n^2+2}{6n(n-1)}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{6n}, & \text{if } n \text{ is odd.} \end{cases} \tag{19}$$

Hence according to (19)

$$WF_{2n} = \frac{2n^2+1}{6n(2n-1)} \tag{20}$$

$$WF_{2n+2} = \frac{2(n+1)^2+1}{3(2n+1)^2} \tag{21}$$

$$WF_{2n-1} = \frac{n}{3(2n-1)} \tag{22}$$

$$WF_{2n+1} = \frac{n+1}{3(2n+1)} \tag{23}$$

(a) Let us show $WF_{2n} > WF_{2n-1}$. According to (20) and (22)

$$WF_{2n} = \frac{2n^2+1}{6n(2n-1)} = \frac{n+\frac{1}{2n}}{3(2n-1)} > \frac{n}{3(2n-1)} = WF_{2n-1}$$

(b) Let us show $WF_{2n-1} > WF_{2n+1}$. According to (22) and (23)

$$WF_{2n-1} = \frac{n}{3(2n-1)} = \frac{n+\frac{2n}{2n-1}}{3(2n-1)} > \frac{n+1}{3(2n+1)} = WF_{2n+1}$$

(c) Let us show $WF_{2n} > WF_{2n+2}$. According to (20) and (21)

$$WF_{2n} = \frac{2n^2+1}{6n(2n-1)} = WF_{2n+2} \cdot \frac{2n^2+1}{2n(2n-1)} \frac{(2n+1)^2}{2(n+1)^2+1} > WF_{2n+2}$$

The statement is proved. $\square$

Proof of Theorem 3

1. Let us prove the first statement of Theorem 3

$$\forall \alpha \in (0, 1.6]\ ER_n, AG_n \to 0, \text{with } n \to \infty$$

According to (6)

$$\lim_{n\to\infty} ER_n = \frac{2^{1+\alpha}}{3} \lim_{n\to\infty} \frac{n+1}{n^{\alpha+1}} = 0$$

because $\alpha > 0$.

According to (7)

$$\lim_{n\to\infty} AG_n = \lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

The statement is proved.

2. Let us prove the second statement of Theorem 3

$$\forall r \in (0,1)\ WT_n \to f(r) > 0, \quad n \to \infty$$

According to (5) in the case of equal distribution

$$WT_n = 2^r \sum_{i=1}^{n} \frac{1}{n} \left| \frac{i-1}{n-1} - \frac{1}{2} \right|^r \tag{24}$$

Because of $r \in (0,1)$

$$WT_n > \frac{2^r}{n} \left(\sum_{i=1}^{n} \frac{1}{n} \left| \frac{i-1}{n-1} - \frac{1}{2} \right| \right)^r = \begin{cases} \frac{2^r n}{4(n-1)}, & \text{if } n \text{ is even} \\ \frac{2^r (n+1)}{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

Because of $\lim_{n\to\infty} \frac{2^r n}{4(n-1)} = \lim_{n\to\infty} \frac{2^r(n+1)}{4n} = 2^{r-2}$

$$\lim_{n\to\infty} WT_n > 2^{r-2}$$

3. The third statement of Theorem follows from (19)

$$\lim_{n\to\infty} WF_n = \lim_{n\to\infty} \frac{n^2+2}{6n(n-1)} = \lim_{n\to\infty} \frac{n+1}{6n} = \frac{1}{6}$$

□

Proof of Theorem 4

According to Theorem 1, AG_n and ER_n decrease monotonically with n growing. Therefore, $\Delta(n) = AG_n - ER_n$ has a constant sign on $\mathbb{N}$, or there is n^* such that $\Delta(n)$ changes its sign at $n = n^*$.

Assume that $\forall n\ \Delta_n \geq 0$, i.e. $AG_n \geq ER_n$. Then according to (6) and (7)

$$\frac{1}{n-1} > \frac{n+1}{n^2} \geq \left(\frac{2}{n}\right)^{1+\alpha} \frac{n+1}{3} \Rightarrow n > \left(\frac{2^{1+\alpha}}{3}\right)^{\frac{1}{1+\alpha}},$$

and this does not hold for $\alpha = 1.6, n = 2$.

Assume that $\forall n\ \Delta_n \leq 0 \Rightarrow AG_n \leq ER_n$. Then

$$\left(\frac{2}{n}\right)^{1+\alpha} \frac{n+1}{3} \geq \frac{1}{n-1} > \frac{n+1}{n^2} \Rightarrow \frac{n^{1+\alpha}}{n^2-1} \leq \frac{2^{1+\alpha}}{3},$$

and this does not hold for $\alpha = 1.6, n = 4$.

Hence there is n^* such that Δ_n changes its sign at $n = n^*$. From $AG_3 = \left(\frac{2}{3}\right)^2$, $ER_3 = 2\left(\frac{2}{3}\right)^{2+\alpha} \Rightarrow AG_3 < ER_3$, it follows that for $3 \leq n \leq \leq n^*$, $\Delta_n \leq 0 \Rightarrow$ for $n > n^*$, $\Delta_n > 0$. Theorem is proved. $\square$

Proof of Theorem 5

Let us show that $\forall n \geq 1\ WT_{2n} > WF_{2n}$. According to (9) and (20)

$$WT_{2n} = \frac{1}{2n(n-1)^r} \sum_{i=1}^{n} (2n-2i+1)^r > \frac{3\sum_{i=1}^{n}(2n-2i+1)^r}{3n(2n-1)} >$$

$$> \frac{3n(n+1)}{6n(2n-1)} > \frac{2n^2+1}{6n(2n-1)} = WF_{2n}$$

Let us show that $\forall n \geq 1\ WT_{2n+1} > WF_{2n+1}$ According to (15) and (23)

$$WT_{2n+1} = \frac{2}{2n(n+1)n^r} \sum_{i=1}^{n+1} (n-i+1)^r - \frac{1}{2n+1}\left(\frac{n+1}{n}\right)^r >$$

$$> \frac{(n+1)(n+2)-(n+1)^r}{n^r(2n+1)} = \frac{(n+1)^2}{3n(2n+1)} > \frac{n+1}{3(2n+1)} = WF_{2n+1}$$

Theorem is proved. $\square$

Proof of Theorem 6

In the case of n equal groups $\forall i = 1, ...n\ p_i = \dfrac{1}{n}$. Then according to (1)

$$ER = \frac{2^{1+\alpha}}{n^{2+\alpha}} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \tag{25}$$

Let us express $S = \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|$ through $S_{equal} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \dfrac{i-1}{n-1} - \dfrac{j-1}{n-1} \right|$.

In the case under study $\forall i \neq \frac{n+1}{2}\ y_i = \dfrac{i-1}{n-1}$. Then

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| = S_{equal} - 2 \sum_{i=1}^{n} \left| \frac{i-1}{n-1} - \frac{1}{2} \right| + 2 \sum_{i=1}^{n} \left| \frac{i-1}{n-1} - x \right| - 2 \left(x - \frac{1}{2} \right) =$$

$$= S_{equal} - 2 \frac{n+1}{2(n-1)} \frac{n-1}{2} + 2 \frac{n+1}{2(n-1)} \frac{n-1}{2} + 2x - 1 - 2 \left(x - \frac{1}{2} \right) = S_{equal}$$

Thereby $S = S_{equal}$ regardless of middle group position and $ER = ER_{equal} =$

$= \left(\dfrac{2}{n} \right)^{1+\alpha} \dfrac{n+1}{3}$. Theorem is proved. $\square$

Proof of Theorem 7

In the case of n equal groups $\forall i = 1, ...n\ p_i = \dfrac{1}{n}$. Then according to (3)

$$AG = \frac{4}{n^2} \sum_{i=1}^{n} |y_i - c| \tag{26}$$

In the case under study $\forall i \neq \frac{n+1}{2}\ y_i = \dfrac{i-1}{n-1}$. Hence

$$AG = \frac{4}{n^2} \left(\sum_{i=1}^{\frac{n+1}{2}} (c - y_i) + |x - c| + \sum_{i=\frac{n+1}{2}+1}^{n} (y_i - c) \right) = \frac{4}{n^2} \left(\frac{n+1}{4} + |x - c| \right),$$

where

$$c = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{2} - \frac{1}{2n} + \frac{x}{n}$$

Thus Aleskerov-Golubenko index value

$$AG = \frac{4}{n^2}\left(\frac{n+1}{4} + \frac{n-1}{n}\left|x - \frac{1}{2}\right|\right) \tag{27}$$

Then the derivative of AG with respect to x is

$$\frac{dAG}{dx} = \frac{4(n-1)}{n^3} sign\left(x - \frac{1}{2}\right) \tag{28}$$

So, AG takes minimal value at $x = \frac{1}{2}$, for $x > \frac{1}{2}$ AG increases $\left(\frac{dAG}{dx} > 0\right)$, and for $x < \frac{1}{2}$ AG decreases $\left(\frac{dAG}{dx} < 0\right)$. According to (27)

$$AG\Big|_{x=\frac{n-3}{2(n-1)}} = AG\Big|_{x=\frac{n+1}{2(n-1)}} = \frac{4}{n^2}\left(\frac{n+1}{4} + \frac{1}{n}\right),$$

therefore, Aleskerov-Golubenko index value is maximal at $x = \frac{n-3}{2(n-1)}$ and $x = \frac{n+1}{2(n-1)}$. Theorem is proved. □

Proof of Theorem 8

1. Let us prove Theorem 8 for Wang-Tsui index. In the case of n equal groups $p_i = \frac{1}{n}$, $m = x$. Then according to (5)

$$WT = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{y_i}{x} - 1\right|^r \tag{29}$$

$$\frac{dWT}{dx} = -\frac{r}{nx^2}\sum_{i=1}^{n}\left|\frac{y_i}{x} - 1\right|^{r-1} y_i \cdot sign|y_i - x| \tag{30}$$

Let us show that $S = \sum_{i=1}^{n}\left|\frac{y_i}{x} - 1\right|^{r-1} y_i \cdot sign|y_i - x| > 0$.

In the case under study $\forall i \neq \frac{n+1}{2}$ $y_i = \frac{i-1}{n-1}$.

$$S = -\sum_{i=1}^{\frac{n-1}{2}}\left(x - \frac{i-1}{n-1}\right)^{r-1}\frac{i-1}{n-1} + \sum_{i=\frac{n+1}{2}}^{n}\left(\frac{i-1}{n-1} - x\right)^{r-1}\frac{i-1}{n-1} =$$

$$= \sum_{i=1}^{\frac{n-1}{2}} \left(\left(\frac{i + \frac{n-1}{2} - 1}{n-1} - x \right)^{r-1} \frac{i + \frac{n-1}{2} - 1}{n-1} - \left(x - \frac{i-1}{n-1} \right)^{r-1} \frac{i-1}{n-1} \right) +$$

$$+(1-x)^r$$

Let $x - \frac{i-1}{n-1} = \delta$. Because of $0 \le \delta \le \frac{1}{2}$ i-th summand of S is

$$\left(\frac{i + \frac{n-1}{2} - 1}{n-1} - x \right)^{r-1} \frac{i + \frac{n-1}{2} - 1}{n-1} - \left(x - \frac{i-1}{n-1} \right)^{r-1} \frac{i-1}{n-1} =$$

$$= \left(\frac{1}{2} - \delta \right)^{r-1} \left(\frac{i-1}{n-1} + \frac{1}{2} \right) - \delta^{r-1} \frac{i-1}{n-1} \ge \left(\frac{1}{2} \right)^{r-1} \left(\frac{i-1}{n-1} + \frac{1}{2} \right) > 0$$

Therefore, $S > 0 \Rightarrow \frac{dWT}{dx} = -\frac{r}{nx^2} S < 0 \Rightarrow WT$ monotonically decreases with x increasing. The statement is proved.

2. Let us prove Theorem 8 for Wolfson-Foster index (4). In considered case

$$T = \frac{3(n+1)}{4n} + \frac{x}{n} \tag{31}$$

$$G = \frac{n+1}{6nc} \tag{32}$$

where $c = \frac{1}{2} - \frac{1}{2n} + \frac{x}{n}$ - the mean value of the attribute. If middle group position is x, the median value of attribute $m = x$, and according to (4)

$$WF = \frac{3(n+1)}{4n} \cdot \frac{c}{x} - \frac{c}{n} - \frac{n+1}{6n} \cdot \frac{1}{x} \tag{33}$$

Then

$$\frac{dWF}{dx} = \frac{3(n+1)}{4n} \cdot \frac{c'_x x - c}{x^2} - \frac{c'_x}{n} + \frac{n+1}{6n} \cdot \frac{1}{x^2} = -\frac{(n+1)(5n-9)}{24n^2x^2} - \frac{1}{n^2} < 0,$$

Hence WF monotonically decrease with x increasing. Theorem is proved. $\square$

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Задачи оценки поляризованности возникают в разных областях социологии, политологии и экономики. В работе проводится исследование четырех известных индексов поляризованности: Эстебана – Рэя, Алескерова – Голубенко, Вольфсона – Фостера и Ванга – Цуи. Сформулированы новые свойства для индексов поляризованности, выявлены важные особенности и отличия в поведении индексов.

Ключевые слова: индексы поляризованности, поляризованность, биполяризованность, индекс Эстебана – Рэя, индекс Алескерова – Голубенко, индекс Вольфсона – Фостера, индекс Ванга – Цуи

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(*на английском языке*)

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