Isomonodromic τ-functions and W_N-conformal blocks

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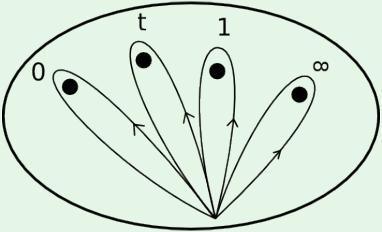
Isomonodromic deformations

Fuchsian system

$$\frac{d}{dz}\Phi(z) = \sum_{\nu=1}^{n} \frac{A_\nu}{z - z_\nu}\Phi(z) = A(z)\Phi(z) \qquad \sum_\nu A_\nu = 0$$

Monodromies



$$\gamma_\nu : \Phi(z) \mapsto \Phi(z) M_\nu$$

$$\Phi(z) = \left(1 + O(z - z_\nu)\right)(z - z_\nu)^{A_\nu} C_\nu$$

$$M_\nu = C_\nu^{-1} e^{2\pi i A_\nu} C_\nu \sim e^{2\pi i A_\nu}$$

Riemann-Hilbert problem

Connection

$$\{A_\nu\}, \quad A_\nu \sim \boldsymbol{\theta}_\nu$$

$$\sum_{\nu=1}^{n} A_\nu = 0$$

$$\Phi(z) \mapsto g^{-1}\Phi(z)$$

$$g : A_\nu \mapsto g^{-1} A_\nu g$$

$$\{A_\nu\}/SL_N$$

Monodromies

$$\{M_\nu\}, \quad M_\nu \sim e^{2\pi i \boldsymbol{\theta}_\nu}$$

$$\prod_{\nu=1}^{n} M_\nu = 1$$

$$\Phi(z) \mapsto \Phi(z) h$$

$$M_\nu \mapsto h^{-1} M_\nu h$$

$$\{M_\nu\}/SL_N = \mathcal{M}_n^{\mathfrak{sl}_N}(\boldsymbol{\theta}_\nu)$$

$$\{A_\nu\}/SL_N \leftrightarrow \{M_\nu\}/SL_N$$

Isomonodromic deformations

Gauge transformation

$$\Phi(z) \mapsto \left(1 + \epsilon \frac{A_\nu}{z - z_\nu}\right) \Phi(z)$$

$$A(z) \mapsto A(z) + \epsilon \frac{A_\nu}{(z - z_\nu)^2} - \epsilon \left[\frac{A_\nu}{z - z_\nu}, A(z)\right]$$

$$z_\nu \mapsto z_\nu + \epsilon\,, \quad A_{\mu\neq\nu} \mapsto A_\mu + \epsilon \frac{[A_\nu, A_\mu]}{z_\nu - z_\mu}\,, \quad A_\nu \mapsto A_\nu - \epsilon \sum_{\mu\neq\nu} \frac{[A_\nu, A_\mu]}{z_\nu - z_\mu}$$

Schlesinger system

$$\frac{\partial A_\mu}{\partial z_\nu} = \frac{[A_\mu, A_\nu]}{z_\mu - z_\nu}\,, \qquad \frac{\partial A_\nu}{\partial z_\nu} = -\sum_{\mu\neq\nu} \frac{[A_\mu, A_\nu]}{z_\mu - z_\nu}$$

τ-function

$$\frac{\partial}{\partial z_i} \operatorname{res}_{z_j} \operatorname{tr} A(z)^2 = \frac{\partial}{\partial z_j} \operatorname{res}_{z_i} \operatorname{tr} A(z)^2$$

$$\frac{1}{2} \operatorname{res}_{z_j} \operatorname{tr} A(z)^2 = \frac{\partial}{\partial z_i} \log \tau(\{z_\nu\})$$

$$\frac{1}{2} \operatorname{tr} A(z)^2 \tau(\{z_\nu\}) = \left(\sum_{\nu=1}^{n} \frac{\Delta_\nu}{(z - z_\nu)^2} + \frac{\partial_\nu}{z - z_\nu} \right) \tau(\{z_\nu\})$$

$$\Delta_\nu = \frac{1}{2} \operatorname{tr} A_\nu^2$$

Actually:

$$\tau = \tau(\{z_\nu\}, \{M_\nu\}/SL_N)$$

Moduli spaces of flat connections

$G, \mathfrak{g}$ – Lie group, algebra
$H, \mathfrak{h}$ – Cartan torus

3 points

$$\mathcal{M}_3^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \{(M_1, M_2, M_3)\}/G = \{(M_1, M_2, e^{2\pi i \boldsymbol{\theta}_3})\}/H\,,$$

4 points

$$\mathcal{M}_4^{\mathfrak{g}} = \{(M_1, M_2, M_3, M_4)\}/G, \quad M_1 M_2 = S \sim e^{2\pi i \sigma}$$

$$\mathcal{M}_4^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2; \boldsymbol{\sigma}; \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) = \{(M_1, M_2, M_3, M_4)\}/G =$$
$$= \{(M_1, M_2, e^{-2\pi i \boldsymbol{\sigma}}), (e^{2\pi i \boldsymbol{\sigma}}, M_3, M_4)\}/H$$

Extra torus action (twist): $h : \{(M_1, M_2, e^{-2\pi i \boldsymbol{\sigma}}), (e^{2\pi i \boldsymbol{\sigma}}, M_3, M_4)\} \mapsto$
$$\mapsto \{h^{-1}(M_1, M_2, e^{-2\pi i \boldsymbol{\sigma}})h, (e^{2\pi i \boldsymbol{\sigma}}, M_3, M_4)\}$$

Moduli spaces of flat connection

$$\mathcal{M}_4^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2; \boldsymbol{\sigma}; \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)/H = \{(M_1, M_2, e^{-2\pi i \boldsymbol{\sigma}}), (e^{2\pi i \boldsymbol{\sigma}}, M_3, M_4)\}/H \times H = \\ = \mathcal{M}_3^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, -\boldsymbol{\sigma}) \times \mathcal{M}_3^{\mathfrak{g}}(\boldsymbol{\sigma}, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)$$

$$\mathcal{M}_4^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2; \boldsymbol{\sigma}; \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) \approx \mathcal{M}_3^{\mathfrak{g}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, -\boldsymbol{\sigma}) \times H_{\boldsymbol{\beta}} \times \mathcal{M}_3^{\mathfrak{g}}(\boldsymbol{\sigma}, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)$$

Coordinates

- $\boldsymbol{\sigma}$ – gluing parameters
- $\boldsymbol{\beta}$ – relative twist parameters depending on $\operatorname{tr} M_1 M_3^{-1}$, $\operatorname{tr} M_1^{-1} M_3$
- Coordinates on $\mathcal{M}_3$: twist-invariants like $\operatorname{tr} M_1^{-1} M_3$, $\operatorname{tr} M_1 M_3^{-1}$

Coordinates for SL_3

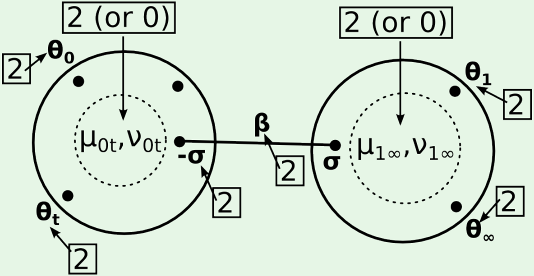
$e_1 = diag(\frac{N-1}{N}, -\frac{1}{N}, ..., -\frac{1}{N})$ – weight of the 1-st fundamental representation.

3 points

$$\dim \mathcal{M}_3^{\mathfrak{sl}_3}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = 2$$

$$\dim \mathcal{M}_3^{\mathfrak{sl}_3}(\boldsymbol{\theta}_1, ae_1, \boldsymbol{\theta}_3) = 0$$

General (or degenerate) case



Gamayun-Iorgov-Lisovyy formula for $N = 2$ (2012)

$$\tau(t) = \sum_{n\in\mathbb{Z}} s^n C_n^{(0t)}(\theta_0, \theta_t, \sigma_{0t}) C_n^{(1\infty)}(\theta_1, \theta_\infty, \sigma_{0t}) \times$$
$$\times t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2} \mathcal{B}(\{\theta_i\}, \sigma_{0t}+n; t)$$

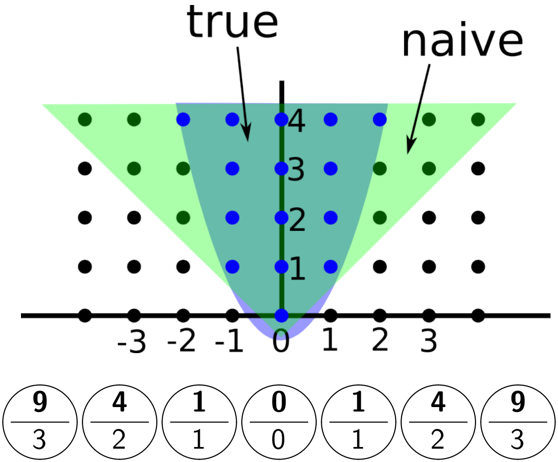
- s – twist parameter
- σ_{0t} – gluing parameter
- $\mathcal{B}(\{\theta_i\}, \sigma_{0t}+n; t) = 1 + \mathcal{B}_1 t + \ldots$ – Virasoro conformal block at $c = 1$

$$C_n^{(0t)}(\theta_0, \theta_t, \sigma_{0t}) C_n^{(1\infty)}(\theta_1, \theta_\infty, \sigma_{0t}) =$$
$$= \frac{\prod\limits_{\epsilon=\pm,\epsilon'=\pm} G(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)) G(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n))}{G(1-2\sigma_{0t}) G(1+2\sigma_{0t})}$$

$G(x)$ – Barnes function. $G(x+1) = \Gamma(x)G(x)$

Proofs by [Iorgov, Teschner, Lisovyy'14] and [Bershtein, Shchechkin'14]

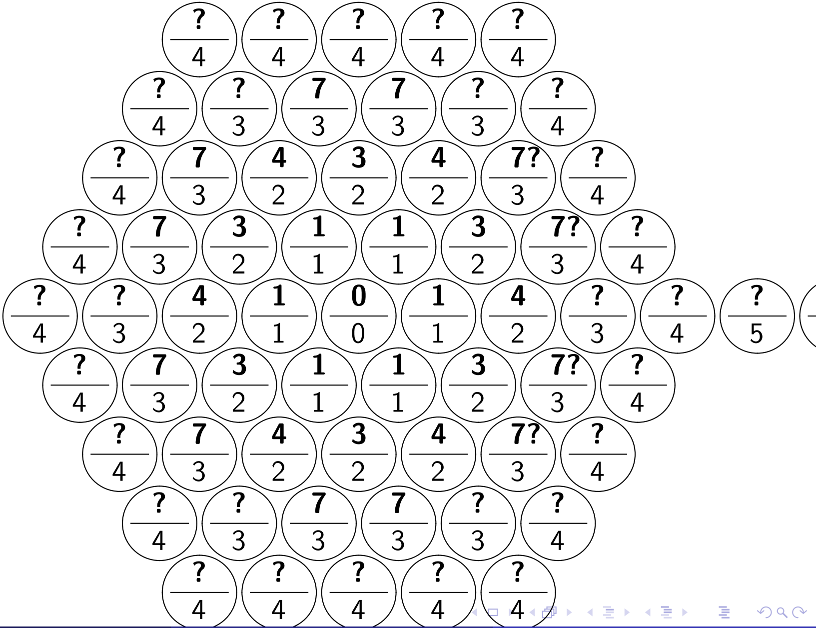
Support of $\tau(t)$ and $A_i(t)$, $\log\tau(t)$



1. Quadratic support of $\tau(t)$
2. Factorization of the structure constants
3. Relative twist enters only s.

$$\log\tau(t) = \chi\log t + \sum_n t^{2\sigma_{0t}n} t^{|n|} c_n \qquad \tau(t) = t^{\chi}\sum_n t^{2\sigma_{0t}n} t^{n^2} c'_n$$

Generalization to SL_3. Parabolic support.



Generalzation to SL_3

Q – $\mathfrak{sl}_3$ root lattice

$$\tau(t) = \sum_{\mathbf{w}\in Q} e^{(\boldsymbol{\beta},\mathbf{w})} C^{(0t)}_{\mathbf{w}}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_t, \boldsymbol{\sigma}_{0t}, \mu_{0t}, \nu_{0t}) C^{(1\infty)}_{\mathbf{w}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_\infty, \boldsymbol{\sigma}_{0t}, \mu_{1t}, \nu_{1t}) \times$$
$$\times t^{\frac{1}{2}(\boldsymbol{\sigma}_{0t}+\mathbf{w},\boldsymbol{\sigma}_{0t}+\mathbf{w}) - \frac{1}{2}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) - \frac{1}{2}(\boldsymbol{\theta}_t,\boldsymbol{\theta}_t)} \mathcal{B}_{\mathbf{w}}(\{\boldsymbol{\theta}_i\}, \boldsymbol{\sigma}_{0t}, \mu_{0t}, \nu_{0t}, \mu_{1\infty}, \nu_{1\infty}; t)$$

Properties:

1. Parabolic support $t^{\frac{1}{2}(\mathbf{w},\mathbf{w})}$.
2. All twist parameters enters only $\boldsymbol{\beta} \in \mathfrak{h}$
3. 3-point functions are factorized
4. The first terms in the expansion of conformal block have the form $\mathcal{B}_0 = 1 + [\alpha + \beta C_1(\mu_{0t}, \nu_{0t}) + \gamma \tilde{C}_1(\mu_{1\infty}, \nu_{1\infty}) + \delta C_1(\mu_{0t}, \nu_{0t}) \tilde{C}_1(\mu_{1\infty}, \nu_{1\infty})] t + \ldots$

Generalzation to SL_N. Degenerate case

$$\mathcal{B}_{\mathbf{w}}(\boldsymbol{\theta}_\infty, a_1, \boldsymbol{\sigma}, a_t, \boldsymbol{\theta}_0; t) = \mathcal{B}(\boldsymbol{\theta}_\infty, a_1, \boldsymbol{\sigma} + \mathbf{w}, a_t, \boldsymbol{\theta}_0; t)$$

$\mathcal{B}(\boldsymbol{\theta}_\infty, a_1, \boldsymbol{\sigma} + \mathbf{w}, a_t, \boldsymbol{\theta}_0; t)$ – conformal block of the W_N algebra at $c = N - 1$

$$C_{\mathbf{w}}^{(0t)}(\boldsymbol{\theta}_0, a_t, \boldsymbol{\sigma}) C_{\mathbf{w}}^{(1\infty)}(\boldsymbol{\sigma}, a_1, \boldsymbol{\theta}_\infty) =$$
$$= \frac{\prod_{ij} G[1 - \frac{a_t}{N} + (e_i, \boldsymbol{\theta}_0) - (e_j, \boldsymbol{\sigma} + \mathbf{w})] G[1 - \frac{a_1}{N} + (e_i, \boldsymbol{\sigma} + \mathbf{w}) + (e_j, \boldsymbol{\theta}_\infty)]}{\prod\limits_i G[1 + (\alpha_i, \boldsymbol{\sigma} + \mathbf{w})]}$$

e_i – weights of the first fundamental representation
α_i – all roots of $\mathfrak{sl}_N$
Proof by [G., Iorgov, Lisovyy], unpublished.

Idea of generalization

$$\begin{array}{ccccc}
SL_2 & \longrightarrow & \text{degenerate } SL_N & \longrightarrow & \text{general } SL_N \\
\downarrow & & \downarrow & & \downarrow \\
W_2 \text{ conf. blocks} & \longrightarrow & \text{degenerate } W_N \text{ conf. block} & \longrightarrow &
\end{array}$$

Empty place is left for the general W_N conformal block.

W_3 algebra

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T\left(\frac{z+w}{2}\right)}{(z-w)^2} + \text{reg.}\,,$$

$$T(z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{reg.}\,,$$

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{2T\left(\frac{z+w}{2}\right)}{(z-w)^4} +$$

$$+\frac{1}{(z-w)^2}\left(\frac{32}{22+5c}\,\Lambda\left(\frac{z+w}{2}\right) + \frac{1}{20}\partial^2 T\left(\frac{z+w}{2}\right)\right) + \text{reg.}$$

$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z)$

Vertex operator

$$T(z)\phi(w) = \frac{\Delta\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{reg.}$$

$$W(z)\phi(w) = \frac{\mathrm{w}\phi(w)}{(z-w)^3} + \frac{(\mathcal{W}_{-1}\phi)(w)}{(z-w)^2} + \frac{(\mathcal{W}_{-2}\phi)(w)}{z-w} + \text{reg.}$$

Representation

Basis in the representation

$$L_{-m_1} L_{-m_2} \dots L_{-m_k} W_{-n_1} W_{-n_2} \dots |\Delta, \mathrm{w}\rangle$$

$$m_1 \geq m_2 \geq \dots \geq m_k, n_1 \geq n_2 \geq \dots \geq n_k$$

$$W_0 |\Delta, \mathrm{w}\rangle = \mathrm{w} |\Delta, \mathrm{w}\rangle, \quad L_0 |\Delta, \mathrm{w}\rangle = \Delta |\Delta, \mathrm{w}\rangle$$

$$W_{k>0} |\Delta, \mathrm{w}\rangle = 0, \quad L_{k>0} |\Delta, \mathrm{w}\rangle = 0$$

$$\Delta_\nu = \frac{1}{2}(\boldsymbol{\theta}_\nu, \boldsymbol{\theta}_\nu)$$

$$\mathrm{w}_\nu = \sqrt{\frac{3}{2}} \prod_i (\boldsymbol{\theta}_\nu, e_i)$$

Ambiguity in the conformal blocks

$\mathcal{W}_{-1}\phi(z)$ and $\mathcal{W}_{-2}\phi(z)$ are not known.

Extra parameters

$$C_k = \langle \Delta_\infty, \mathrm{w}_\infty | \phi(1) W_{-1}^k | \Delta_0, \mathrm{w}_0 \rangle, \qquad k = 1, 2, \ldots$$

Partly degenerate field

$\boldsymbol{\theta} = a e_1$ – the same as for $\dim \mathcal{M}_3^{\mathfrak{sl}_3} = 0$

$\mathcal{W}_{-1}\phi_{\boldsymbol{\theta}}(z) = \frac{3\mathrm{w}}{2\Delta}\partial\phi_{\boldsymbol{\theta}}(z)$
All C_k are fixed

Resolution of the ambiguity for the general case

Completely degenerate field

$$\phi_{e_1}(z)\phi_{\boldsymbol{\theta}_1}(w) = \sum_k C_{e_1,\boldsymbol{\theta}_1}^{\boldsymbol{\theta}_1+e_k} \cdot (z-w)^{(\boldsymbol{\theta}_1,e_k)} \left(\phi_{\boldsymbol{\theta}_1+e_k}(w) + \text{descendants}\right)$$

Solution of the Riemann-Hilbert problem:

$$t^{(e_1,e_1)}\langle -\boldsymbol{\theta}_\infty|\phi_{\boldsymbol{\theta}_1}(1)\phi_{e_1}(t)|\boldsymbol{\theta}_0 - e_k\rangle$$

$$\langle -\boldsymbol{\theta}_\infty|\phi_{\boldsymbol{\theta}_1}(1)P_n\phi_{e_1}(t)|\boldsymbol{\theta}_0 - e_k\rangle = \sum_{\vec{Y}}\langle -\boldsymbol{\theta}_\infty|\phi_{\boldsymbol{\theta}_1}(1)\widetilde{|\vec{Y}\rangle}\langle\vec{Y}|\phi_{e_1}(t)|\boldsymbol{\theta}_0 - e_k\rangle$$

All $\langle\boldsymbol{\theta}_0 \widetilde{- e_k + e_n}, \vec{Y}|\phi_{e_1}(t)|\boldsymbol{\theta}_0 - e_k\rangle$ are known $\Rightarrow$ system of the linear equations for C_k resulting in $C_k = C_k(\mu,\nu)$.

From the conceptual point of view we define $\phi_{\boldsymbol{\theta},\mu,\nu}(z)$ as the eigenvector of Verlinde loop operators.

Unsolved problems

1. To find the general solution of the 3-point 3×3 Riemann-Hilbert problem.

Thank you for your attention!