Game theoretic aspects of of matching problems under preferences (2nd talk)

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Summer school on matchings
Moscow
5-8 October 2015

Outline

- Matching with couples
- Stable fractional solutions for capacitated NTU-games
- Scarf's algorithm as a new heuristic
- Matching with payments

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National Resident Matching Program - Konqueror

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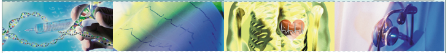

The National Resident Matching Program (NRMP) is a private, not-for-profit corporation established in 1952 to provide a uniform date of appointment to positions in graduate medical education (GME) in the United States.

News from the NRMP!

New>NRMP TO IMPLEMENT MATCH WEEK CHANGES

The NRMP Board of Directors has voted to proceed with changes to Match Week 2012. A new Supplemental Offer and Acceptance Program will be implemented for unmatched applicants and unfilled programs.

New>LARYNGOLOGY JOINS THE NRMP!

The NRMP is pleased to welcome Laryngology as a new fellowship match for the 2012 appointment year. Sponsored by the American Laryngological Association (ALA), the Laryngology Fellowship Match will open for registration on September 29, 2010 with Match Day on February 2, 2011. For more information about the Laryngology Fellowship Match, including the Schedule of Dates, click on Fellowship Matches at the top of this page or contact our Helpdesk Specialists toll free at 1-866-617-5834.

MEDICAL GENETICS JOINS THE NRMP

To participate in a NRMP match, click Register/Login above.

Main Residency Match

Registration for the 2011 Match opens on August 15th for applicants and September 1st for institutions and programs.

The 2010 Main Residency Match was the largest in NRMP history, encompassing more than 37,000 applicants, 4,100 graduate medical education programs, and 25,500 residency training positions. For more information, read the press release and listen to an interview with NRMP Executive Director Mona M. Signer.

Communications

Visit the Communications page for more information about and access to recent NRMP web conferences and webcasts.

Data and Reports

Visit the Data and Reports section for recent reports and historic NRMP match data.

New>Results of the 2010 NRMP Program Director Survey *(PDF, 164 pages)* This report presents the results of selected items from the 2010 NRMP Program Director Survey. Data are reported for 19 specialties and include: (1) factors

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NRMP: Independent Applicants - Konqueror

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members of the couple must be active applicants in the Match.

Step 1	
Each partner should first arrange an individual preference list on separate sheets of paper. In the example, the letters refer to a specific program in a particular hospital in that city.	
Partner I	Partner II
1) New York City - A	1) Chicago - X
2) Chicago - A	2) Chicago - Y
3) Evanston - B	3) Boston - X
4) Los Angeles - A	4) Chicago - Z
5) New York City - B	5) New York City - X
	6) New York City - Y

Step 2	
Next, both partners must decide together how to prepare their lists as pairs of programs. For example, they could consider all the possible pairings where the hospital programs are in the same general location, as indicated in the list below. In some cases one rank in the pair may be designated "No Match" to indicate that one partner is willing to go unmatched if the other is matched to a position. Note that the list below is not necessarily in the order that will eventually be submitted.	
Partner I	Partner II
New York City - A	New York City -X
New York City - A	New York City -Y
Chicago - A	Chicago -X
Chicago -A	Chicago -Y
Chicago -A	Chicago -Z
Evanston -B	Chicago -X
Evanston -B	Chicago -Y
Evanston -B	Chicago -Z
New York City -B	New York City -X
New York City -B	New York City -Y
New York City -A	No Match

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Hospitals / Residents problem with couples

Applicants:	Bill	Adam and Eve
1st choice:	Queens	(Memorial, Queens)
2nd choice:	Memorial	

the ranking of NY Queens Hospital: Eve, Bill
the ranking of NY Memorial Hospital: Bill, Adam

- P. Biró, R.W. Irving and I. Schlotter, Stable matching with couples – an empirical study. ACM Journal of Experimental Algorithmics, 16: Article number 1.2, 2011.

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2nd choice:	Memorial	

the ranking of NY Queens Hospital: Eve, Bill
the ranking of NY Memorial Hospital: Bill, Adam

Roth (1984): Stable solution may not exist.

Ronn (1990): The related decision problem is NP-complete.

B.-Irving-Schlotter (2011): NP-complete even for master lists.

B.-Manlove-McBride (2014): NP-complete even for preference lists of length 2 on both sides.

Heuristics are used in the applications...

- P. Biró, R.W. Irving and I. Schlotter, Stable matching with couples – an empirical study. ACM Journal of Experimental Algorithmics, 16: Article number 1.2, 2011.

Some more examples...

Applicants:	Adam and Eve	Romeo and Julia
1st choice:	(NY Memorial, NY Queens)	(NY Memorial, NY Queens)

NY Memorial: Romeo, Adam
NY Queens: Eve, Julia

Note 1: No applicant-optimal solution

- P. Biró and F. Klijn, Matching with Couples: a Multidisciplinary Survey. International Game Theory Review 15(2), 1340008 (2013).

Some more examples...

Applicants:	Adam and Eve	Romeo and Julia
1st choice:	(NY Memorial, NY Queens)	(NY Memorial, NY Queens)
2nd choice:		(SF General, SF Kaiser)

NY Memorial:	Romeo, Adam
NY Queens:	Eve, Julia
SF General:	Julia
SF Kaiser:	Romeo

Note 2: No rural hospital theorem

- P. Biró and F. Klijn, Matching with Couples: a Multidisciplinary Survey. International Game Theory Review 15(2), 1340008 (2013).

Some more examples...

Adam and Eve	Romeo and Julia	Bill
(NY Memorial, NY Queens)	(NY Memorial, NY Queens) (SF General, SF Kaiser)	SF Kaiser SF General

NY Memorial: Romeo, Adam
NY Queens: Eve, Julia
SF General: Romeo, Bill
SF Kaiser: Bill, Julia

Note 3: No path to stability

- P. Biró and F. Klijn, Matching with Couples: a Multidisciplinary Survey. International Game Theory Review 15(2), 1340008 (2013).

Some more examples...

Adam and Eve	Romeo and Julia	Bill
(NY Memorial, NY Queens)	(NY Memorial, NY Queens) (SF General, SF Kaiser)	NY Queens SF General

NY Memorial:	Romeo, Adam
NY Queens:	Eve, Bill, Julia
SF General:	Romeo, Bill
SF Kaiser:	Julia
common ranking:	Eve, Romeo, Bill, Julia, Adam

Note 4: No strategy proof mechanism that always outputs a stable matching if there exists one

- P. Biró and F. Klijn, Matching with Couples: a Multidisciplinary Survey. International Game Theory Review 15(2), 1340008 (2013).

Some more examples...

Adam and Eve	Romeo and Julia	Bill
(NY Memorial, NY Queens)	(NY Memorial, NY Queens) (SF General, SF Kaiser)	NY Memorial SF General

NY Memorial:	Romeo, Bill, Adam
NY Queens:	Eve, Julia
SF General:	Romeo, Bill
SF Kaiser:	Julia
common ranking:	Eve, Romeo, Bill, Julia, Adam

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Economics/Game Theory literature

- A. E. Roth. The evolution of the labor market for medical interns and residents: a case study in game theory. Journal of Political Economy, 6(4):991-1016, 1984.
- A. E. Roth and E. Peranson. The redesign of the matching market for American physicians: Some engineering aspects of economic design. American Economic Review, 89(4):748-780, 1999.
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- B. Klaus and F. Klijn. Stable matchings and preferences of couples. Journal of Economic Theory, 121:75-106, 2005.
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- F. Kojima, P.A. Pathak, and A.E. Roth. Matching with Couples: Stability and Incentives in Large Markets. Quarterly Journal of Economics, 128(4): 1585-1632, 2013.

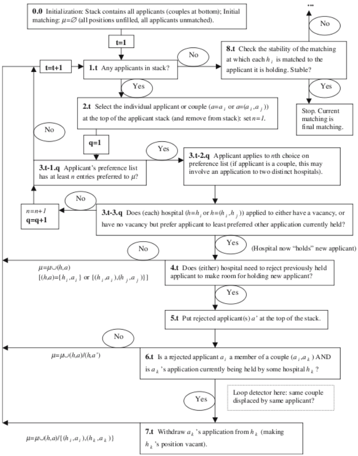
The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design

By ALVIN E. ROTH AND ELLIOTT PERANSON*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains this; that each applicant interviews only a small fraction of available positions is important. We also describe engineering aspects of the design process. (JEL C78, B41, J44)

The entry-level labor market for new physicians in the United States is organized via a centralized clearinghouse called the National Resident Matching Program (NRMP). Each year, approximately 20,000 jobs are filled in a process in which graduating physicians and other applicants interview at residency programs throughout the country and then compose and submit Rank Order Lists (ROLs) to the NRMP, each indicating an applicant's prefer-

employment, rather than waiting to participate in the larger market. (By the 1940's, contracts were typically being signed two years in advance of employment.) Although the matching algorithm has been adapted over time to meet changes in the structure of medical employment, roughly the same form of clearinghouse market mechanism has been used since 1951 (see Roth, 1984). The kind of market failure that gave rise to this clearinghouse has since been



Flowchart 1: The analyzed part of the Applicant Proposing Couples Algorithm (APCA)

(Appendix, Table 2), which, because of responsiveness, is the outcome of the deferred acceptance algorithm.

Next, we apply the Applicant Proposing Couples Algorithm to this couples

Maths / Computer Science literature

- E. Ronn. NP-complete stable matching problems. Journal of Algorithms, 11:285-304, 1990.
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- D. Marx and I. Schlotter. Stable assignment with couples: parameterized complexity and local search. Discrete Optimization, 8:25-40, 2011.
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- I. Ashlagi, M. Braverman, and A. Hassidim. Stability in large matching markets with complementarities. Forthcoming in Operations Research, 2014.

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SFAS 2010 Timetable - click to view

Post Compatibility (xls) - click to view (Compatible Posts are marked in Blue) for linked applications

Scottish Scoring Comparison Chart (xls)

The SFAS matching scheme

The SFAS Matching Scheme uses a computer program that aims to produce a matching that best satisfies the applicants' preferences. The algorithm that underlies this program was developed in the Department of Computing Science at the University of Glasgow, and is based on state-of-the-art research into optimal matching.

Introduction

The matching algorithm takes account of the following factors:

- the number of places in each programme
- the preference list of each individual applicant
- the score of each applicant
- which pairs of applicants are linked
- the compatibility information on programmes (from the viewpoint of linked applicants).

The algorithm is complicated by the need to deal with linked pairs in a fair way, giving them neither an advantage nor a disadvantage over single applicants, and ensuring that, if they are matched, then it is to compatible programmes. The description below is initially in terms of single applicants, and then an indication is given of the adaptations needed to accommodate linked pairs of applicants.

The algorithm – main idea

The first step is a *tie-breaking* step in which applicants with equal scores are randomly ordered. In effect, each applicant is given a unique score, but if applicant *a* had a higher original score than applicant *b* this will still be true for the revised scores.

The main body of the algorithm can be viewed as a sequence of attempts to match an applicant to a programme. At any point during the progress of the algorithm, an applicant is either matched (at least temporarily) or unmatched. Initially, each applicant's *best achievable* preference is the first entry on his/her preference list. At each step of the algorithm, a *random* applicant is chosen from those who are unmatched, and an attempt is made to match this applicant to his/her best achievable preference. If the programme has at least one free place then the match is accepted. Otherwise, the match is only accepted if a lower scoring applicant can be displaced from the programme – in this case the assigned applicant with the lowest score is displaced; if not the match is rejected. A rejection, or a displacement, results in the best achievable preference being advanced by one position in the list of the applicant concerned. The process terminates when each applicant is either matched or has been rejected by, or displaced from, all of the programmes on his/her preference list.

The resulting matching has the crucial *stability* property, namely:

- there can be no applicant *a* who would prefer to be matched to programme *p*, and at the same time *p* has an unfilled place or an assigned

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The resulting matching has the crucial *stability* property, namely:

- there can be no applicant *a* who would prefer to be matched to programme *p*, and at the same time *p* has an unfilled place or an assigned applicant with a lower score than *a*.

In other words, no private 'deal' could be made by an applicant and a programme that would be to the benefit of both.

Linked applicants

To accommodate linked applicants, a *joint* preference list is formed for each such pair, using their individual preference lists and the programme compatibility information. If such a pair, *a* and *b*, have individual preferences $p1, p2, \ldots, p10$ and $q1, q2, \ldots, q10$ respectively (with *a* the higher scoring applicant), then the joint preference list of the pair (a,b) is $(p1,q1)$, $(p1,q2)$, $(p2,q1)$, $(p2,q2)$, $(p1,q3)$, $(p3,q1)$, $(p2,q3)$, $(p3,q2)$, $\ldots$, $(p9,q10)$, $(p10,q9)$, $(p10,q10)$ (except that incompatible pairs of programmes are omitted)

In the main body of the algorithm, the members of a linked pair are handled together, so the match of the pair (a,b) to the programmes (p,q) will be accepted only if each of these programmes either has an unfilled place or a lower scoring applicant who can be displaced. A complication arises when one member *x* of a linked pair has to be withdrawn from a programme *p* because his/her partner was displaced from their current assigned programme. In this case, some other applicants may have been rejected by *p* because of the presence of *x*, and any such applicant *a* must be withdrawn from their current programme, if any, and have their best achievable preference reset to *p*. (A corresponding, but more complex reset operation is needed if *a* is a member of a linked pair). This reset operation thereby allows a further opportunity for applicant *a* to be matched to programme *p*.

The algorithm terminates when every single applicant and linked pair is either matched or has been rejected by, or displaced from, every entry in their preference list with no possibility of reconsideration by a programme that has had a withdrawal.

The final matching is stable for single applicants, as before, but also for linked pairs, in the sense that:

- there can be no linked pair (a,b) of applicants who would prefer to be matched to compatible programmes (p,q), and at the same time, each of *p* and *q* has an unfilled place or an assigned applicant with a lower score than *a* and *b* respectively.

Frequently Asked Questions

back to top

Stable matching with couples – theory and practice

Péter Biró[1,*,†], Robert W. Irving[2,*] and Ildikó Schlotter[3,‡]

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Abstract

In practical applications, algorithms for the classical version of the Hospitals Residents problem (the many-one version of the Stable Marriage problem) may have to be extended to accommodate the needs of couples who wish to be allocated to (geographically) compatible places. Such an extension has been in operation in the NRMP matching scheme in the US for a number of years. In this setting, a stable matching need not exist, and it is an NP-complete problem to decide if one does. However, the only previous empirical study in this context (focused on the NRMP algorithm), together with information from NRMP, suggest that, in practice, stable matchings do exist and that an appropriate heuristic can be used to find such a matching.

The study presented here was motivated by the recent decision to accommodate couples in the Scottish Foundation Allocation Scheme (SFAS), the Scottish equivalent of the NRMP. Here, the problem is a special case, since hospital preferences are derived from a 'master list' of resident scores, but we show that the existence problem remains NP-complete in this case. We describe the algorithm used in SFAS, and contrast it with a version of the algorithm that forms the basis of the NRMP approach. We

	Number of couples										
Algorithm	12	25	50	75	100	125	150	175	200	225	250
C-RAN	**976**	**958**	908	862	**811**	729	586	352	163	40	5
C-STA	965	925	807	745	660	588	481	331	191	41	**10**
C-SGL	**976**	957	904	861	801	**752**	677	504	244	61	4
C-CPL	964	908	804	767	709	580	426	253	122	30	5
C-RLP	962	922	805	546	271	92	19	3	1	0	0
BB-RAN	**976**	**958**	**911**	**870**	800	655	412	169	51	14	0
BB-SCO	958	914	793	663	498	342	230	122	65	29	8
BB-USE	**976**	957	909	867	799	696	501	254	81	27	4
BB-USS	963	934	880	825	764	716	**682**	**546**	**281**	**71**	4
BB-SGL	963	934	879	828	773	720	680	529	232	44	0
BB-CPL	974	943	783	482	215	95	25	8	0	1	2
RP-RAN	888	771	579	453	320	188	119	67	35	16	4
RP-SGL	952	897	701	547	395	277	170	83	41	9	3
RP-CPL	872	778	585	424	306	183	115	63	28	11	1
Total	976	958	911	871	820	775	739	642	401	143	29

Table 2: Instances of size 500 (5 seconds per instance)

Some more examples...

Adam and Eve	Romeo and Julia	Bill
(NY Memorial, NY Queens) (LA Lincoln, LA Hollywood)	(SF General, SF Kaiser) (NY Memorial, NY Queens) (LA Lincoln, LA Pacific)	NY Queens SF General

and

David and Victoria	Cliff
(LA Hollywood, LA Sunset)	LA Hollywood LA Sunset

common ranking: Eve, Julia, Bill, Romeo, Adam, David, Cliff, Victoria

Note 5: Inevitable failure of heuristics based on best applications

- P. Biró, R.W. Irving and I. Schlotter, Stable matching with couples – an empirical study. ACM Journal of Experimental Algorithmics, 16: Article number 1.2, 2011.

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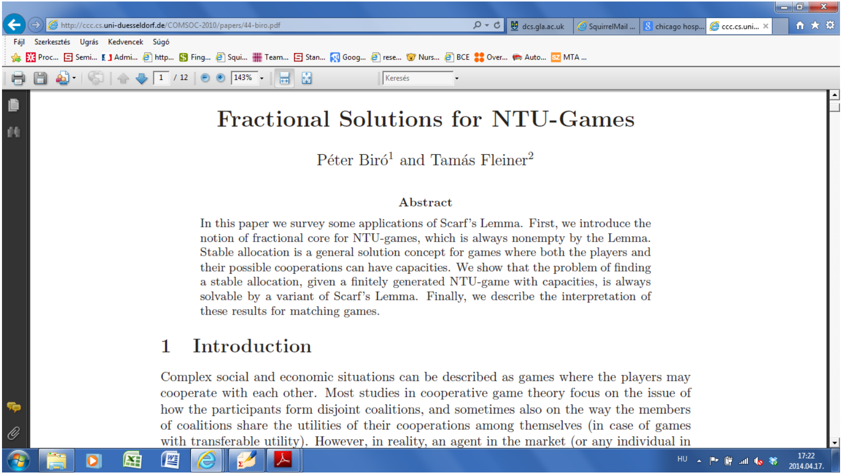
Now, something completely different... (!?)



Fractional Solutions for NTU-Games

Péter Biró[1] and Tamás Fleiner[2]

Abstract

In this paper we survey some applications of Scarf's Lemma. First, we introduce the notion of fractional core for NTU-games, which is always nonempty by the Lemma. Stable allocation is a general solution concept for games where both the players and their possible cooperations can have capacities. We show that the problem of finding a stable allocation, given a finitely generated NTU-game with capacities, is always solvable by a variant of Scarf's Lemma. Finally, we describe the interpretation of these results for matching games.

1 Introduction

Complex social and economic situations can be described as games where the players may cooperate with each other. Most studies in cooperative game theory focus on the issue of how the participants form disjoint coalitions, and sometimes also on the way the members of coalitions share the utilities of their cooperations among themselves (in case of games with transferable utility). However, in reality, an agent in the market (or any individual in

Definitions: a general setting

Set of residents: $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$, where $\mathcal{A} = \mathcal{S} \cup \mathcal{C}$, i.e., single residents and couples. Set of hospitals: $\mathcal{H} = \{h_1, h_2, \dots, h_m\}$ with $c(h_p)$ denoting the capacity of hospital h_p.
Set of applications, E, has three types ($E = E^S \cup E^J \cup E^C$)

- E^S: single application from a single resident to a hospital
- E^J: joint application from a couple to a pair of hospitals
- E^C: combined application from a couple to a hospital

Each application specifies one or two employments, respectively.

A **matching** M is a set of employments specified by a set of (accepted) applications E_M, where no resident is employed in more than one hospital and no hospital employs more residents than its quota.

Preferences:
- the single residents and couples have strict preferences over the applications
- the hospitals have strict rankings over the residents, which generates choice functions over the set of applications (and thus over the set of residents).

Stability: no blocking application, which would be chosen by each party involved in the application when offered together with the currently accepted applications of that party.

Definitions: specific model used in SFAS

Easy to check fairness (for single and joint applications) with cutoff scores:

- If a single application $[a_i \rightarrow h_p]$ is rejected then h_p filled its quota with better residents than a_i (i.e., the resident did not meet the cutoff score).
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The creation of the hospitals' choice functions:
- Each hospital h_p has a strict ranking $\succ_{h_p}$ over the residents.
- This defines weak preferences $\geq_{h_p}$ over the applications according to the corresponding residents making single or joint applications or the **weakest members** of couples making combined applications.
(- Ties: one resident can submit several joint applications to a hospital).
- Refined strict preference $>_{h_p}$ is where the above ties are broken according to the residents' preferences.
- Choice function Ch_{h_p} over the set of applications is derived as follows:
h_p accepts each application from $X \subseteq E$ in the order of $>_{h_p}$ such that no two applications from the same couple are accepted and its quota is not violated.
- We call this type of choice functions, derived from refined strict preferences over applications, **quota-responsive**.

Notes on Cooperative Game Theory

For Stable Marriage problem,
set of stable matchings = core of the corresponding CFG

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Scarf (1967): Every balanced NTU-game has nonempty core.
(Scarf's algorithm always returns a core element for such games)

But what if an NTU-game is not balanced? The Scarf algorithm still returns a (fractional) core solution...

Stable (fractional) matchings

bipartite graph		
Marriage problem Gale-Shapley ‘62: $\exists$ stable matching		

Stable (fractional) matchings

bipartite graph	**nonbipartite graph**	
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For every vertex v, let $<_v$ be a linear order on the edges incident with v. A weight-function $x : E(G) \to \{0,1\}$ is a **matching** if $\sum_{v \in e} x(e) \leq 1$ for every $v \in V(G)$.

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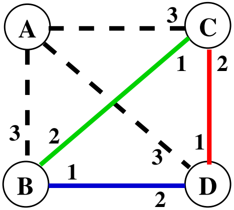

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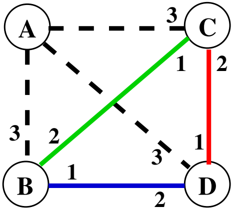

- Gale-Shapley (1962):
  Stable matching may not exist!
- Tan (1990): **Stable half-matching** always exists! i.e. $x(e) \in \{0, \frac{1}{2}, 1\}$.
  Here: $x(\{B,C\}) = x(\{C,D\}) = x(\{B,D\}) = \frac{1}{2}$

Stable (fractional) matchings

bipartite graph	nonbipartite graph	**hypergraph**
Marriage problem Gale-Shapley '62: $\exists$ stable matching	Roommates problem Tan '90: $\exists$ stable half-matching	Coalition Formation Game Aharoni-Fleiner '03 (Scarf '67): $\exists$ stable fractional matching

For every vertex v, let $<_v$ be a linear order on the hyper− edges incident with v. A weight-function $x : E(G) \to \{0,1\}$ is a **matching** if $\sum_{v \in e} x(e) \leq 1$ for every $v \in V(G)$.

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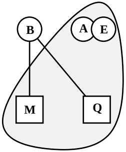
Aharoni-Fleiner (2003): Scarf's algorithm returns a **stable fractional matching**, as defined above with $x(e) \in [0,1]$.

An example: stable fractional matching

Applicants:	Bill	Adam and Eve
1st choice:	Queens	(Memorial, Queens)
2nd choice:	Memorial	

ranking of **NY Queens:** Eve, Bill
ranking of **NY Memorial:** Bill, Adam



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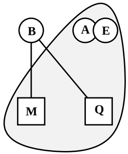

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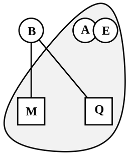

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-These can be seen as part-time contracts...

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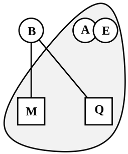

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What is the meaning of a fractional solution?
-These can be seen as part-time contracts...

What if the fractional solution obtained is integral?
-Then it corresponds to a stable matching (or a core element).
Thus the Scarf algorithm can be used as a heuristic to find a stable matching (or to find a core element in any NTU-game).

Stable b-matchings: agents with capacities

bipartite graph		
College Admissions Gale-Shapley '62: $\exists$ stable matching		

Let $b : V(G) \to \mathbb{Z}_+$ be vertex-bounds.

A weight-function $x : E(G) \to \{0, 1\}$ is a **(b-)matching** if $\sum_{v \in e} x(e) \leq b(v)$ for every $v \in V(G)$.

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Biró-Fleiner (2003): A stable half-matching can be found efficiently for nonbipartite graphs.

Cechlárová-Fleiner (2005), Irving-Scott (2007): A stable matching can be found in linear time, if one exists ("Stable Multiple Activities" or "Stable Fixtures").

Stable b-matchings: agents with capacities

bipartite graph	nonbipartite graph	**hypergraph**
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This can be used for the Hospitals Residents problem with couples!
In the case when hospitals have capacities, but no couple may apply for a pair of positions at the same hospital.
The stable matchings as defined here are stable matchings for the matching with couples problem, and vice versa.

A motivating example for stable schedules

Researchers' contributions in projects sponsored by the **Hungarian Scientific Research Fund**:

Each researcher can be involved in several running projects, but she has to declare her contribution in each project, and her total contribution cannot exceed 1.0 at any time.

Similar requirements apply for the grant applications of the **French National Research Agency**.

Stable schedules

Let $r_v(e)$ denote v's contribution in contract e, and

let $b : V(G) \to \mathbb{Z}_+$ be vertex-bounds.
A weight-function $x : E(G) \to \{0, 1\}$ is a **schedule** if
$\sum_{v \in e} r_v(e) \cdot x(e) \leq b(v)$ for every $v \in V(G)$.

- P. Biró and T. Fleiner, Fractional solutions for capacitated NTU-games, with applications to stable matchings. To appear in Discrete Optimization.

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This can be used for the Hospitals Residents problem with couples!
In the general case, where each combined applications is a contract with 1 contribution for the couple and 2 for the hospital.
Stable schedules correspond to stable matchings for the couples' market, but not the other way!

- P. Biró and T. Fleiner, Fractional solutions for capacitated NTU-games, with applications to stable matchings. To appear in Discrete Optimization.

Experiments on random samples with 500 applicants

	Number of couples										
Algorithm	12	25	50	75	100	125	150	175	200	225	250
Roth-Perantson	952	897	701	547	395	277	170	83	41	9	3
Best heuristics in B-I-S	976	958	911	870	811	752	682	546	281	71	10
Scarf (int. solution)	895	813	649	532	426	356	316	261	202	174	158
Scarf half-int. solution	999	997	978	958	918	859	816	777	692	695	588
Scarf frac. solution	105	187	351	468	574	644	684	739	798	826	842
Av. # of frac. weights	3.9	4.8	5.7	6.7	7.6	8.8	10.0	10.8	12.8	13.5	15.7
# of frac. weights = 1	41	61	104	127	119	126	106	114	97	85	69
# of frac. weights = 2	2	9	21	30	36	41	43	43	44	48	41
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Scarf's algorithm performs very well for high proportion of couples!

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Biró-Manlove-McBride: Experiments by IP techniques show that around 70% of these instances with couples only are solvable.

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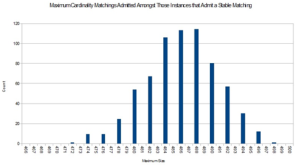
Integer programming techniques (David Manlove's talk)

University of Glasgow

Integer Programming for HRC

- Model developed by Iain McBride (2013)
- Solved using CPLEX IP solver
- Random instances, scalability (preference lists of length between 5 and 10):
  - 5000 residents, 500 hospitals, 500 couples, 5000 posts (x25)
    - solved in 99.6 seconds on average
  - 10000 residents, 1000 hospitals, 1000 couples, 10000 posts (x1)
    - solved in 10 minutes
- Random instances, solvability / sizes of largest stable matchings found:
  - 500 residents, 50 hospitals, 250 couples, 500 posts (x1000)
    - around 70% of instances were solvable
    - Average time taken 75s per instance



- SFAS instances:
  - 2012: 710 residents, stable matching of size 681 found in 16s
  - 2011: 736 residents, stable matching of size 688 found in 17s
  - 2010: 734 residents, stable matching of size 681 found in 65s

39

- P. Biró, I. McBride and D.F. Manlove. The Hospitals / Residents problem with Couples: Complexity and Integer Programming models. To appear in Proceedings of SEA 2014: the 13th International Symposium on Experimental Algorithms, Lecture Notes in Computer Science, Springer, 2014.

Matching **with payments**

marriage = one-to-one market with no transfers (?)

implicit assumptions on 'marriage':

1. Everybody can have at most one partner
2. Only men and women can marry each other
3. No dowry (no transfer)

marriage = one-to-one market with no transfers (?)

The relaxation of the implicit assumptions on 'marriage':

1. *Everybody can have at most one partner*
   $\rightarrow$ stable b-matching for bipartite graph
   =College Admissions (resident allocation)
2. Only men and women can marry each other
3. No dowry (no transfer)

marriage = one-to-one market with no transfers (?)

The relaxation of the implicit assumptions on 'marriage':

1. Everybody can have at most one partner
2. *Only men and women can marry each other*
   → stable matching for nonbipartite graphs
   =Roommates problem (kidney exchange)
3. No dowry (no transfer)

marriage = one-to-one market with no transfers (?)

The relaxation of the
implicit assumptions on 'marriage':

1. Everybody can have at most one partner
2. Only men and women can marry each other
3. *No dowry (no transfer)*
   → stable matching for bipartite graphs with TU =Assignment Game ("the market")

All possible models with relaxations

		two-sided	one-sided
	NTU	stable marriage problem	stable roommates problem
	TU	assignment game	matching game
capa-city	NTU	college admissions problem	stable fixtures problem
	TU	multiple partners assignment game	**this paper**

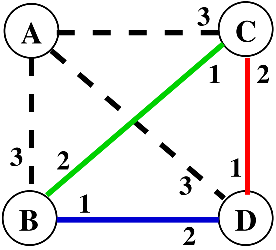
Notes on the problems with no payments

	two-sided	one-sided
	stable marriage	
capacity	college admissions	

Gale-Shapley (1962): A stable matching always exists for the marriage problem, and the same result holds for the many-to-one college admissions problem.

Notes on the problems with no payments

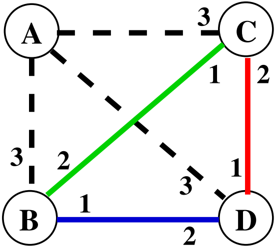
	two-sided	one-sided
	stable marriage	stable roommates
capacity	college admissions	



- Gale-Shapley (1962):
  Stable matching may not exist!
- Irving (1985): A stable matching can be found in $O(m)$ time, if one exists.
- Tan (1990): Stable *half-matching* always exists. +The same odd cycles are formed in every stable solution.
- Diamantoudi et al. (2004): Path to stability result.

Notes on the problems with no payments

	two-sided	one-sided
	stable marriage	stable roommates
capacity	college admissions	stable fixtures



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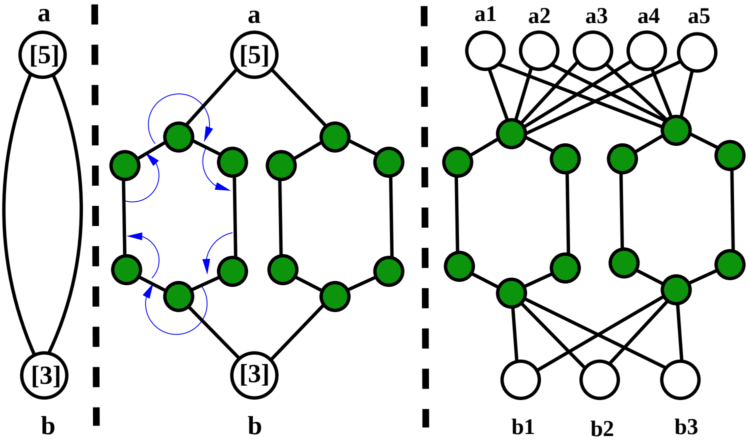
- Diamantoudi et al. (2004): Path to stability result.

- Irving-Scott (2007): The stable fixtures problem can be solved efficiently.
- Cechlárová-Fleiner (2005): The problem can be reduced to the stable roommates problem with a simple graph construction.

Graph reduction by Cechlárová-Fleiner 2005



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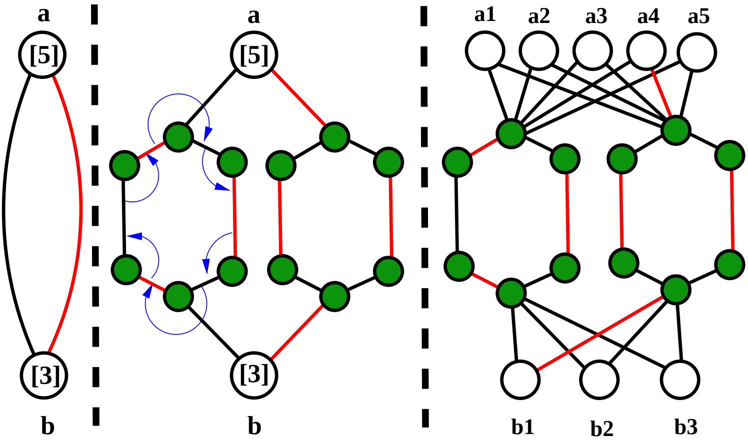
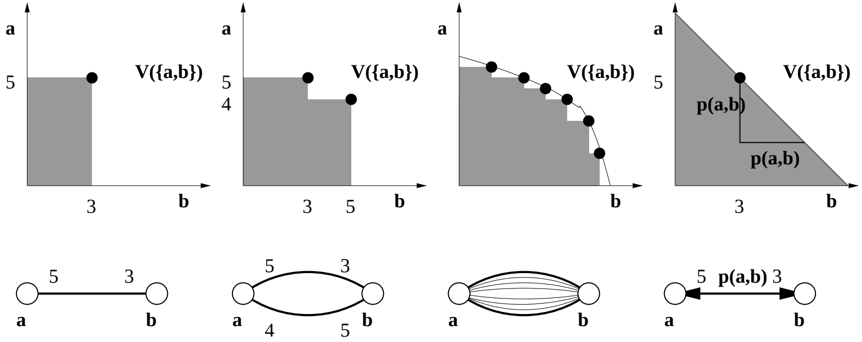
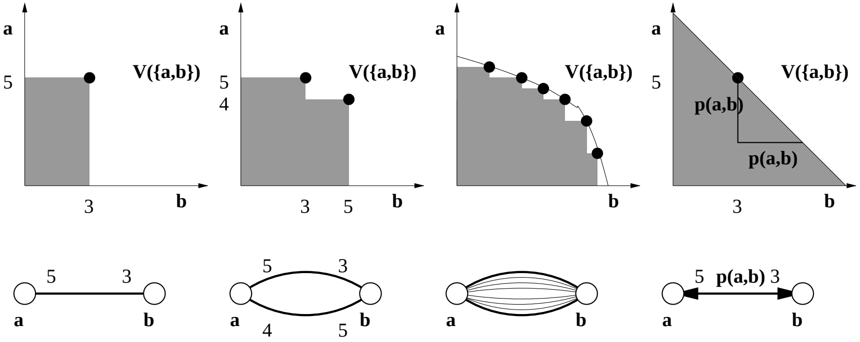


- stable matchings of the capacitated market correspond to stable matchings in the reduced non-capacitated market...

Stable matchings with or without payments



Stable matchings with or without payments



- Stable matching problems with payments can be seen as stable matching problems with contracts.
- Stable matchings with contracts can be reduced to stable matching problems (with the Cechlárová-Fleiner construction).


APA

Basic graph theoretical notions

$G(N, E)$ graph, nodes: $N = \{\ldots, i, \ldots, j, \ldots\}$, edges: $E = \{\ldots, ij, \ldots\}$

A matching is a set of independent edges $M \subseteq E$, i.e., it can be described with its characteristic function: $x : E \to \{0, 1\}$: for each $i \in N$, $\sum_{j \in N} x(ij) \leq 1$.

For given edge-weights $w : E \to \mathbb{R}_+$, $c : N \to \mathbb{R}_+$ is a cover, if for each $ij \in E$, $c(i) + c(j) \geq w_{ij}$.

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maximum weight of a matching $=$ minimum value of a cover

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Egerváry 1931: If G is bipartite then
maximum weight of a matching $=$ minimum value of a cover

Balinski 1965: If G is nonbipartite then
maximum weight of a **half**-matching $=$ minimum value of a cover

linear programming, duality theorem

LP: max. weight frac. matching

$$max \sum_{ij \in E} w_{ij} x(ij)$$

s.t. $\sum_{j:ij \in E} x(ij) \leq 1$ for each $i \in N$

where $0 \leq x(ij)$ for each $ij \in E$

DLP: minimum value cover

$$min \sum_{i \in N} y(i)$$

s.t. $y(i) + y(j) \geq w_{ij}$ for each $ij \in E$

where $0 \leq y(i)$ for each $i \in N$,

maximum weight matching $= max_{IP} \leq max_{hIP} \leq max_{LP} =$

$= min_{DLP} =$ minimum value cover

Note: The theorem of Egerváry is implied by the fact the incidence matrix of any bipartite graph is **totally unimodular**.

Nonbipartite graphs: the role of half-matching

Balinski (1965): The maximum weight of a half-matching is equal to the minimum value of a cover.

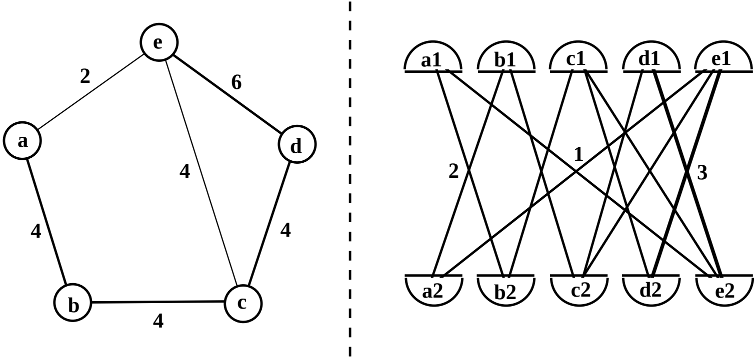
Simple proof: duplication technique (Nemhauser-Trotter, 1975):
$G(N, E) \to G^d(N^d, E^d)$, where $N^d = N_1 \cup N_2$,
$i \in N \to i_1 \in N_1, i_2 \in N_2$
$ij \in E \to i_1 j_2, i_2 j_1 \in E^d$, and $w^d(i_1 j_2) = w^d(i_2 j_1) = \frac{1}{2} w(ij)$.

Let x^d be a maximum weight matching and c^d a minimum value cover in G^d. Let us define $x(ij) = \frac{x^d(i_1 j_2) + x^d(i_2 j_1)}{2}$ for each edge and $c(i) = c(i_1) + c(i_2)$ for each vertex.
We can verify that x is a half-matching, c is a minimum cover in G s.t.

$$w(x) = w^d(x^d) = c^d(N^d) = c(N)$$

Corollary: We can compute a maximum weight half-matching (and also a minimum cover) efficiently by the Hungarian method.

Example for the duplication construction



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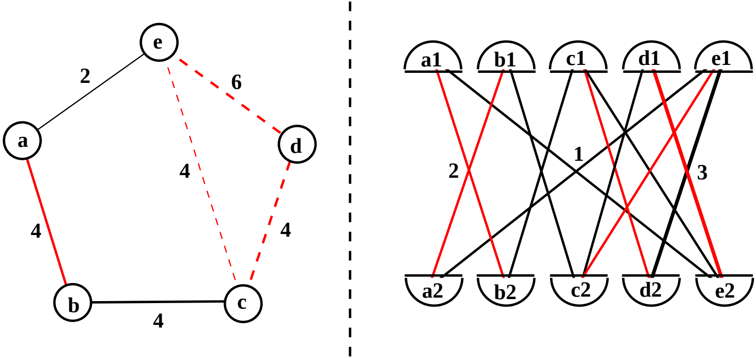

$$max_{IP}(G^d) \leq max_{hIP}(G)$$

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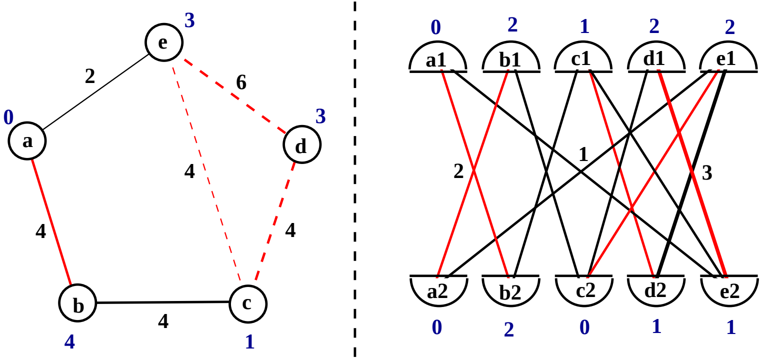

$$max_{IP}(G^d) \leq max_{hIP}(G) \leq min_{DLP}(G) \leq min_{DLP}(G^d)$$

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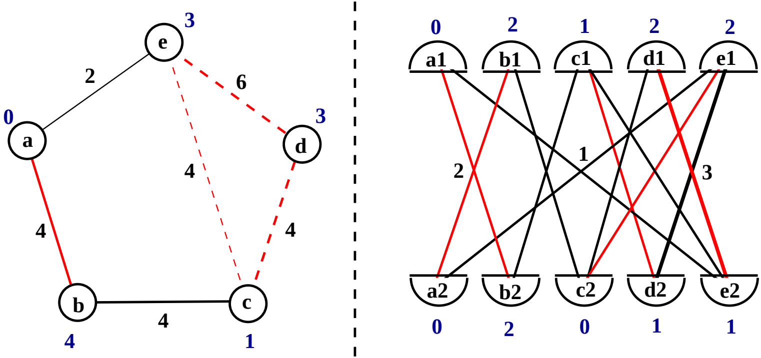

$$max_{IP}(G^d) \leq max_{hIP}(G) \leq min_{DLP}(G) \leq min_{DLP}(G^d)$$

but $max_{IP}(G^d) = min_{DLP}(G^d)$ so we have $=$ everywhere!

Game theory: Koopmans-Beckmann (1957, Econometrica)

stable matching with payments:

Let $G(N, E)$ be a bipartite graph, where $N = I \cup J$ (buyers-sellers), and $w : E \to \mathbb{R}_+$ edge-weights (value of pairs).

A solution is a pair (M, p), where $M \subseteq E$ is a matching and $p : N \to \mathbb{R}_+$ are the payments of the agents such that

- $ij \in M \to p(i) + p(j) = w_{ij}$ and
- i is not covered by $M \to p(i) = 0$.

A solution is **stable** if for each $ij \in E \setminus M$: $p(i) + p(j) \geq w_{ij}$.

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Observation: (M, p) is stable $\iff$ M is a maximum weight matching and p is a minimum cover.

So **the Egerváry thm implies the Koopmans-Beckmann thm**:

The stable matching problem with payments is always solvable.

Game theory: Shapley-Shubik (1971, IJGT)

assignment game:
Let $G(N, E)$ be a bipartite graph, where $N = I \cup J$ (buyers-sellers), and $w : E \to \mathbb{R}_+$ edge-weights (value of pairs).

We define a TU-game (N, v) as follows. For any coalition $S \subseteq N$, let $v(S)$ = maximum weight of a matching on S, the value of S.
$u : N \to \mathbb{R}_+$ is an *imputation* if $\sum_{i \in N} u(i) = u(N) = v(N)$.
u is in the *core* of the game if for each $S \subseteq N$, $v(S) \leq u(S)$.

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Observation: u is in the core $\iff$ u is not blocked by any pair.
u is in the core $\iff$ (M, u) is a stable matching with payments

Koopmans-Beckmann thm implies Shapley-Shubik thm:

The assignment game has a nonempty core.

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Koopmans-Beckmann thm implies Shapley-Shubik thm:

The assignment game has a nonempty core.

+Shapley-Shubik 1971: The set of stable solutions forms a lattice with a buyer-optimal and a seller-optimal solution.

Generalisations of the assignment game

	bipartite graph	nonbipartite graph
non-capacitated	assignment game	
capacitated		

Generalisations of the assignment game

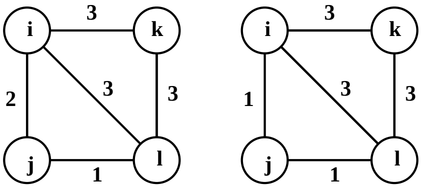
	bipartite graph	nonbipartite graph
non-capacitated	assignment game	
capacitated	multiple partners a.g.	

Sotomayor: *multiple partners assignment game*
1992: stable solution exists
1999, IJGT: the stable solutions form a lattice
2007, JET: competitive equilibria exist and form a sub-lattice
(competitive equilibrium: each seller gets the same payment from any of her buyers, which can be seen as the price of her goods)

Generalisations of the assignment game

	bipartite graph	nonbipartite graph
non-capacitated	assignment game	matching game
capacitated	multiple partners a.g.	

Biró-Kern-Paulusma 2012: A matching game has a stable solution $\iff$ the maximum weight of a matching is equal to the maximum weight of a half-matching. (Thus it can be decided efficiently with Edmonds' algorithm and with the Hungarian method.)



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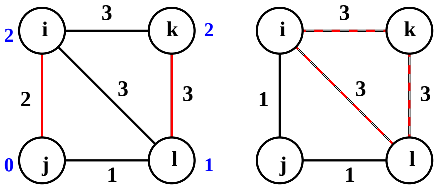
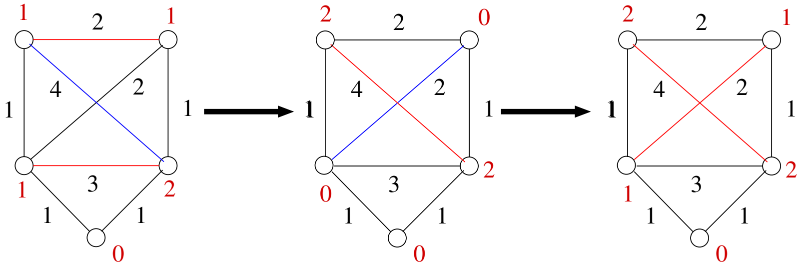


Path to stability for assignment games

For an unstable state (M, p), satisfying a blocking pair $ij \notin M$ means that we get a new state (M', p') such that
- $ij \in M'$, $p'(i) + p'(j) = w(ij)$, $p'(i) \geq p(i)$ and $p'(j) \geq p(j)$
- if i was matched in M then $M(i)$ is unmatched in M'
- agents outside i, j and their partners in M are not affected.



Biró-Bomhoff-Golovach-Kern-Paulusma (2014, TCS): If a stable solution exists then one can be reached in at most $2n$ steps.

References

Matching with payments, assignment game

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(many-to-many) multiple partners assignment game:

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- M. Sotomayor. The lattice structure of the set of outcomes of the multiple partners assignment game. International Journal of Game Theory, 28, 567-583 (1999)
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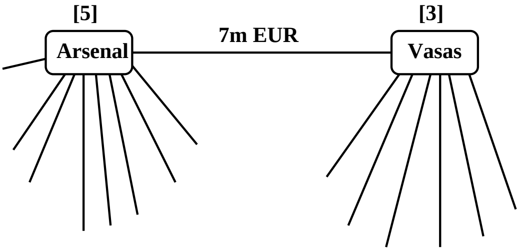
(one-sided) matching game:

- K. Eriksson and J. Karlander. Stable outcomes of the roommate game with transferable utility. International Journal of Game Theory, 29, 555-569 (2001)
- P. Biró, W. Kern and D. Paulusma. Computing solutions for matching games. International Journal of Game Theory, 41, 75-90 (2012)
- P. Biró, M. Bomhoff, P.A. Golovach, W. Kern and D. Paulusma. Solutions for the Stable Roommates Problem with Payments. Theoretical Computer Science, 540-541, 53-61 (2014)

Stable Fixtures problem with Payments (SFP)

	bipartite graph	nonbipartite graph
non-capacitated	assignment game	matching game
capacitated	multiple partners a.g.	SFP

A motivating example: soccer teams looking for opponents in the summer training season...



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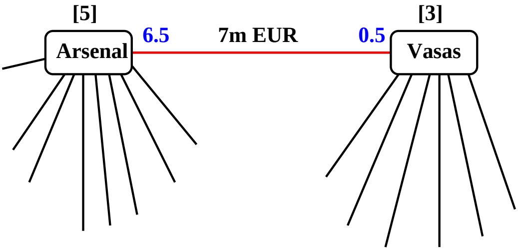

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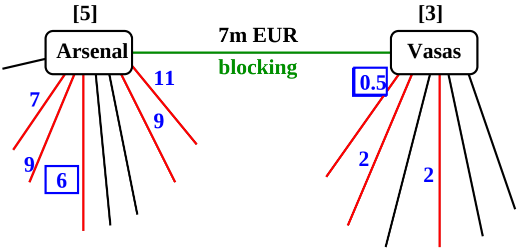

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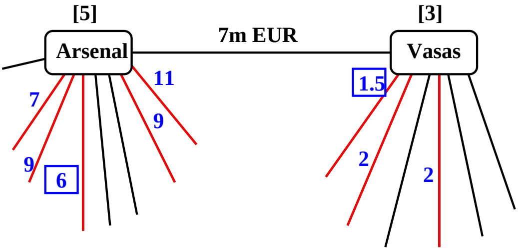


Stable Fixtures problem with Payments (SFP)

$G(N, E)$ nonbipartite, with $w : E \to \mathbb{R}_+$ edge-weights and $b : N \to \mathbb{Z}_+$ node-capacities.
A solution is a pair (M, p), where

1. $M \subseteq E$ is a b-matching, i.e. for each $i \in N$ $|\{j : ij \in M\}| \leq b_i$, and
2. $p : E \to \mathbb{R}^2_+$ are the payments, such that
   a) $ij \in M \to p(i,j) + p(j,i) = w_{ij}$ and
   b) $ij \notin M \to p(i,j) = p(j,i) = 0$.

Let $u_p(i) = 0$ if $|\{j : ij \in M\}| < b_i$ and $u_p(i) = min\{p(i,j) : ij \in M\}$ otherwise.

A solution is (M, p) **stable**, if for each $ij \in E \setminus M$, $u_p(i) + u_p(j) \geq w_{ij}$.

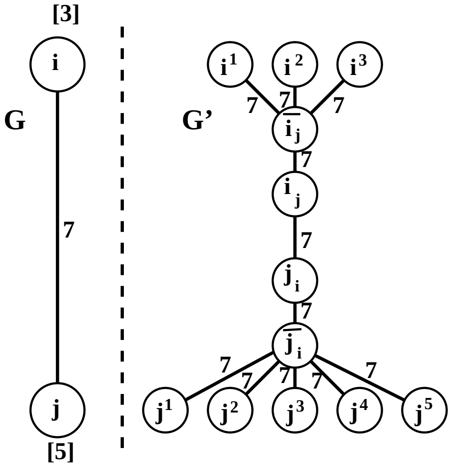
Simple reduction with a graph construction

We can reduce

multiple partners a.g. to a.g.

and

SFP to matching game
(construction by Tutte 1954)



$$ij \in M \iff \{i^k, \overline{i_j}\}, \{i_j, j_i\}, \{\overline{j_i}, j^l\} \in M'$$

$$ij \notin M \iff \{\overline{i_j}, i_j\}, \{\overline{j_i}, j_i\} \in M'$$

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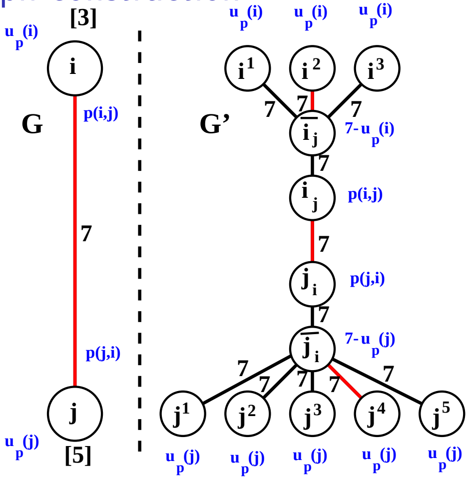

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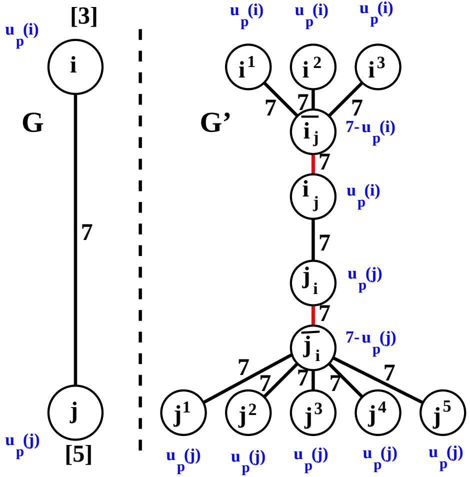

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Consequence for two-sided markets

Alternative proofs for Sotomayor's theorems:

1992: stable solution exists
(from the reduction + Koopmans-Beckhamm 1957)

1999: the stable solutions form a lattice
(from the lattice prop. on the 'middle agents' in the reduction)

2007: competitive equilibria exist and form a sub-lattice
(from the lattice prop. on the 'copied sellers' in the reduction)

LP model, where dual solutions $\iff$ payments

PRIMAL:

$$max \sum_{ij \in E} w_{ij} x(ij)$$

s.t.

$$\sum_{j:ij \in E} x(ij) \leq b_i \text{ for each } i \in N$$

where

$$0 \leq x(ij) \leq 1 \text{ for each } ij \in E$$

DUAL:

$$min \sum_{i \in N} b_i y(i) + \sum_{ij \in E} d(ij)$$

s.t.

$$y(i) + y(j) + d(ij) \geq w_{ij} \text{ for each } ij \in E$$

where $0 \leq y(i)$ for each $i \in N$,

and $0 \leq d(ij)$ for each $ij \in E$

LP model, where dual solutions $\iff$ payments

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s.t.

$$y(i)+y(j)+d(ij) \ge w_{ij} \text{ for each } ij \in E$$

where $0 \le y(i)$ for each $i \in N$,

and $0 \le d(ij)$ for each $ij \in E$

Thm 1: If (M, u) is a stable solution for an instance of SFP then $y(i) = u_p(i)$, $d(ij) = w_{ij} - u_p(i) - u_p(j)$ is opt. solution for DUAL.

LP model, where dual solutions $\iff$ payments

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$$max \sum_{ij \in E} w_{ij} x(ij)$$

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where $0 \leq y(i)$ for each $i \in N$,

and $0 \leq d(ij)$ for each $ij \in E$

Thm 2: (M', u') is a stable solution for the reduced instance IFF $y(i) = u'(i^s)$, $d(ij) = (u'(i_j) - u'(i^s)) + (u'(j_i) - u'(j^t))$ is opt. solution for DUAL.

Solving SFP efficiently

Theorem: An instance (G, b, w) of SFP admits a stable solution if and only if the maximum weight of a b-matching in G is equal to the maximum weight of a half-b-matching in G. So this can be decided in $O(n^2 mlog(n^2/m))$ time.

Proof: again by the duplication technique:

$max_{IP}(G^d) \leq max_{hIP}(G) \leq min_{DLP}(G) \leq min_{DLP}(G^d)$

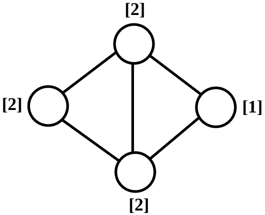
but $max_{IP}(G^d) = min_{DLP}(G^d)$ so we have $=$ everywhere!

Core of Multiple Partners Matching Game

We define the TU-game (N, v) that corresponds with a multiple partners matching game (G, b, w) by setting, for every $S \subseteq N$,

$$v(S) = w(M_S) = \sum_{e \in M_S} w(e),$$

where M_S is a maximum weight b-matching in S.



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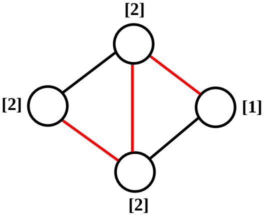

maximum weight of a matching: 3

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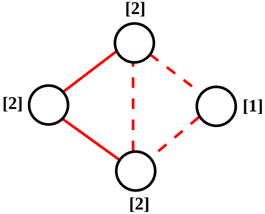

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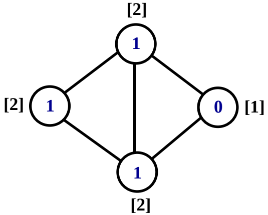

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maximum weight of a half-matching: 3.5

yet, core allocation exists

Core of Multiple Partners Matching Game

Theorem: The payoff vector of every stable solution of a multiple partners matching game is a core allocation.

Proof: Let (M, p) be a stable solution, with total payoff vector $p^t \in \mathbb{R}^n$ defined by $p^t(i) = \sum_{ij \in E} p(i,j)$ for all $i \in N$. Let M' be a maximum-weight b-matching in S...

$$
\begin{aligned}
p^t(S) &= \sum_{i \in S} p^t(i) \\
&= \sum_{i \in S} \Big(\sum_{j:ij \in M \cap M'} p(i,j) + \sum_{j:ij \in M \setminus M'} p(i,j)] \Big) \\
&= \sum_{ij \in M \cap M'} (p(i,j) + p(j,i)) + \sum_{i \in S} \sum_{j:ij \in M \setminus M'} p(i,j) \\
&= \sum_{ij \in M \cap M'} w(ij) + \sum_{i \in S} \sum_{j:ij \in M \setminus M'} p(i,j) \\
&\geq \sum_{ij \in M \cap M'} w(ij) + \sum_{i \in S} \sum_{j:ij \in M' \setminus M} u_p(i) \\
&= \sum_{ij \in M \cap M'} w(ij) + \sum_{ij \in M' \setminus M} u_p(i) + u_p(j) \\
&\geq \sum_{ij \in M \cap M'} w(ij) + \sum_{ij \in M' \setminus M} w(ij) \\
&= w(M') = v(S).
\end{aligned}
$$

Core of Multiple Partners Matching Game

Theorem: It is possible to test in polynomial time if an allocation is in the core of a multiple partners matching game defined on a triple (G, b, w) with $b \leq 2$.

Proof: Let (N, v) be a multiple partners matching game defined on a triple (G, b, w), where $b(i) \leq 2$ for all $i \in N$. Given $S \subseteq N$, a maximum weight b-matching in $G[S]$ is composed of cycles and paths. Hence the core can be alternatively described by the following (slightly smaller) set of constraints:

$$\begin{array}{rcll} p(C) & \geq & w(C), & \text{for all cycles } C \in \mathcal{C} \\ p(P) & \geq & w(P), & \text{for all paths } \ P \in \mathcal{P} \\ p(N) & = & v(N). & \end{array}$$

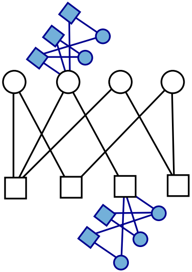
The first condition is testable efficiently by solving the tramp steamer problem. The second is testable by solving $O(n^3)$ instances of the shortest path problem.

Core of Multiple Partners Matching Game

Theorem: It is co-NP-complete to test if an allocation is in the core of a multiple partners matching game defined on a triple (G, b, w) with $b = 3$.

Proof: reduction from BIPARTITE CUBIC SUBGRAPH problem: Testing whether a bipartite graph has a 3-regular subgraph.

We add new vertices and create $K_{3,3}$ subgraphs in G':
original agent gets: $\frac{3}{2} - \frac{1}{n}$
new agents get: $\frac{3}{2} + \frac{1}{5n}$



Blocking coalition exists $\iff$ G has a 3-regular subgraph

Conclusions

- Half-matchings are crucial in solving and characterising the roommates problems.
- The 'basic' capacitated stable matching problems can be reduced to non-capacitated problems by simple graph constructions, thus their properties are similar.
- The basic models with payments are not much different from the corresponding models without payments (although we still need to understand the exact connections)

Further references on generalised roommates problems:

- A. Alkan and A. Tuncay. Pairing games and markets. Working paper, August 2013.
- P. Biró, and T. Fleiner. The Integral Stable Allocation Problem on Graphs. Discrete Optimization 7(1-2), pp: 64-73, 2010.
- P. Biró, and T. Fleiner. Fractional solutions for capacitated NTU-games, with applications to stable matchings. To appear in Discrete Optimization, 2015.
- T. Fleiner. The stable roommates problem with choice functions. In proceedings of IPCO 2008, LNCS, vol. 5035, pp:385-400, 2008.

Open questions

- Any further result of non-capacited models that can be generalised to capacitated models? (e.g. the path to stability result)
- More general models, e.g. stable fixtures **with contributions**? Motivation: a friendly game might take 1 day for the home team but 3 days for the visitors...
- Other TU-games with capacities and contributions?

References on capacitated TU-games with contributions:

- G. Chalkiadakis, E. Elkind, E. Markakis, M. Polukarov and N. R. Jennings. Cooperative Games with Overlapping Coalitions. Journal of Artificial Intelligence Research, 39:179–216, 2010.
- Y. Zick, E. Elkind. Arbitrators in overlapping coalition formation games. Proceedings of AAMAS 2011.
- Y. Zick, G. Chalkiadakis, E. Elkind. Overlapping coalition formation games: Charting the tractability frontier. Proceedings of AAMAS 2012.

Further references

New book on the algorithmic aspects:
David F. Manlove: Algorithmics of matching under preferences. World Scientific, 2013.

Summer school talks by Manlove and others:
`http://econ.core.hu/english/res/MatchingSchool.html`

COST Action on Computational Social Choice:
`http://www.illc.uva.nl/COST-IC1205/`

The Matching in Practice network website:
`http://www.matching-in-practice.eu/`

My research website:
`http://www.cs.bme.hu/~pbiro/research.html`