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**NORMATIVE PROPERTIES
OF MULTI-CRITERIA CHOICE
PROCEDURES AND THEIR
SUPERPOSITIONS: II**

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Two-stage superposition choice procedures, which sequentially apply two choice procedures so that the result of the first choice procedure is the input for the second choice procedure, are studied. We define which of them satisfy given normative conditions, showing how a final choice is changed due to the changes of preferences or a set of feasible alternatives. A theorem is proved showing which normative conditions are satisfied for two-stage choice procedures based on different scoring rules, rules, using majority relation, value function and tournament matrix. A complexity of two-stage choice procedures as well as its runtime on real data are evaluated.

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Introduction

The choice of the best alternatives among a set of all possible alternatives has been a matter for study, analysis and debates for a long time. It is hardly to find any sphere where this problem did not occur.

There are a lot of different choice procedures that allow to choose and rank alternatives from the initial set [1-10]. In this part, we consider choice procedures of a special type based on the superposition principle. Let us remind that by superposition of two choice functions $C_1(\cdot)$ and $C_2(\cdot)$ we mean a binary operation \odot , the result of which is a new function $C^*(\cdot)=C_2(\cdot)\odot C_1(\cdot)$, having the form $\forall X \in 2^A C^*(X)=C_2(C_1(X))$ [1]. In other words, superposition consists in sequential application of choice functions where the result of the previous choice function C_1 is the input for the next choice function C_2 . It is necessary to mention that the change of the order of functions may lead to completely different results, as the superposition operation is not commutative.

The interest in superposition of choice procedures can be explained by several reasons. First, most existing accurate choice procedures have a high computational complexity so they cannot be applied in the cases when the number of alternatives or/and criteria is very large. The use of superposition allows to reduce the complexity by applying choice procedures with a low computational complexity on first stages and more accurate choice procedures on final stages. Thus, the results can be obtained in a reasonable time. Second, there are a lot of situations when after applying some choice procedures the remaining set of alternatives is too large. The use of superposition allows to avoid such situations through the use of additional choice procedures.

The change of presentation, a set of criteria or criterial values of some alternatives can affect the final choice. Consequently, there is a need for more detailed study of existing choice procedures and for understanding which of them can be used in a particular case.

Thus, we consider the two-stage superposition choice procedures based on scoring rules, rules, using majority relation, value function and tournament matrix. The main focus of the paper is the study of its properties, its computational complexity and its runtime on real data. The study of the properties of two-stage superposition choice procedures is based on the study of the properties of different multi-criteria choice procedures which is done in the first part of the study [11].

This part is divided into several sections. First, some background information on two-stage choice procedures is given. Second, we form a list of two-stage choice procedures. Then, we study the properties of two-stage superposition choice procedures. Finally, a computational and run-time complexity of studied procedures is given.

A survey of the literature

In [1] two-stage extremizational choice procedures that consist of scalar or vector choice procedures were studied. The description of such choice procedures is given below.

Definition 1. A choice function $C(\cdot)$ is called rationalized by scalar criterion φ (or simply scalar), if $\forall X \subseteq A$

$$C(X) = \{y \in X | \nexists x \in X: \varphi(x) > \varphi(y)\}.$$

Suppose now that all alternatives from A are mapped into n criterial scales instead of one criterial scale. Denote by $\vec{\varphi}$ a set of n criterial scales $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$, where $\vec{\varphi}$ is a «vector criterion», and φ_i , where $i = 1, 2, \dots, n$, - its component.

Definition 2. A choice function $C(\cdot)$ is called rationalized by vector criterion $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$ or simply vector, if $\forall X \subseteq A$

$$C(X) = \{y \in X | \nexists x \in X: \vec{\varphi}(x) > \vec{\varphi}(y)\},$$

where

$$\vec{\varphi}(x) > \vec{\varphi}(y) \Leftrightarrow \forall i \in \{1, \dots, n\} \varphi_i(y) > \varphi_i(x)$$

Thus, there are 4 main types of two-stage extremizational choice procedures

1. scalar-scalar choice procedure;
2. scalar-vector choice procedure;
3. vector-scalar choice procedure;
4. vector-vector choice procedure.

As the single-criterion extremizational choice procedure is a special case of the vector choice procedure, it is clear that the choice procedure 1 is a special case of choice procedures 2 and 3, which are special cases of the procedure 4. In [1] it was also shown that scalar-scalar choice procedure (type 1) is equivalently reducible to the usual single-criterion extremizational choice procedure and two-stage scalar-vector choice procedure - to the usual multi-criteria extremizational choice procedure. To define under what circumstances the procedure of type 2 is equivalent to the scalar procedure, a notion of ψ -triad was introduced in [1].

Definition 3. Let $u, v, w \in A$, where A is a set of alternatives. These alternatives are said to form a ψ -triad if $\psi(u) < \psi(v)$, $\psi(u) \chi \psi(w)$ and $\psi(v) \chi \psi(w)$, where the inequality is understood as some vector (component-wise) inequality, and χ stands for the independent relation introduced as

$$\psi(u) \chi \psi(w) \Leftrightarrow \exists i, j \in I = \{1, \dots, n\}: \psi_i(u) \geq \psi_i(w) \& \psi_j(u) \leq \psi_j(w),$$

where I - a set of criteria, ψ - vector criterion.

The scalar-vector two-stage choice procedure (type 2) is equivalent to the one-stage single-criterion extremizational choice procedure if and only if the set X lacks ψ -triads for any $X \subseteq A$ and the criterion φ has the same value over all alternatives from this set, that is $\forall x \in X \varphi(x) = const$ [1].

As for the vector-vector choice procedure of type 4, it was shown in [1] that this procedure is far from being always reducible to the usual multi-criteria (all the more so, to single-criterion) choice procedure. Reducibility is possible only for some special relative positions of the alternatives in criterial spaces φ and ψ .

In [1] there were also defined under which conditions choice procedures of type 3 and 4 can be reduced to pair-dominant choice procedures. Let us remind that a choice procedure $C(\cdot)$ is called pair-dominant or rationalized by binary relation P if $\forall X \subseteq A$

$$C(X) = \{y \in X | \nexists x \in X: xPy\} \quad (1)$$

In other words, pair-dominant choice procedure is a procedure for which the rational choice consists in the choice of undominated by their pairwise comparisons alternatives.

To define under which conditions choice procedures of type 4 can be reduced to pair-dominant choice procedure, a notion of uncoordinated triad was introduced.

Definition 4. Let $u, v, w \in A$ make up a φ -triad referred to as a φ, ψ -uncoordinated triad of the first, or second, or third type if correspondingly

1. $\psi(w) < \psi(u), \psi(w) \not\leq \psi(v)$ or
2. $\psi(u) \chi \psi(w), \psi(v) < \psi(w)$ or
3. $\psi(u) \chi \psi(w), \psi(v) \chi \psi(w),$

where φ, ψ – vector criteria, χ – independence relation, $\not\leq$ is a violation of the vector inequality $<$, i.e.,

$$\psi(u) \not\leq \psi(v) \Leftrightarrow \exists i \in I: \psi_i(u) \geq \psi_i(v).$$

It was also proved in [1] that for the two-stage vector-vector choice procedure an equivalent pair-dominant choice procedure exists if and only if the alternatives from A do not make up φ, ψ -uncoordinated triads of the first type.

As for a pair-dominant choice procedure, it is necessary to define its levels. It was defined in [1] that a pair-dominant choice procedure has the level $\hat{1}$ if a preference P on alternatives in (1) is acyclic, level $\hat{2}$ if P is acyclic and transitive, i.e., it is a partial order, level $\hat{3}$ if P is acyclic, transitive and negatively transitive, i.e., it is a weak order.

It turned out that for the two-stage vector-vector choice procedure to generate a choice function of level $\hat{2}$ (level $\hat{3}$), it is necessary and sufficient that no alternatives from set A make up φ, ψ -noncoordinated triads of a second type (correspondingly φ, ψ -noncoordinated triads of a second and third type, and also ψ, φ -noncoordinated triads of third type).

Finally, it was proved in [1] that two-stage vector-scalar choice procedure (type 3) is equivalent to

- a) a pair-dominant level $\hat{1}$ choice procedure;
- b) a one-stage multi-criteria extremizational choice procedure; and
- c) a one-stage single-criterion extremizational choice procedure.

if and only if there are no φ -triads $\langle u, v, w \rangle$ (φ – vector criterion) in A satisfying, respectively, the following conditions

- a') $\psi(u) > \psi(w) \geq \psi(v)$, where ψ is a scalar criterion;
- b') $\psi(u) \geq \psi(w) \geq \psi(v)$ with at least one strict inequality;
- c') $\psi(u) \geq \psi(w) \geq \psi(v)$.

In [1] an example of a two-stage choice procedure was provided. This two-stage choice procedure operates as follows: at the first stage a Pareto set is chosen from initial set X from which, at the second stage, an alternative is isolated by Euclidean metric to some «ideal» point $\omega_0 = (\omega_1, \dots, \omega_n)$ in criterial space, where $\omega_1, \dots, \omega_n$ – maximum values for each criterion. It was shown that this choice procedure can be reduced to its second stage iff

$$\forall i \in I \omega_i \geq \max_{x \in A} \varphi_i(x).$$

In [12] a two-stage sequential choice procedure was studied, the first stage being defined by q-Pareto multicriterial choice rule, and the second stage being defined by scalar extremization choice procedure.

Let us define a q-Pareto choice rule that was studied in [2,3]. The main idea of this rule is to choose alternatives which are dominated by no more than q alternatives. Hence, it allows to avoid situations when the measurement by criterion has not been made very accurately and, consequently, it allows not to miss almost optimal elements from the chosen set of alternatives.

Below we consider q-Pareto choice rule more precisely, but first we need an additional notion of upper contour set.

The upper contour set $D_i(x)$ for an alternative x in the i -th criterion is

$$D_i(x) = \{y \in A \mid \varphi_i(y) > \varphi_i(x)\},$$

where φ is a vector criterion.

In other words, the upper contour set for x is a set of all alternatives which have higher values than x in the criterion i .

The upper contour set for an alternative x in a set X is defined as $[D_i(x) \cap X]$.

Now, we re-define the Pareto rule as

$$x \in C(X) \Leftrightarrow \text{card}(\cap_{i \in N} [D_i(x) \cap X]) = 0.$$

Obviously, Pareto rule is a special case of q-Pareto rule when parameter q is equal to 0. The generalization of q-Pareto choice rule can be defined as

$$x \in C(X) \Leftrightarrow \text{card}(\cap_{i \in N} [D_i(x) \cap X]) \leq q.$$

Thus, the choice rule consists in choice of alternatives which are dominated by no more than q alternatives.

Example. Consider the following situation. $A = \{a, b, c, d, e, f, g, h, k, l, m\}$.

A	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>	<i>l</i>	<i>m</i>
φ_1	1	3	5	0	5	4	4	5	2	4	1
φ_2	5	3	0	4	1	2	5	4	4	4	3

The results of applying q-Pareto choice rule depending on parameter value q are the following

q	C(A)
0	$\{g, h\}$
1	$\{a, e, g, h\}$
2	$\{a, c, e, g, h, l\}$
3	$\{a, b, c, e, f, g, h, k, l\}$
4	$\{a, b, c, e, f, g, h, k, l\}$
5	$\{a, b, c, d, e, f, g, h, k, l\}$
6	$\{a, b, c, d, e, f, g, h, k, l, m\}$

In [13] it was studied which rationality conditions are satisfied for q-Pareto choice rule. It was found that q-Pareto choice rule satisfies only condition **C** (see below).

In [12] q-Pareto choice rule is used on the first stage of two-stage choice procedure. However, since the choice set of the first stage usually contains too many elements, obtained set is used as a presentation for the second stage constructed by a scalar extremization choice procedure. Thus, q-Pareto-scalar choice procedure can be presented as a superposition of two choice functions $C(\cdot) = C_2(C_1(\cdot))$, where C_1 is defined by q-Pareto rule, C_2 is defined by scalar choice procedure.

q-Pareto-scalar choice procedure was firstly studied in [12]. There were found necessary and sufficient conditions when this procedure can be reduced to the choice on scalar criterion.

Two-stage choice procedure can be reduced to one-stage choice procedure if $C_1(X) \subseteq C_2(X)$, i.e., when for any subset $X \subseteq A$ the remaining set after the first stage of the rule always contains alternatives which are chosen on the second stage of the rule. In addition, necessary conditions were defined in [12] under which rationality conditions are satisfied for two-stage q-Pareto-scalar choice procedure.

Before moving to the study of the properties of two-stage choice procedures, it is necessary to consider the notion of choice functions closedness with respect to superposition operator. In order to do it some definitions were given in [1].

Any particular choice function $C(\cdot)$ can be treated as a point of the abstract space \mathcal{C} , consisting of all the possible choice functions on A .

A set of choice functions $C_1(\cdot), \dots, C_n(\cdot)$ is called a functional profile and denoted by $\{C_i(\cdot)\}$.

Let a domain $Q \subseteq \mathcal{C}$ and number n be fixed. Any mapping from $Q^n = \underbrace{Q \times Q \times \dots \times Q}_n$ times into \mathcal{C} is called a functional operator. A functional operator assigns to any profile $\{C_i(\cdot)\}$ consisting of n choice functions from Q a single choice function denoted by $C^*(\cdot) = F(\{C_i(\cdot)\})$, where F stands for the functional operator.

As usual, the set Q^n is called the domain of F , and the set of all functions $C^*(\cdot) \in \mathcal{C}$ such that $\exists \{C_i(\cdot)\} \in Q^n: F(\{C_i(\cdot)\}) = C^*(\cdot)$ as the range of operator.

The set $Q_{def} \equiv Q$ is called the domain of the functional operator $F: Q^n \rightarrow \mathcal{C}$. Any domain $Q' \subseteq \mathcal{C}$ such that $\forall \{C_i(\cdot)\} \in Q^n F(\{C_i(\cdot)\}) \in Q'$ will be called the range of $F(Q' = Q_{def})$. According to the definition, the domain Q_{def} of the functional operator F is any subset from \mathcal{C} which includes the range F .

If $C^*(\cdot) = F(\{C_i(\cdot)\}) \in Q$ for any functional profile $\{C_i(\cdot)\} \in Q^n$, the domain Q is called closed with respect to the operator F .

Now introduce functional operator of ‘‘superposition’’ (F^\odot):

$$F^\odot: \forall X \in 2^A C^*(X) = C_2(C_1(X)).$$

Let us define in which cases superposition, as applied to the choice functions $C_1(\cdot)$ and $C_2(\cdot)$ from different domains of \mathcal{C} , ‘preserves’ the result, i.e., the function $C^*(\cdot)$ in any of the domains. With such an approach, it is only natural to speak about ‘superposition’ mapping the domains Q_1 and Q_2 to which, respectively, $C_1(\cdot)$ and $C_2(\cdot)$ belong into the domain Q' , denoting it symbolically as $\mathfrak{R}^\odot(Q_1, Q_2) = Q'$.

By closedness (conditional and unconditional) of domains, we consider the following. The domain $Q_1 \subseteq \mathcal{C}$ is called *conditionally closed* at the first superposition stage at fixed domain Q_2 of the second stage if $\mathfrak{R}^\odot(Q_1, Q_2) \subseteq Q_1$. Similarly, the domain $Q_2 \subseteq \mathcal{C}$ is called conditionally closed at the second superposition stage at fixed domain Q_1 of the first stage if $\mathfrak{R}^\odot(Q_1, Q_2) \subseteq Q_2$. Finally, the domain Q is called unconditionally closed with respect to the ‘superposition’ operator if $\mathfrak{R}^\odot(Q_1, Q) \subseteq Q'$.

Theorem 1 [1]. There are the following conditions of domains closedness in \mathcal{C} with respect to the superposition operator \odot :

1°. The domain \mathbf{ACA} is closed (unconditionally) with respect to superposition operator, i.e., $\mathfrak{R}^\odot(K, K) \subseteq \mathbf{ACA}$; none of the domains $Q = \mathbf{H}, \mathbf{C}, \mathbf{O}, \mathbf{H} \cap \mathbf{C}, \mathbf{H} \cap \mathbf{O}, \mathbf{C} \cap \mathbf{O}, \mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$ is closed unconditionally with respect to the superposition operator;

2°. For $Q_1 = \mathbf{ACA}$, the domains $Q_2 = \mathbf{H}, \mathbf{C}, \mathbf{H} \cap \mathbf{C}, \widehat{\mathbf{O}}, \widehat{\mathbf{H}} \cap \widehat{\mathbf{O}}, \widehat{\mathbf{C}} \cap \widehat{\mathbf{O}}$ and $\widehat{\mathbf{H}} \cap \widehat{\mathbf{C}} \cap \widehat{\mathbf{O}}$ are conditionally closed at the second superposition stage. For $Q_1 = \mathbf{ACA}$, the domains $Q_2 = \mathbf{O}, \mathbf{H} \cap \mathbf{O}, \mathbf{C} \cap \mathbf{O}, \mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$ are not closed. None of the domains $Q_2 = \mathbf{H}$ or \mathbf{C} or... or $\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$ is closed for none $Q_1 = \mathbf{H}$, or \mathbf{C} , or... or $\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$, where the notion of $\widehat{\mathbf{H}}, \widehat{\mathbf{C}}, \widehat{\mathbf{O}}$ is used for a class of choice functions that satisfies current condition $(\mathbf{H}, \mathbf{C}, \mathbf{O})$ as well as the condition of choice non-emptiness.

3°. None of the fundamental domains and their intersections $\mathbf{H}, \mathbf{C}, \dots, \mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$ is conditionally closed as Q_1 , at the first superposition stage for $Q_2 = \mathbf{ACA}$ (and all the more so for $Q_2 = \mathbf{H}, \mathbf{C}, \dots, \mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$ at the second one).

Remark. Although generally going out of the classical domain $\mathbf{H} \cap \mathbf{C}$, the superposition of two classical choice functions may be shown to stay within the fundamental domain \mathbf{C} . It means that $\mathfrak{R}^\odot(\mathbf{H} \cap \mathbf{C}, \mathbf{H} \cap \mathbf{C}) \subset \mathbf{C}$.

Thus, the results on choice functions closedness with respect to superposition operator solve the inverse problem that helps to determine will the function resulting from application of this operator to the elements of the set satisfy the rationality conditions \mathbf{H}, \mathbf{C} or \mathbf{O} .

Now let us study the properties of two-stage choice procedures.

Two-stage superposition choice procedures

We consider the two-stage superposition choice procedures based on scoring rules, rules, using majority relation, value function and tournament matrix. A full description of studied choice procedures is given in [11].

A list of two-stage superposition choice procedures is provided in Table 1.

Table 1. Two-stage choice procedures

№	Stage 1	Stage 2
1-121	Scoring rules (11 procedures)	Scoring rules (11 procedures)
122-231		Rules, using majority relation (10 procedures)
232-286		Rules, using value function (5 procedures)
287-308		Rules, using tournament matrix (2 procedures)
309-418	Rules, using majority relation (10 procedures)	Scoring rules (11 procedures)
419-518		Rules, using majority relation (10 procedures)
519-568		Rules, using value function (5 procedures)
569-588		Rules, using tournament matrix (2 procedures)
589-643	Rules, using value function (5 procedures)	Scoring rules (11 procedures)
644-693		Rules, using majority relation (10 procedures)
694-718		Rules, using value function (5 procedures)
719-728		Rules, using tournament matrix (2 procedures)
729-750	Rules, using tournament matrix (2 procedures)	Scoring rules (11 procedures)
751-770		Rules, using majority relation (10 procedures)
771-780		Rules, using value function (5 procedures)
781-784		Rules, using tournament matrix (2 procedures)

Thus, 784 two-stage procedures of 16 different types are studied.

As properties of two-stage choice procedures should be studied separately, let us assign an identification number to each two-stage choice procedure by the following formula

$$id = 28 * (i - 1) + j,$$

where id is an identification number of two-stage choice procedure, i is a number of choice procedure from [11] used on the first stage, j is a number of choice procedure from [11] used on the second stage.

Before proceeding to the study of the properties of two-stage choice procedures, it is necessary to make some notes.

Note 1. Table 2 provides a list of two-stage procedures that does not make any sense, i.e., the second stage of which does not change the choice.

Table 2. Two-stage choice procedures that does not make any sense («...» is any studied choice procedure)

Two-stage procedure		
№	Stage 1	Stage 2
1-28	Simple majority rule	...
113-140	Run-off procedure	...
141-168	Hare rule (Ware procedure)	...
231	Inverse Borda rule	Borda rule
233	Inverse Borda rule	Inverse Borda rule
234	Inverse Borda rule	Nanson rule
259	Nanson rule	Borda rule
261	Nanson rule	Inverse Borda rule
262	Nanson rule	Nanson rule
281-308	Coombs procedure	...
320	Minimal dominant set	Minimal dominant set
348	Minimal undominant set	Minimal dominant set
349	Minimal undominant set	Minimal undominant set
505-532	Condorcet winner	...
533	Core	Simple majority rule
539	Core	Borda rule
540	Core	Black procedure
541	Core	Inverse Borda rule
542	Core	Nanson rule
544	Core	Minimal dominant set
545	Core	Minimal undominant set
546	Core	Minimal weakly stable set
547	Core	Fishburn rule
548	Core	Uncovered set I
549	Core	Uncovered set II
550	Core	Richelson rule
552	Core	Core
553	Core	k-stable set (k>1)
555	Core	Copeland rule 1
556	Core	Copeland rule 2
557	Core	Copeland rule 3
559	Core	Minimax procedure
560	Core	Simpson procedure
Total number of choice procedures - 168		

The simple majority rule, the run-off procedure, the Hare procedure, the Coombs procedure and the Condorcet winner choose a single best alternative, consequently, two-stage choice procedures 1-28, 113-168, 281-308, 505-532 which use such procedures on the first stage do not make any sense as their second stage does not affect the final choice.

Also two-stage procedures 231, 233, 234, 259, 261, 262 do not make any sense as the first stage of such procedures gives a set of alternatives with the same Borda count. Similarly, two-stage procedures 283, 291 do not make any sense as the first stage of such procedures gives a set of alternatives which are considered as the worst by the same number of criteria.

Two-stage procedures 320, 348, 349 do not make any sense as by the definition the remaining after the first stage set of alternatives is already minimal. As for two-stage choice procedure 553, it does not make any sense as the second stage of such procedures does not change the choice.

Finally, two-stage procedures 533, 539-542, 544-550, 552, 555-557, 559-560 do not make any sense as $\forall x, y \in C_1(\vec{P}_X, X) \rightarrow card\{i \in N | xP_i y\} = card\{i \in N | yP_i x\}$, where $C_1(\vec{P}_X, X)$ is a first-stage procedure (the core).

Thus, choice procedures provided in Table 2 are excluded from further consideration.

Note 2. Properties of two-stage choice procedures which use the Black procedure on the first stage completely coincide with properties of two-stage choice procedures which use the Borda rule if there is no Condorcet winner.

Note 3. Table 3 provides a list of two-stage procedures which are equivalent to existing choice procedures.

Table 3. Two-stage procedures equivalent to existing choice procedure

Two-stage choice procedure			What procedure is equivalent to
№	Stage 1	Stage 2	
309	Minimal dominant set	Simple majority rule	Condorcet winner
321	Minimal dominant set	Minimal undominant set	Minimal undominant set
322	Minimal dominant set	Minimal weakly stable set	Minimal weakly stable set
323	Minimal dominant set	Fishburn rule	Fishburn rule
324	Minimal dominant set	Uncovered set I	Uncovered set I
325	Minimal dominant set	Uncovered set II	Uncovered set II
326	Minimal dominant set	Richelson rule	Richelson rule
327	Minimal dominant set	Condorcet winner	Condorcet winner
328	Minimal dominant set	Core	Core
331	Minimal dominant set	Copeland rule 1	Copeland rule 1
332	Minimal dominant set	Copeland rule 2	Copeland rule 2
333	Minimal dominant set	Copeland rule 3	Copeland rule 3
337	Minimal undominant set	Simple majority rule	Core (single chosen alternative)
350	Minimal undominant set	Minimal weakly stable set	Minimal weakly stable set
355	Minimal undominant set	Condorcet winner	Condorcet winner
356	Minimal undominant set	Core	Core
393	Fishburn rule	Simple majority rule	Core (single chosen alternative)
411	Fishburn rule	Condorcet winner	Core (single chosen alternative)
421	Uncovered set I	Simple majority rule	Core (single chosen alternative)
439	Uncovered set I	Condorcet winner	Core (single chosen alternative)
449	Uncovered set II	Simple majority rule	Core (single chosen alternative)
467	Uncovered set II	Condorcet winner	Core (single chosen alternative)
477	Richelson rule	Simple majority rule	Core (single chosen alternative)
495	Richelson rule	Condorcet winner	Core (single chosen alternative)
551	Core	Condorcet winner	Condorcet winner
Total number of choice procedures - 25			

Two-stage choice procedures provided in previous table are equivalent to some existing choice procedures. Thus, their properties fully coincided with properties of such existing procedures. However, these two-stage procedures are not excluded from further consideration as the computational complexity of some of them can be lower than the complexity of existing procedures.

Thus, it remains to study properties of 591 two-stage choice procedures.

A study of the properties of two-stage choice procedures

A list of studied normative properties is given in [11].

A study of the properties is conducted as follows. If a two-stage choice procedure does not satisfy given normative condition, a counter-example is provided. On the contrary, if a two-stage choice procedure satisfies given normative condition a necessary proof is followed. The study of the properties of two-stage choice procedures is based on the study of the properties of multi-criteria choice procedures which is done in [11].

The results of the study of the properties of 591 two-stage choice procedures are given in Theorem 2.

Theorem 2. Information on which choice procedures satisfy given normative conditions is provided in Table 4.

Table 4. Properties of two-stage choice procedures («+» - choice procedure satisfies given normative condition, «-» - choice procedure does not satisfy given normative condition)

Two-stage choice procedure				Normative conditions								
				Rationality				Monotonicity			Non-compensability	
				Condition H	Condition C	Condition O	Condition ACA	Monotonicity 1	Monotonicity 2	Strict monotonicity		
Stage 1		Stage 2										
№	Name	№	Name									
2	Plurality rule											
3	Inverse plurality rule											
4	q-Approval rule ($q > 1$)											
12	Minimal dominant set											
13	Minimal undominant set											
15	Fishburn rule											
16	Uncovered set I											
17	Uncovered set II	5	Run-off procedure									
18	Richelson rule	6	Hare rule (Ware procedure)	-	-	-	-	-	+	-	-	
20	Core	11	Coombs procedure									
23	Copeland rule 1											
24	Copeland rule 2											
25	Copeland rule 3											
26	Super-threshold rule (threshold depends on X)											
27	Minimax procedure											
28	Simpson procedure											
9	Inverse Borda rule	1	Simple majority rule									
10	Nanson rule	5	Run-off procedure									
14	Minimal weakly stable set	6	Hare rule (Ware procedure)	-	-	-	-	-	+	-	-	
21	k-stable set ($k > 1$)	11	Coombs procedure									
		19	Condorcet winner									
2	Plurality rule											
3	Inverse plurality rule											
4	q-Approval rule ($q > 1$)											
23	Copeland rule 1											
24	Copeland rule 2	1	Simple majority rule						+	+	-	-
25	Copeland rule 3	19	Condorcet winner	-	-	-	-	+	+	-	-	
26	Super-threshold rule (threshold depends on X)											
27	Minimax procedure											
28	Simpson procedure											

Two-stage choice procedure				Normative conditions							
				Rationality			Monotonicity				
				Condition H	Condition C	Condition O	Condition ACA	Monotonicity 1	Monotonicity 2	Strict monotonicity	Non-compensability
Stage 1		Stage 2									
№	Name	№	Name								
7	Borda rule	1	Simple majority rule	-	-	-	-	+	+	-	-
8	Black procedure	5	Run-off procedure								
22	Threshold rule	6	Hare rule (Ware procedure)								
		19	Condorcet winner								
13	Minimal undominant set	1	Simple majority rule	+	+	-	-	+	+	-	-
20	Core										
15	Fishburn rule	1	Simple majority rule	+	+	-	-	+	+	-	-
16	Uncovered set I	19	Condorcet winner								
17	Uncovered set II										
18	Richelson rule										
20	Core	7	Borda rule								
		8	Black procedure								
		9	Inverse Borda rule								
		10	Nanson rule								
		12	Minimal dominant set								
		13	Minimal undominant set								
		14	Minimal weakly stable set								
		15	Fishburn rule	+	+	-	-	+	+	-	-
		16	Uncovered set I								
		17	Uncovered set II								
		18	Richelson rule								
		23	Copeland rule 1								
		24	Copeland rule 2								
		25	Copeland rule 3								
		27	Minimax procedure								
		28	Simpson procedure								
17	Uncovered set II	12	Minimal dominant set	-	+	-	-	+	-	-	-
2	Plurality rule										
3	Inverse plurality rule										
4	q-Approval rule (q>1)	21	k-stable set (k>1)	-	-	-	-	+	-	-	-
23	Copeland rule 1										
24	Copeland rule 2										
25	Copeland rule 3										
16	Uncovered set I	20	Core	-	-	-	-	+	-	-	-

Two-stage choice procedure				Normative conditions						
				Rationality			Monotonicity			
				Condition H	Condition C	Condition O	Condition ACA	Monotonicity 1	Monotonicity 2	Strict monotonicity
Stage 1		Stage 2								
№	Name	№	Name							
		2	Plurality rule							
		3	Inverse plurality rule							
		4	q-Approval rule ($q > 1$)							
		7	Borda rule							
		8	Black procedure							
		12	Minimal dominant set							
		13	Minimal undominant set							
		14	Minimal weakly stable set							
		15	Fishburn rule							
2	Plurality rule	16	Uncovered set I	-	-	-	-	+	-	-
3	Inverse plurality rule	17	Uncovered set II							
4	q-Approval rule ($q > 1$)	18	Richelson rule							
7	Borda rule	20	Core							
8	Black procedure	22	Threshold rule							
22	Threshold rule	23	Copeland rule 1							
23	Copeland rule 1	24	Copeland rule 2							
24	Copeland rule 2	25	Copeland rule 3							
25	Copeland rule 3	26	Super-threshold rule (threshold depends on X)							
		27	Minimax procedure							
		28	Simpson procedure							
7	Borda rule	9	Inverse Borda rule	-	-	-	-	+	-	-
8	Black procedure	10	Nanson rule							
22	Threshold rule	21	k-stable set ($k > 1$)							
13	Minimal undominant set	16	Uncovered set I	-	-	-	-	+	-	-
17	Uncovered set II	18	Richelson rule							
		15	Fishburn rule							
		17	Uncovered set II							
		21	k-stable set ($k > 1$)	-	-	-	-	+	-	-
13	Minimal undominant set	23	Copeland rule 1							
		24	Copeland rule 2							
		25	Copeland rule 3							
12	Minimal dominant set	26	Super-threshold rule (threshold depends on X)	-	-	-	-	+	-	-
13	Minimal undominant set	27	Minimax procedure							
		28	Simpson procedure							

Two-stage choice procedure				Normative conditions							
				Rationality				Monotonicity			Non-compensability
				Condition H	Condition C	Condition O	Condition ACA	Monotonicity 1	Monotonicity 2	Strict monotonicity	
Stage 1		Stage 2									
№	Name	№	Name								
15	Fishburn rule	12	Minimal dominant set								
16	Richelson rule	16	Uncovered set I	-	-	-	-	+	-	-	
18	Uncovered set I	18	Richelson rule								
26	Super-threshold rule (threshold depends on X)	12	Minimal dominant set								
27	Minimax procedure	13	Minimal undominant set	-	-	-	-	+	-	-	
28	Simpson procedure	20	Core								
2	Plurality rule										
3	Inverse plurality rule										
4	q-Approval rule ($q > 1$)										
12	Minimal dominant set	9	Inverse Borda rule								
13	Minimal undominant set	10	Nanson rule	-	-	-	-	-	-	-	
23	Copeland rule 1										
24	Copeland rule 2										
25	Copeland rule 3										
9	Inverse Borda rule	12	Minimal dominant set								
10	Nanson rule	13	Minimal undominant set								
14	Minimal weakly stable set	16	Uncovered set I	-	-	-	-	-	-	-	
21	k-stable set ($k > 1$)	18	Richelson rule								
		20	Core								
9	Inverse Borda rule	2	Plurality rule								
10	Nanson rule	4	q-Approval rule ($q > 1$)								
14	Minimal weakly stable set	8	Black procedure								
15	Fishburn rule	14	Minimal weakly stable set								
16	Uncovered set I	15	Fishburn rule								
17	Uncovered set II	17	Uncovered set II								
18	Richelson rule	21	k-stable set ($k > 1$)	-	-	-	-	-	-	-	
21	k-stable set ($k > 1$)	22	Threshold rule								
26	Super-threshold rule (threshold depends on X)	23	Copeland rule 1								
27	Minimax procedure	24	Copeland rule 2								
28	Simpson procedure	25	Copeland rule 3								
		26	Super-threshold rule (threshold depends on X)								
		27	Minimax procedure								
		28	Simpson procedure								
16	Uncovered set I	13	Minimal undominant set	-	-	-	-	-	-	-	

Two-stage choice procedure				Normative conditions							
				Rationality				Monotonicity			Non-compensability
				Condition H	Condition C	Condition O	Condition ACA	Monotonicity 1	Monotonicity 2	Strict monotonicity	
Stage 1		Stage 2									
№	Name	№	Name								
9	Inverse Borda rule	3	Inverse plurality rule	-	-	-	-	-	-	-	-
10	Nanson rule	11	Inverse plurality rule	-	-	-	-	-	-	-	-
12	Minimal dominant set	2	Plurality rule	-	-	-	-	-	-	-	-
13	Minimal undominant set	3	Inverse plurality rule	-	-	-	-	-	-	-	-
		4	q-Approval rule ($q > 1$)	-	-	-	-	-	-	-	-
		7	Borda rule	-	-	-	-	-	-	-	-
		8	Black procedure	-	-	-	-	-	-	-	-
		22	Threshold rule	-	-	-	-	-	-	-	-
14	Minimal weakly stable set			-	-	-	-	-	-	-	-
15	Fishburn rule			-	-	-	-	-	-	-	-
16	Uncovered set I	3	Inverse plurality rule	-	-	-	-	-	-	-	-
17	Uncovered set II	7	Borda rule	-	-	-	-	-	-	-	-
18	Richelson rule	9	Inverse Borda rule	-	-	-	-	-	-	-	-
21	k-stable set ($k > 1$)	10	Nanson rule	-	-	-	-	-	-	-	-
26	Super-threshold rule (threshold depends on X)			-	-	-	-	-	-	-	-
27	Minimax procedure			-	-	-	-	-	-	-	-
28	Simpson procedure			-	-	-	-	-	-	-	-
15	Fishburn rule	13	Minimal undominant set	-	-	-	-	-	-	-	-
17	Uncovered set II	20	Core	-	-	-	-	-	-	-	-
18	Richelson rule			-	-	-	-	-	-	-	-
20	Core	2	Plurality rule	-	-	-	-	-	-	-	-
		3	Inverse plurality rule	-	-	-	-	-	-	-	-
		4	q-Approval rule ($q > 1$)	-	-	-	-	-	-	-	-
		22	Threshold rule	-	-	-	-	-	-	-	-
		26	Super-threshold rule (threshold depends on X)	-	-	-	-	-	-	-	-
26	Super-threshold rule (threshold depends on X)	16	Uncovered set I	-	-	-	-	-	-	-	-
27	Minimax procedure	18	Richelson rule	-	-	-	-	-	-	-	-
28	Simpson procedure			-	-	-	-	-	-	-	-

The proof of the theorem is provided in Appendix 1.

Computational complexity of choice procedures

A computational complexity of choice procedures used in two-stage superpositions is provided in Table 5.

Table 5. Theoretical computational complexity of existing choice procedures (M – cardinality of initial set of alternatives, n – number of criteria, q, k – parameters of choice procedures, l, d – parameters which depend on initial set of alternatives, $d \geq 1, 1 \leq l \leq M$)

№	Name of choice procedure	Computational complexity (theoretical)	Maximum number of remaining alternatives after applying the choice procedure
1	Simple majority rule	$O(M \cdot n)$	1
2	Plurality rule	$O(M \cdot n)$	n
3	Inverse plurality rule	$O(M \cdot n)$	$M - n$
4	q-Approval rule ($q > 1$)	$O(M \cdot n \cdot \log_2(q + 1))$	$q \cdot n$
5	Run-off procedure	$O(M \cdot n)$	1
6	Hare rule (Ware procedure)	$O(M \cdot n)$	1
7	Borda rule	$O(M \cdot \log_2(M) \cdot n)$	M
8	Black procedure	$O(M \cdot \log_2(M) \cdot n)$	M
9	Inverse Borda rule	$O(M^2 \cdot n)$	M
10	Nanson rule	$O(M^2 \cdot n)$	M
11	Coombs procedure	$O(M^2 \cdot n)$	n
12	Minimal dominant set	$O(M^{2,37 \cdot d})$	M
13	Minimal undominant set	$O(M^{2,37 \cdot d})$	M
14	Minimal weakly stable set	$O(M^2 \cdot n + \sum_{i=1}^l C_M^i \cdot (M - i))$	M
15	Fishburn rule	$O(M^3)$	M
16	Uncovered set I	$O(M^3)$	M
17	Uncovered set II	$O(M^3)$	M
18	Richelson rule	$O(M^3)$	M
19	Condorcet winner	$O(M \cdot n)$	1
20	Core	$O(M^2 \cdot n)$	M
21	k-stable set ($k > 1$)	$O(M^{2,37 \cdot k})$	M
22	Threshold rule	$O(M \cdot n)$	M
23	Copeland rule 1	$O(M^2 \cdot n)$	M
24	Copeland rule 2	$O(M^2 \cdot n)$	M
25	Copeland rule 3	$O(M^2 \cdot n)$	M
26	Super-threshold rule	$O(M)$	M
27	Minimax procedure	$O(M^2 \cdot n)$	M
28	Simpson procedure	$O(M^2 \cdot n)$	M

A computational complexity of choice procedures was calculated by the author of the paper.

Based on information provided in Table 5 we can divide all two-stage procedures in several groups in accordance with their computational complexity. The results are provided in Table 6.

Table 6. A computational complexity of two-stage choice procedures («...» is any choice procedure)

Two-stage choice procedure	
Stage 1	Stage 2
Choice procedures with a low computational complexity	
Plurality rule Threshold rule q-Approval rule ($q > 1$)	...
Inverse plurality rule Super-threshold rule Borda rule Black procedure	Simple majority rule Run-off procedure Hare rule (Ware procedure) Borda rule Black procedure Condorcet winner Plurality rule Threshold rule Inverse plurality rule q-Approval rule ($q > 1$) Super-threshold rule
Computational complexity depends on initial set of alternatives	
Inverse plurality rule Super-threshold rule Borda rule Black procedure	Inverse Borda rule Nanson rule Core Copeland rules 1-3 Minimax procedure Simpson procedure Coombs procedure
Choice procedures with average computational complexity	
Inverse Borda rule Nanson rule Core Copeland rules 1-3 Minimax procedure Simpson procedure	Simple majority rule Run-off procedure Hare rule (Ware procedure) Borda rule Black procedure Condorcet winner Plurality rule Threshold rule Inverse plurality rule q-Approval rule ($q > 1$) Super-threshold rule Inverse Borda rule Nanson rule Core Copeland rules 1-3 Minimax procedure Simpson procedure Coombs procedure
Coombs procedure	...
Choice procedures with a high computational complexity	
Inverse plurality rule Super-threshold rule Borda rule Black procedure Inverse Borda rule	Minimal dominant set Minimal undominant set Minimal weakly stable set Fishburn rule Uncovered set I, II

Two-stage choice procedure	
Stage 1	Stage 2
Nanson rule Core Copeland rules 1-3 Minimax procedure Simpson procedure	Richelson rule k-stable set (k>1)
Minimal dominant set Minimal undominant set Minimal weakly stable set Fishburn rule Uncovered set I, II Richelson rule k-stable set (k>1)	...

It is necessary to mention that by the group called “Computational complexity depends on initial set of alternatives” we mean that the computational complexity of such two-stage choice procedures depends on how many alternatives are eliminated at the first stage.

The run-time complexity for two-stage superposition choice procedures

Let the initial set M_1 contains 300 thousands of alternatives and the total number of criteria is equal to 10. Suppose, a computer can process around 3 billion instructions per second. Consider situations when the remaining set M_2 after applying the first-stage choice procedure contains 10, 50 and 100 thousands of alternatives.

The run-time complexity of two-stage choice procedures from different groups (according to Table 6) is provided in Table 7.

Table 7. The run-time complexity for two-stage choice procedures (M_2 – number of alternatives remained after applying the first stage of the choice procedure)

Two-stage choice procedures		Run time		
Stage 1	Stage 2	$M_2=10000$	$M_2=50000$	$M_2=100000$
Choice procedures with a low computational complexity				
Plurality rule	Uncovered set I	≈50 ms		
Inverse plurality rule	Condorcet winner	≈52 ms	≈58 ms	≈66 ms
Super-threshold rule	Threshold rule	≈6 ms	≈13 ms	≈21 ms
Borda rule	Plurality rule	≈910 ms	≈918 ms	≈926 ms
Computational complexity depends on initial set of alternatives				
Inverse plurality rule	Minimax procedure	≈16 s	≈7 min	≈27 min
Choice procedures with average computational complexity				
Inverse Borda rule	Simple majority rule	≈4 h 12 min		
Minimax procedure	Simpson procedure	≈4 h 15 min	≈4 h 21 min	≈4 h 36 min
Choice procedures with a high computational complexity				
Inverse plurality rule	Richelson rule	≈46 h 20 min	≈241 days	≈5 years
Uncovered set I	Borda rule	≈140 years		

The results obtained from Tables 6 and 7 give us information on which procedures can be applied when we deal with Big Data and which procedures cannot be applied in such problems as they are not allow to obtain any results in a sufficient time.

Conclusion

We have studied the properties of 592 two-stage choice procedures, which can be used in various multi-criteria problems. It was defined which choice procedures satisfy given normative conditions, showing how a final choice is changed due to the changes of preferences or a set of feasible alternatives. Such information leads to a better understanding of different choice procedures and how stable and sensible is a set of alternatives obtained after applying some choice procedure.

The results show that most of the two-stage procedures do not satisfy any normative conditions. Only some of them satisfy the Monotonicity condition 1. More detailed information is provided in Table 4.

To compute run-time complexity of choice procedures the average computational complexity was used. All choice procedures were divided into different groups (see Table 6). It was shown that two-stage choice procedures which use choice procedures with a high computational complexity on the first stage require more time than other procedures. It means that such procedures are not recommended to use in applications to Big Data. Two-stage choice procedures which use on the first stage choice procedures with a low computational complexity and on the second stage - with a high computational complexity can be used in applications to Big Data, however, their application depends on the number of alternatives remained after the first stage. Two-stage choice procedures, which use on both stages choice procedures with a low computational complexity, can be used in applications to Big Data with no restrictions.

Appendix 1. Properties of two-stage choice procedures

Two-stage choice procedure 29. ‘Plurality rule – Simple majority rule’

29.1 Heredity condition (H)

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃	P₄	P₅
<i>a</i>	<i>a</i>	<i>c</i>	<i>b</i>	<i>b</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>a</i>
<i>b</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>c</i>

According to the two-stage choice procedure $C(\vec{P}_X, X) = \{b\}$.

Consider now the subset $X' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃	P₄	P₅
<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>

According to the rule the alternative *c* will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{c\}$.

Then $C(\vec{P}_{X'}, X') \not\subseteq C(\vec{P}_X, X) \cap X'$. Thus, the condition **H** is not satisfied.

29.2 Concordance condition (C)

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃
<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>a</i>	<i>b</i>
<i>c</i>	<i>c</i>	<i>a</i>

According to the two-stage choice procedure the choice is empty, i.e., $C(\vec{P}_X, X) = \emptyset$.

Now let us consider the subset $X' = X \setminus \{c\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃
<i>a</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>a</i>

According to the two-stage choice procedure the alternative *b* will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{b\}$.

Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃
<i>b</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>

According to the two-stage choice procedure the alternative *b* will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

29.3 Outcast condition (O)

The condition **O** is not satisfied (see paragraph 29.1 of Appendix 1).

29.4 Arrow’s choice axiom (ACA)

The condition **ACA** is not satisfied since the two-stage choice procedure does not satisfy the condition **H**.

29.5 Monotonicity condition 1

Let $C(\vec{P}_X, X) = \{a\}$. It means that

1. $\forall x \in X \setminus \{a\} n^+(a, \vec{P}_X) > n^+(x, \vec{P}_X)$ or
2. $\exists y \in X \setminus \{a\} n^+(a, \vec{P}_X) = n^+(y, \vec{P}_X)$ and $n^+(a, \vec{P}_{X'}) > \frac{1}{2} \cdot |\vec{P}_{X'}|$, where X' is a set of alternatives remained for the second stage of the choice procedure.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative a . Then

- $n^+(a, \vec{P}'_X) > n^+(a, \vec{P}_X)$, $n^+(x, \vec{P}'_X) \leq n^+(x, \vec{P}_X)$ or
- $n^+(a, \vec{P}'_X) = n^+(a, \vec{P}_X)$, $n^+(a, \vec{P}'_X) \geq n^+(a, \vec{P}_{X'})$ and $|\vec{P}'_X| = |\vec{P}_X|$.

Thus, $C(\vec{P}'_X, X) = \{a\}$.

Then $a \in C(\vec{P}_X, X)$ and $a \in C(\vec{P}'_X, X)$. Consequently, the Monotonicity condition 1 is satisfied.

29.6 Monotonicity condition 2

Since given two-stage choice procedure chooses no more than one best alternative, the Monotonicity condition 2 is not applicable to it as it considers the choice of more than two alternatives. In other words, such two-stage choice procedure obeys the Monotonicity condition 2 trivially.

29.7 Strict monotonicity condition

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3
a	a	b
c	b	c
b	c	a

According to the two-stage choice procedure the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative c in P'_1 :

P'_1	P'_2	P'_3
c	a	b
a	b	c
b	c	a

According to the two-stage choice procedure the choice is empty, i.e., $C(\vec{P}'_X, X) = \emptyset$.

$$C(\vec{P}'_X, X) \neq \begin{cases} C(\vec{P}_X, X) \text{ or} \\ \{c\} \text{ or} \\ C(\vec{P}_X, X) \cup \{c\}. \end{cases}$$

Thus, the strict monotonicity condition is not satisfied.

29.8 Non-compensatory condition

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3
a	a	b
b	b	c
c	c	a

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Let us write the profile \vec{P}_X in the following form

X	φ_1	φ_2	φ_3
a	3	3	1
b	2	2	3
c	1	1	2

According to the non-compensatory condition the alternative b is better than the alternative a and the alternative a is better than the alternative c . Thus, the non-compensatory condition is not satisfied as $\{b\} \neq C(\vec{P}_X, X)$.

Two-stage choice procedures 30-46

Two-stage choice procedures 30-46 do not satisfy the same conditions as choice procedures which are used on the second stage. To prove it we can use the same examples but with larger number of alternatives omitted on the first stage of the choice procedure [11].

Example. Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

	P_1	P_2	P_3	P_4	P_5
a	a	a	c	b	b
c	c	c	b	a	a
b	b	b	a	c	c

Let us transform this example to the following form

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
d	e	f	g	h	a	b	c	a	b	c	
a	a	c	b	b	b	c	a	c	a	b	
c	c	b	a	a	c	a	b	b	c	a	
b	b	a	c	c					...		

According to the plurality rule the alternatives d, e, f, g, h are omitted on the first stage of the procedure and the alternatives a, b, c are presented for choice on the second stage.

Thus, it is necessary to check those normative conditions of two-stage choice procedures which are satisfied for the choice procedures of the second stage.

- Two-stage choice procedures 30-32,35-36,40-56 satisfy the Monotonicity condition 1 (the proof follows from the paragraph 29.5 of Appendix 1 and properties of second stage choice procedures).
- Two-stage choice procedures 40,45,47,48 do not satisfy the condition C. To proof it the following example is used.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
c	c	c	b	b	b	d	d	d	d	a	a	a	a	a
a	a	a	d	d	d	b	b	c	c	b	b	b	b	c
d	d	d	a	a	a	c	c	b	b	c	c	c	c	b
b	b	b	c	c	c	a	a	a	a	d	d	d	d	d

Let us calculate the value $n^+(x, \vec{P}_X)$ for each alternative $x \in X$: $n^+(a, \vec{P}_X) = 4$, $n^+(b, \vec{P}_X) = 3$, $n^+(c, \vec{P}_X) = 3$, $n^+(d, \vec{P}_X) = 4$. According to the rule the alternatives a and d will be chosen, i.e., $C(\vec{P}_X, X) = \{a, d\}$.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P ₁	P ₂	P ₃	P ₄	P ₅	P ₆	P ₇	P ₈	P ₉	P ₁₀	P ₁₁	P ₁₂	P ₁₃	P ₁₄
<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>
<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>

Let us calculate the value $n^+(x, \vec{P}_{X'})$ for each alternative $x \in X'$: $n^+(a, \vec{P}_{X'}) = 4$, $n^+(b, \vec{P}_{X'}) = 5$, $n^+(c, \vec{P}_{X'}) = 5$. Thus, the alternative a will be omitted on the first stage and the alternative b will be chosen on the second stage, i.e., $C(\vec{P}_{X'}, X') = \{b\}$.

Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P ₁	P ₂	P ₃	P ₄	P ₅	P ₆	P ₇	P ₈	P ₉	P ₁₀	P ₁₁	P ₁₂	P ₁₃	P ₁₄
<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>
<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>

Let us calculate the value $n^+(x, \vec{P}_{X''})$ for each alternative $x \in X''$: $n^+(b, \vec{P}_{X''}) = 6$, $n^+(c, \vec{P}_{X''}) = 5$, $n^+(d, \vec{P}_{X''}) = 4$. According to the rule the alternative b will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b\} \not\subseteq C(\vec{P}_X, X)$. Thus, the two-stage choice procedure does not satisfy the condition **C**.

- Two-stage choice procedures 40,47 do not satisfy the condition **O** (see paragraph 29.1 of Appendix 1).
- Two-stage choice procedures 47,48 do not satisfy the condition **H** (see paragraph 29.1 of Appendix 1).
- Two-stage choice procedures 48,54 do not satisfy the Monotonicity condition 2. To prove it the following example is used.

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P ₁	P ₂	P ₃	P ₄	P ₅	P ₆
<i>a</i>	<i>a</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>a</i>

Let us calculate the value $n^+(x, \vec{P}_X)$ for each alternative $x \in X$: $n^+(a, \vec{P}_X) = 2$, $n^+(b, \vec{P}_X) = 2$, $n^+(c, \vec{P}_X) = 1$. According to the plurality rule used on the first stage the alternatives a and b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{b\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P ₁	P ₂	P ₃	P ₄	P ₅	P ₆
<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>

According to the plurality rule the alternative c will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{c\}$.

Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃	P₄	P₅	P₅
c	c	c	b	b	c
b	b	b	c	c	b

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c\}$.

Then $\{a, b\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}_{X'}, X')$ and $\{b\} \notin C(\vec{P}_{X''}, X'')$. Thus, the two-stage choice procedure does not satisfy the Monotonicity condition 2.

- Two-stage choice procedure 50 does not satisfy the non-compensatory condition. To prove it the following example is used.

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃
a	a	b
c	c	c
b	b	a

Let us calculate the value $n^+(x, \vec{P}_X)$ for each alternative $x \in X$: $n^+(a, \vec{P}_X) = 2$, $n^+(b, \vec{P}_X) = 1$, $n^+(c, \vec{P}_X) = 0$. According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

According to the non-compensatory condition the alternative c is better than the alternative a while the alternative a is better than the alternative b . Thus, the non-compensatory condition is not satisfied as $\{c\} \neq C(\vec{P}_X, X)$.

Two-stage choice procedures 57-84

Two-stage choice procedures 57-84 do not satisfy the same conditions as choice procedures which are used on the second stage. To prove it we can use the same examples but with larger number of alternatives which are regarded as the worst by the maximum number of criteria [11].

Thus, it is necessary to check those normative conditions of two-stage choice procedures which were satisfied for the second stage choice procedures.

- Two-stage choice procedures 57,75,76 do not satisfy the condition **H**. To prove it the following example is used.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃
b	d	a
a	b	d
c	a	c
d	c	b

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Consider now the subset $X' = X \setminus \{c, d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃
b	b	a
a	a	b

According to the rule the alternative b will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \not\subseteq C(\vec{P}_X, X) \cap X'$. Thus, the condition **H** is not satisfied.

- Two-stage choice procedures 57,68,75 do not satisfy the condition **O** (see the previous example).
- Two-stage choice procedures 68,73,75,76 do not satisfy the condition **C**. To prove it the following example is used.

Consider the previous example. According to the rule $C(\vec{P}_X, X) = \{a\}$.

Consider now the subset $X' = X \setminus \{c\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃
<i>b</i>	<i>d</i>	<i>a</i>
<i>a</i>	<i>b</i>	<i>d</i>
<i>d</i>	<i>a</i>	<i>b</i>

According to the rule the alternatives a, b, d will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{a, b, d\}$.

Finally, let us consider the subset $X'' = X \setminus \{a, d\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃
<i>b</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>

According to the rule the alternative b will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

- Two-stage choice procedures 57-60, 63, 64, 68-84 satisfy the Monotonicity condition 1 (the proof follows from the properties of the inverse plurality rule and second stage choice procedures).
- Two-stage choice procedures 76, 82 do not satisfy the Monotonicity condition 2. To prove it the following example is used.

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃	P₄	P₅	P₆	P₇	P₈	P₉	P₁₀
<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>a</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>

According to the rule the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{b\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃	P₄	P₅	P₆	P₇	P₈	P₉	P₁₀
<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>a</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>

According to the inverse plurality rule used on the first stage the alternative c will be chosen, i.e.,

$C(\vec{P}_{X'}, X') = \{c\}$. Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃	P₄	P₅	P₆	P₇	P₈	P₉	P₁₀
<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c\}$.

Then $\{a, b\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}_{X'}, X')$ and $\{b\} \notin C(\vec{P}_{X''}, X'')$. Thus, the Monotonicity condition 2 is not satisfied.

- Two-stage choice procedure 78 does not satisfy the non-compensatory condition. To prove it let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4
a	b	c	b
b	a	a	c
c	c	b	a

According to the non-compensatory condition the alternative b is better than the alternative a which is better than the alternative c . However, according to the two-stage choice procedure 78, the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$. Thus, the non-compensatory condition is not satisfied as $\{b\} \neq C(\vec{P}_X, X)$.

Two-stage choice procedures 85-112

Two-stage choice procedures 85-112 satisfy the same conditions as two-stage choice procedures 30-46 which use the plurality rule on the first stage. The proof follows from properties of q-Approval rule [11].

Two-stage choice procedures 169-196

Two-stage choice procedures 169-196 do not satisfy the same conditions as choice procedures which are used on the second stage. To prove it we can use the same examples that were used to check the properties of the Borda rule [11] but with larger number of alternatives which are omitted on the first stage.

Thus, it is necessary to check those normative conditions of two-stage choice procedures which were satisfied for the second stage choice procedures.

- Two-stage choice procedures 169,187,188 do not satisfy the condition **H**. To prove it the following example is provided.

Let $X = \{a, b, c, d, e\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5
e	e	a	b	b
a	a	d	c	c
b	b	c	e	a
c	c	b	a	d
d	d	e	d	e

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = r(b, \vec{P}_X) = 13, r(c, \vec{P}_X) = 10, r(d, \vec{P}_X) = 4, r(e, \vec{P}_X) = 10$. According to the rule the alternatives c, d, e will be omitted on the first stage and the alternative a will be chosen on the second stage, i.e., $C(\vec{P}_X, X) = \{a\}$.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3	P_4	P_5
e	e	a	b	b
a	a	c	c	c
b	b	b	e	a
c	c	e	a	e

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_{X'}) = 8, r(b, \vec{P}_{X'}) = 9, r(c, \vec{P}_{X'}) = 6, r(e, \vec{P}_{X'}) = 7$. According to the rule the alternative b will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \not\subseteq C(\vec{P}_X, X) \cap X'$. Thus, the condition **H** is not satisfied.

- Two-stage choice procedures 169, 180, 187 do not satisfy the condition **O** (see the previous example).
- Two-stage choice procedures 180,185,187,188 do not satisfy the condition **C**. To prove it the following example is provided.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃
<i>a</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>a</i>	<i>b</i>
<i>d</i>	<i>d</i>	<i>a</i>
<i>b</i>	<i>c</i>	<i>d</i>

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = 6, r(b, \vec{P}_X) = r(c, \vec{P}_X) = 5, r(d, \vec{P}_X) = 2$. According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃
<i>a</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>a</i>	<i>b</i>
<i>b</i>	<i>c</i>	<i>a</i>

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_{X'}) = r(b, \vec{P}_{X'}) = r(c, \vec{P}_{X'}) = 3$.

According to the rule the alternatives a, b, c will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{a, b, c\}$.

Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃
<i>c</i>	<i>b</i>	<i>c</i>
<i>d</i>	<i>d</i>	<i>b</i>
<i>b</i>	<i>c</i>	<i>d</i>

Let us calculate the Borda count for each alternative: $r(b, \vec{P}_{X''}) = 3, r(c, \vec{P}_{X''}) = 4, r(d, \vec{P}_{X''}) = 2$.

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{c\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

- Two-stage choice procedures 169-196 satisfy the Monotonicity condition 1 (the proof follows from properties of the Borda rule).
- Two-stage choice procedures 188, 194 do not satisfy the Monotonicity condition 2. To prove it the following example is provided.

Let $X = \{a, b, c, d, e, f\}$ and the profile \vec{P}_X is the following

P₁	P₂	P₃	P₄	P₅	P₆	P₇	P₈
<i>a</i>	<i>a</i>	<i>a</i>	<i>f</i>	<i>f</i>	<i>f</i>	<i>f</i>	<i>c</i>
<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>f</i>
<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>e</i>	<i>e</i>
<i>d</i>	<i>d</i>	<i>d</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>d</i>
<i>e</i>	<i>e</i>	<i>e</i>	<i>d</i>	<i>d</i>	<i>d</i>	<i>b</i>	<i>a</i>
<i>f</i>	<i>f</i>	<i>f</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>a</i>	<i>b</i>

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = r(b, \vec{P}_X) = 25$, $r(c, \vec{P}_X) = 24$, $r(d, \vec{P}_X) = 13$, $r(e, \vec{P}_X) = 9$, $r(f, \vec{P}_X) = 24$. According to the rule the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{b\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a	a	a	f	f	f	f	c
c	c	c	a	a	a	c	f
d	d	d	c	c	c	e	e
e	e	e	d	d	d	d	d
f	f	f	e	e	e	a	a

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_{X'}) = 21$, $r(c, \vec{P}_{X'}) = 22$, $r(d, \vec{P}_{X'}) = 13$, $r(e, \vec{P}_{X'}) = 7$, $r(f, \vec{P}_{X'}) = 19$. According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{c\}$.

Finally, let us consider the subset $X'' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
b	b	b	f	f	f	f	c
c	c	c	b	b	b	c	f
d	d	d	c	c	c	e	e
e	e	e	d	d	d	d	d
f	f	f	e	e	e	b	b

Let us calculate the Borda count for each alternative: $r(b, \vec{P}_{X''}) = 21$, $r(c, \vec{P}_{X''}) = 22$, $r(d, \vec{P}_{X''}) = 13$, $r(e, \vec{P}_{X''}) = 7$, $r(f, \vec{P}_{X''}) = 19$. According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c\}$.

Then $\{a, b\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}_{X'}, X')$ and $\{b\} \notin C(\vec{P}_{X''}, X'')$. Thus, the Monotonicity condition 2 is not satisfied.

- Two-stage choice procedure 190 does not satisfy the non-compensatory condition. To prove it let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3
a	a	b
b	b	c
c	c	a

According to the two-stage choice procedure the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$. However, according to the non-compensatory condition the alternative b is better than the alternative a which is better than the alternative c .

Thus, the non-compensatory condition is not satisfied as $\{b\} \neq C(\vec{P}_X, X)$.

Two-stage choice procedures 197-224

To check the properties of two-stage choice procedures 197-224 similar examples, which were given to check the properties of existing choice procedures [11], can be used.

- Two-stage choice procedures 197-224 do not satisfy the conditions **H** and **O**. To prove it the following example is provided.

Let $X = \{a, b, c\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5
a	a	d	b	b
b	b	a	d	d
d	d	b	a	a

In this example none of alternatives is a Condorcet winner ($a\mu b, d\mu a, b\mu d$). Thus, the Borda rule is used for this case. Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = 5, r(b, \vec{P}_X) = 6, r(c, \vec{P}_X) = 4$. According to the rule the alternative b will be chosen, i.e., $C(\vec{P}_X, X) = \{b\}$.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3	P_4	P_5
a	a	a	b	b
b	b	b	a	a

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{a\}$.

$C(\vec{P}_{X'}, X') \not\subseteq C(\vec{P}_X, X) \cap X'$. Thus, the condition **H** is not satisfied.

$C(\vec{P}_{X'}, X') \neq C(\vec{P}_X, X)$. Thus, the condition **O** is not satisfied.

- Two-stage choice procedures 197-224 do not satisfy the condition **C**. To prove it the following example is provided.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5
a	a	c	d	d
d	d	b	b	b
c	c	a	a	a
b	b	d	c	c

In this example none of alternatives is a Condorcet winner ($a\mu c, a\mu d, b\mu a, c\mu b, d\mu b, d\mu c$). Thus, the Borda rule is used for this case. Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = 9, r(b, \vec{P}_X) = 6, r(c, \vec{P}_X) = 5, r(d, \vec{P}_X) = 10$. According to the rule the alternative d will be chosen, i.e., $C(\vec{P}_X, X) = \{d\}$.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3	P_4	P_5
a	a	c	b	b
c	c	b	a	a
b	b	a	c	c

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{a\}$.

Finally, let us consider the subset $X'' = X \setminus \{b, c\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P_1	P_2	P_3	P_4	P_5
a	a	a	d	d
d	d	d	a	a

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{a\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{a\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

- Two-stage choice procedures 197-224 do not satisfy the condition **ACA** since such choice procedures do not satisfy the condition **H**.
- Two-stage choice procedures 197-224 satisfy the Monotonicity condition 1 (the proof follows from the properties of the Black procedure).
- Two-stage choice procedures 198-200, 203-214, 216-224 do not satisfy the Monotonicity condition 2 (the proof follows from the properties of the Black procedure).
- Since two-stage choice procedures 197, 201-202, 215 chooses no more than one best alternative, the Monotonicity condition 2 is not applicable to these procedures as it considers the choice of more than two alternatives. In other words, such two-stage choice procedures obey the Monotonicity condition 2 trivially.
- Two-stage choice procedures 197-224 do not satisfy the strict monotonicity condition (the proof follows from the properties of the Black procedure).
- Two-stage choice procedures 197-224 do not satisfy the non-compensatory condition (the proof follows from the properties of the Black procedure).

Two-stage choice procedures 225-252

Two-stage choice procedures 225-252 do not satisfy the same properties as the Inverse Borda rule. To prove it we can use the same examples that were used to check the properties of the Inverse Borda rule [11].

Two-stage choice procedures 253-280

Two-stage choice procedures 253-280 do not satisfy the same properties as the Nanson rule. To prove it we can use the same examples that were used to check the properties of the Nanson rule [11].

Two-stage choice procedures 281-308

Two-stage choice procedures 281-308 do not satisfy the same properties as the Coombs procedure. To prove it we can use the same examples that were used to check the properties of the Coombs procedure [11].

Two-stage choice procedures 310-319, 330, 334-336

310.5 Monotonicity condition 1

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
b	d	a	a	b	d	b	d	a	c
d	c	b	c	a	c	c	c	b	a
a	a	d	b	c	b	d	a	d	d
c	b	c	d	d	a	a	b	c	b

According to the rule the alternatives a, b, d will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b, d\}$.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative a in P'_7 :

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	d	a	a	b	d	b	d	a	c
d	c	b	c	a	c	c	c	b	a
a	a	d	b	c	b	a	a	d	d
c	b	c	d	d	a	d	b	c	b

According to the two-stage choice procedure the alternative d is omitted on the first stage. Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}'_{X'}$, looks as

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	c	a	a	b	c	b	c	a	c
a	a	b	c	a	b	c	a	b	a
c	b	c	b	c	a	a	b	c	b

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}'_{X'}, X) = \{c\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_{X'}, X)$. Thus, the Monotonicity condition 1 is not satisfied.

311.5 Monotonicity condition 1

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
b	d	c	a	a	d	b	d	a	d
d	a	b	c	b	c	c	c	b	a
c	c	a	b	c	b	a	a	d	b
a	b	d	d	d	a	d	b	c	c

According to the rule the alternatives a, b, c will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b, c\}$.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative a in P'_2 :

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	a	c	a	a	d	b	d	a	d
d	d	b	c	b	c	c	c	b	a
c	c	a	b	c	b	a	a	d	b
a	b	d	d	d	a	d	b	c	c

According to the two-stage choice procedure the alternative d is omitted on the first stage. Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}'_X onto a set X' , i.e., $\vec{P}'_{X'}$, looks as

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	a	c	a	a	c	b	c	a	a
c	c	b	c	b	b	c	a	b	b
a	b	a	b	c	a	a	b	c	c

According to the rule the alternatives b, c will be chosen, i.e., $C(\vec{P}'_{X'}, X) = \{b, c\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_{X'}, X)$. Thus, the Monotonicity condition 1 is not satisfied.

315.5 Monotonicity condition 1

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
b	d	c	a	a	d	b	d	a	d
d	c	b	c	b	c	c	c	b	a
c	a	a	b	c	b	a	a	d	b
a	b	d	d	d	a	d	b	c	c

Let us calculate the Borda count for each alternative: $r(a, \vec{P}_X) = r(b, \vec{P}_X) = r(c, \vec{P}_X) = r(d, \vec{P}_X) = 15$. According to the rule the alternatives a, b, c, d will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b, c, d\}$.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative a in P'_{10} :

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	d	c	a	a	d	b	d	a	a
d	c	b	c	b	c	c	c	b	d
c	a	a	b	c	b	a	a	d	b
a	b	d	d	d	a	d	b	c	c

According to the two-stage choice procedure the alternative d is omitted on the first stage. Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}'_X onto a set X' , i.e., $\vec{P}'_{X'}$, looks as

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}
b	c	c	a	a	c	b	c	a	a
c	a	b	c	b	b	c	a	b	b
a	b	a	b	c	a	a	b	c	c

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}'_X, X) = \{c\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

To check the remaining normative conditions of given two-stage choice procedures similar examples which were given to check the properties of existing choice procedures can be used.

- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the condition **H** (see the properties of minimal dominant set).
- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the condition **C** (see the properties of the procedures used on the second stage).
- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the condition **O** (see the properties of the procedures used on the second stage and paragraphs 310.5, 315.5 of Appendix 1).
- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the condition **ACA** since such choice procedures do not satisfy the condition **H**.
- Two-stage choice procedures 310-319, 330 do not satisfy the Monotonicity condition 1 (see the properties of the Coombs procedure for choice procedure 319 and paragraphs 310.5, 311.5, 315.5 for other choice procedures).
- Two-stage choice procedures 334-336 satisfy the Monotonicity condition 1 (the proof follows from the properties of minimal dominant set and second stage choice procedures).
- Two-stage choice procedures 310-313, 316-319, 330, 334-336 do not satisfy the Monotonicity condition 2 (see the properties of minimal dominant set).
- Since two-stage choice procedures 314-315 chooses no more than one best alternative, the Monotonicity condition 2 is not applicable to these procedures as it considers the choice of more than two alternatives. In other words, such two-stage choice procedures obey the Monotonicity condition 2 trivially.
- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the strict monotonicity condition (see the properties of minimal dominant set).
- Two-stage choice procedures 310-319, 330, 334-336 do not satisfy the non-compensatory condition (see the properties of minimal dominant set).

Two-stage choice procedures 338-347, 350-364

Since minimal undominated set is equal to minimal dominant set when the number of criteria is odd, two-stage choice procedures 338-347, 358, 362-364 do not satisfy the same conditions as two-stage choice procedures 310-319, 330, 334-336, which use minimal dominant set on the first stage.

Consider now the properties of other two-stage choice procedures.

- Two-stage choice procedures 351-354, 357, 359-361 do not satisfy the condition **H**, **C** and **O**.

To prove it let $X = \{a, b, c, d\}$ and the profile \vec{P}_X looks as

P₁	P₂	P₃	P₄	P₅	P₆
<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>a</i>	<i>c</i>	<i>d</i>	<i>a</i>	<i>c</i>	<i>b</i>
<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>c</i>
<i>d</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>a</i>

For this case a matrix of majority relation μ is the following

	a	b	c	d
a	-	0	1	1
b	0	-	1	0
c	0	0	-	1
d	0	1	0	-

According to the rule the alternative a is included in minimal undominated set Q . Thus, $C(\vec{P}_X, X) = Q = \{a\}$.

Consider now the subset $X' = X \setminus \{a\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃	P₄	P₅	P₆
<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
<i>d</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>

According to the rule the alternatives b, c, d make the minimal undominated set. Thus, $C(\vec{P}_{X'}, X') = \{b, c, d\}$.

Finally, consider the subset $X'' = X \setminus \{c, d\}$. A contraction of a profile \vec{P}_X onto a set X'' , i.e., $\vec{P}_{X''}$, looks as

P₁	P₂	P₃	P₄	P₅	P₆
<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>
<i>a</i>	<i>b</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>

According to the rule the alternatives a and b will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{a, b\}$.

$C(\vec{P}_{X'}, X') \not\subseteq C(\vec{P}_X, X) \cap X'$. Thus, the condition **H** is not satisfied.

$C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

$C(\vec{P}_{X'}, X') \neq C(\vec{P}_X, X)$. Thus, the condition **O** is not satisfied.

- Two-stage choice procedures 351-354, 357, 359-361 do not satisfy the condition **ACA** since such two-stage choice procedures do not satisfy the condition **H**.
- Two-stage choice procedures 351-354, 357, 359-361 satisfy the Monotonicity condition 1 (the proof follows from properties of minimal undominated set and second stage choice procedures).
- Two-stage choice procedures 351-354, 357, 359-361 do not satisfy the Monotonicity condition 2 (see the properties of minimal undominated set).

- Two-stage choice procedures 351-354, 357, 359-361 do not satisfy the strict monotonicity condition (see the properties of minimal undominated set).
- Two-stage choice procedures 351-354, 357, 359-361 do not satisfy the non-compensatory condition (see the properties of minimal undominated set).

Two-stage choice procedures 365-392

Two-stage choice procedures do not satisfy the same conditions as choice procedures used on the second stage. To prove it we can use the same examples but with larger number of alternatives eliminated on the first stage of the two-stage choice procedure [11].

Thus, it is necessary to check those normative conditions of two-stage choice procedures which were satisfied for the second stage choice procedures.

- Two-stage choice procedures 365,383,384 do not satisfy the condition **H** (see paragraph 365.1 of Appendix 1).
- Two-stage choice procedures 365, 376 do not satisfy the condition **O** (see paragraph 365.1 of Appendix 1).
- Two-stage choice procedures 376, 381, 383, 384 do not satisfy the condition **C**. To prove it the following example is used.

Let $X = \{a, b, c, d, e\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e
a	-	1	0	0	0
b	0	-	1	0	0
c	0	0	-	1	1
d	1	0	0	-	0
e	0	0	0	1	-

According to the rule the alternatives a and c are included in minimal weakly stable set Q . Thus, $C(\vec{P}_X, X) = Q = \{a, c\}$.

Consider now the subset $X' = X \setminus \{e\}$. Then a matrix of majority relation μ is the following

	a	b	c	d
a	-	1	0	0
b	0	-	1	0
c	0	0	-	1
d	1	0	0	-

According to the rule the alternatives a, b, c, d are included in minimal weakly stable set, i.e., $C(\vec{P}_{X'}, X') = \{a, b, c, d\}$.

Finally, consider the subset $X'' = X \setminus \{a\}$. Then a matrix of majority relation μ is the following

	b	c	d	e
b	-	1	0	0
c	0	-	1	1
d	0	0	-	0
e	0	0	1	-

According to the rule the alternative b is included in minimal weakly stable set, i.e., $C(\vec{P}_{X''}, X'') = \{b\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

- Two-stage choice procedures 383, 384 do not satisfy the condition **C** (see paragraph 383.2 of Appendix 1).
- Two-stage choice procedures 365-375, 383-384, 386, 390-392 do not satisfy the Monotonicity condition 1 (the proof follows from the paragraph 365.5 of Appendix 1).
- Two-stage choice procedures 376-382, 385, 387-389 do not satisfy the Monotonicity condition 1 (the proof follows from the paragraph 376.5 of Appendix 1).
- Two-stage choice procedures 365-392 do not satisfy the Monotonicity condition 2 (see the properties of minimal weakly stable set).
- Two-stage choice procedures 365-392 do not satisfy the non-compensatory condition (see the properties of minimal weakly stable set).

365.1 Heredity condition (H)

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	1	0	0	0	1
b	0	-	1	0	0	0
c	1	0	-	0	0	0
d	0	1	0	-	0	0
e	0	0	0	1	-	0
f	0	0	0	0	1	-

Minimal weakly stable set is chosen on the first stage of choice procedure. According to the rule the alternatives a, c are included in minimal weakly stable set Q , i.e., $Q = \{a, c\}$.

Simple majority rule is used on the second stage of the choice procedure. Consider now the remaining set $X' = X \setminus \{b, d, e, f\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P₁	P₂	P₃
c	c	a
a	a	c

According to the simple majority rule, the alternative c will be chosen, i.e., $C(\vec{P}_X, X) = \{c\}$.

Consider now the subset $X' = X \setminus \{d, e, f\}$. Then, according to the two-stage choice procedure, the alternatives a, b, c are included in minimal weakly stable set Q' .

Simple majority rule is used on the second stage of the choice procedure. Consider now the remaining set $Q' = X \setminus \{b, d, e, f\}$. A contraction of a profile \vec{P}_X onto a set Q' , i.e., $\vec{P}_{Q'}$, looks as

P₁	P₂	P₃
c	b	a
a	c	b
b	a	c

According to the rule the choice is empty, i.e., $C(\vec{P}_{X'}, X') = \emptyset$.

Then $C(\vec{P}_X, X) \cap X' = \{c\} \not\subseteq C(\vec{P}_{X'}, X')$. Thus, the condition **H** is not satisfied.

365.5 Monotonicity condition 1

Consider the example from paragraph 365.1. According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_X, X) = \{c\}$.

Suppose that the relative comparison of the alternative c and the alternative d is changed ($c\mu d$) while the relative comparison of any pair of other alternatives remains unchanged. Then a matrix of majority relation looks as

	a	b	c	d	e	f
a	-	1	0	0	0	1
b	0	-	1	0	0	0
c	1	0	-	1	0	0
d	0	1	0	-	0	0
e	0	0	0	1	-	0
f	0	0	0	0	1	-

Minimal weakly stable set is used on the first stage of choice procedure. For this case the alternatives a, b, c form a minimal weakly stable set. Simple majority rule is applied on the second stage of the two-stage choice procedure. Since $c\mu a, a\mu b, b\mu c, C(\vec{P}_X, X) = \emptyset$.

Then $\{c\} \in C(\vec{P}_X, X), \{c\} \notin C(\vec{P}_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

376.5 Monotonicity condition 1

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
a	-	0	1	0	0	0	1	1	0	0	0	0	0	0	0
b	1	-	0	0	0	0	0	0	0	0	1	1	0	0	0
c	0	1	-	1	0	0	0	0	0	0	0	0	0	0	1
d	1	0	0	-	0	0	0	0	0	0	0	0	0	0	0
e	0	0	0	1	-	0	1	0	0	0	0	0	0	0	0
f	0	0	0	0	1	-	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	1	-	0	0	0	0	0	0	0	0
h	0	1	0	0	0	0	0	-	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	1	-	0	1	0	0	0	0
j	0	0	0	0	0	0	0	0	1	-	0	0	0	0	0
k	0	0	0	0	0	0	0	0	0	1	-	0	0	0	0
l	0	0	1	0	0	0	0	0	0	0	0	-	0	0	0
m	0	0	0	0	0	0	0	0	0	0	0	1	-	0	1
n	0	0	0	0	0	0	0	0	0	0	0	0	1	-	0
o	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-

According to the rule the alternatives a, b, c make a minimal weakly stable set. Thus, the alternatives $d, e, f, g, h, i, j, k, l, m, n, o$ will be omitted on the first stage and the alternatives a, b, c will be chosen on the second stage of the two-stage choice procedure, i.e., $C(\vec{P}_X, X) = \{a, b, c\}$.

Suppose that the relative comparison of the alternative a and the alternative d is changed ($a\mu d$) while the relative comparison of any pair of other alternatives remains unchanged. Then a matrix of majority relation looks as

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
a	-	0	1	0	0	0	1	1	0	0	0	0	0	0	0
b	1	-	0	0	0	0	0	0	0	0	1	1	0	0	0
c		1	-	1	0	0	0	0	0	0	0	0	0	0	1
d	1	0	0	-	0	0	0	0	0	0	0	0	0	0	0
e	0	0	0	1	-	0	1	0	0	0	0	0	0	0	0
f	0	0	0	0	1	-	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	1	-	0	0	0	0	0	0	0	0
h	0	1	0	0	0	0	0	-	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	1	-	0	1	0	0	0	0
j	0	0	0	0	0	0	0	0	1	-	0	0	0	0	0
k	0	0	0	0	0	0	0	0	0	1	-	0	0	0	0
l	0	0	1	0	0	0	0	0	0	0	0	-	0	0	0
m	0	0	0	0	0	0	0	0	0	0	0	1	-	0	1
n	0	0	0	0	0	0	0	0	0	0	0	0	1	-	0
o	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-

According to the rule the alternatives a, b make a minimal weakly stable set. Thus, the alternatives $c, d, e, f, g, h, i, j, k, l, m, n, o$ will be omitted on the first stage and the alternative b will be chosen on the second stage of the two-stage choice procedure ($b\mu a$), i.e., $C(\vec{P}_X, X) = \{b\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

383.2 Concordance condition (C)

Let $X = \{a, b, c, d, e, f, g, h\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f	g	h
a	-	1	0	1	0	0	0	0
b	0	-	1	1	1	0	0	0
c	1	0	-	0	0	0	0	1
d	0	0	1	-	0	0	0	0
e	0	0	1	0	-	0	0	0
f	0	0	0	0	1	-	0	0
g	0	0	0	0	0	1	-	0
h	0	0	0	0	0	0	1	-

On the first stage of the two-stage choice procedure minimal weakly stable set is defined. For this case there are 3 minimal weakly stable sets: $\{a, b\}, \{a, d\}$ and $\{b, c\}$. Thus, $Q = \{a, b\} \cup \{a, d\} \cup \{b, c\} = \{a, b, c, d\}$.

On the second stage of the two-stage choice procedure a Condorcet winner rule is used. For this case there is no Condorcet winner ($a\mu b, a\mu d, c\mu a, b\mu c, d\mu c$), Thus, $C(\vec{P}_X, X) = \emptyset$.

Consider now the subset $X' = X \setminus \{c, e, f, g\}$. A matrix of majority relation μ looks as

	a	b	d
a	-	1	1
b	0	-	1
d	0	0	-

On the first stage of the two-stage choice procedure minimal weakly stable set is defined. For this case the alternative a makes a minimal weakly stable set. Thus, $C(\vec{P}_{X'}, X') = \{a\}$.

Finally, consider the subset $X'' = X \setminus \{b\}$. Then a matrix of majority relation μ looks as

	a	c	d	e	f	g	h
a	-	0	1	0	0	0	0
c	1	-	0	0	0	0	1
d	0	1	-	0	0	0	0
e	0	1	0	-	0	0	0
f	0	0	0	1	-	0	0
g	0	0	0	0	1	-	0
h	0	0	0	0	0	1	-

On the first stage of the two-stage choice procedure minimal weakly stable set is defined. For this case the alternatives a and d make a minimal weakly stable set. On the second stage of the two-stage choice procedure a Condorcet winner rule is used. Since aud , $C(\vec{P}_{X''}, X'') = \{a\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{a\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

Two-stage choice procedures 394-410, 412-420

Two-stage choice procedures do not satisfy the same conditions as choice procedures used on the second stage. To prove it we can use the same examples but with larger number of alternatives eliminated on the first stage of the two-stage choice procedure [11].

Thus, it is necessary to check those normative conditions of two-stage choice procedures which were satisfied for the second stage choice procedures.

- Two-stage choice procedure 412 does not satisfy the condition **H** (see paragraph 412.1 of Appendix 1).
- Two-stage choice procedures 404, 409 do not satisfy the condition **C**. To prove it the following example is used.

Let $X = \{a, b, c, d, e\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	1	0	0	1	1
b	0	-	1	1	0	1
c	1	0	-	1	1	0
d	1	0	0	-	0	0
e	0	0	0	1	-	1
f	0	0	1	1	0	-

Let us define the upper contour sets for each alternative. $D(a, \vec{P}_X) = \{c, d\}$, $D(b, \vec{P}_X) = \{a\}$, $D(c, \vec{P}_X) = \{b, f\}$, $D(d, \vec{P}_X) = \{b, c, e, f\}$, $D(e, \vec{P}_X) = \{a, c\}$, $D(f, \vec{P}_X) = \{a, b, e\}$. Then cyd , bye , byf . Thus, $C(\vec{P}_X, X) = \{a, b, c\}$.

Consider now the subset $X' = X \setminus \{c, f\}$. Then a matrix of majority relation μ is the following

	a	b	d	e
a	-	1	0	1
b	0	-	1	0
d	1	0	-	0
e	0	0	1	-

Let us define the upper contour sets for each alternative. $D(a, \vec{P}_{X'}) = \{d\}$, $D(b, \vec{P}_{X'}) = \{a\}$, $D(d, \vec{P}_{X'}) = \{b, e\}$, $D(e, \vec{P}_{X'}) = \{a\}$. Thus, $C(\vec{P}_{X'}, X') = \{a, b, d, e\}$.

Finally, consider the subset $X'' = X \setminus \{a, b\}$. Then a matrix of majority relation μ is the following

	c	d	e	f
c	-	1	1	0
d	0	-	0	0
e	0	1	-	1
f	1	1	0	-

Let us define the upper contour sets for each alternative. $D(c, \vec{P}_{X''}) = \{f\}$, $D(d, \vec{P}_{X''}) = \{c, e, f\}$, $D(e, \vec{P}_{X''}) = \{c\}$, $D(f, \vec{P}_{X''}) = \{e\}$. Then cyd , eyd , fyd . Thus, $C(\vec{P}_{X''}, X'') = \{c, e, f\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{e\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

- Two-stage choice procedures 412 do not satisfy the condition **C** (see paragraph 412.2 of Appendix 1).
- Two-stage choice procedure 404 does not satisfy the condition **O**. To prove it the following example is used.

Let $X = \{a, b, c, d, e\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e
a	-	1	0	1	1
b	0	-	1	1	0
c	1	0	-	1	0
d	0	0	0	-	1
e	0	0	0	0	-

Let us define the upper contour sets for each alternative. $D(a, \vec{P}_X) = \{c\}$, $D(b, \vec{P}_X) = \{a\}$, $D(c, \vec{P}_X) = \{b\}$, $D(d, \vec{P}_X) = \{a, b, c\}$, $D(e, \vec{P}_X) = \{a, d\}$. Then cyd , bye . Thus, $C(\vec{P}_X, X) = \{a, b, c\}$.

Consider now the subset $X' = X \setminus \{d\}$. Then a matrix of majority relation μ is the following

	a	b	c	e
a	-	1	0	1
b	0	-	1	0
c	1	0	-	0
e	0	0	0	-

Let us define the upper contour sets for each alternative. $D(a, \vec{P}_{X'}) = \{c\}$, $D(b, \vec{P}_{X'}) = \{a\}$, $D(c, \vec{P}_{X'}) = \{b\}$, $D(e, \vec{P}_{X'}) = \{a\}$. Thus, $C(\vec{P}_{X'}, X') = \{a, b, c, e\}$.

Then $C(\vec{P}_{X'}, X') \neq C(\vec{P}_X, X)$, Consequently, the condition **O** is not satisfied.

- Two-stage choice procedures 394-403, 405-407, 409, 412, 414-420 do not satisfy the Monotonicity condition 1 (the proof follows from the paragraphs 394.5, 409.5 of Appendix 1 and properties of second stage choice procedures).
- Two-stage choice procedure 412 does not satisfy the Monotonicity condition 2 (see paragraph 412.1 of Appendix 1).
- Two-stage choice procedures 393-420 do not satisfy the non-compensatory condition (see the properties of Fishburn rule).

394.5 Monotonicity condition 1

Consider the example from paragraph 412.1 of Appendix 1. Suppose that $b\mu c$. Then the alternative f is eliminated on the first stage and, consequently, the alternative b is not chosen on the second stage. Since $\{b\} \in C(\vec{P}_X, X)$, $\{b\} \notin C(\vec{P}'_X, X)$, the Monotonicity condition 1 is not satisfied.

409.5 Monotonicity condition 1

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	0	1	0	1	0
b	0	-	0	1	0	1
c	0	0	-	0	0	0
d	0	0	0	-	1	0
e	0	1	0	0	-	1
f	1	0	0	0	0	-

According to the rule the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Suppose that the relative position of the alternative a is improved such that $a\mu b$, $a\mu d$ while the relative comparison of any pair of other alternatives remains unchanged. Then a matrix of majority relation looks as

	a	b	c	d	e	f
a	-	1	1	1	1	0
b	0	-	0	1	0	1
c	0	0	-	0	0	0
d	0	0	0	-	1	0
e	0	1	0	0	-	1
f	1	0	0	0	0	-

For this case the alternatives a, c, f are considered for the second stage of the choice procedure. Since $f\mu a$, $a\mu c$, $C(\vec{P}'_X, X) = \{f, c\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

412.1 Heredity condition (H)

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	1	1	0	1	0
b	0	-	0	1	0	1
c	0	0	-	0	0	1
d	0	0	0	-	1	0
e	0	1	0	0	-	0
f	1	0	0	0	0	-

According to the two-stage choice procedure the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{c, d\}$. A matrix of majority relation μ looks as

	a	b	e	f
a	-	0	1	0
b	0	-	0	1
e	0	1	-	0
f	1	0	0	-

According to the rule the choice is empty, i.e., $C(\vec{P}_{X'}, X') = \emptyset$.

Then $C(\vec{P}_X, X) \cap X' = \{a, b\} \not\subseteq C(\vec{P}_{X'}, X')$. Thus, the condition **H** is not satisfied.

412.2 Concordance condition (C)

Let $X = \{a, b, c, d, e, f, g\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f	g
a	-	0	1	0	1	0	0
b	0	-	0	1	0	1	1
c	0	0	-	0	0	1	0
d	0	0	0	-	1	0	0
e	0	1	0	0	-	0	1
f	1	0	0	0	0	-	0
g	1	0	0	1	0	1	-

According to the two-stage choice procedure the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{b\}$. A matrix of majority relation μ looks as

	a	c	d	e	f	g
a	-	1	0	1	0	0
c	0	-	0	0	1	0
d	0	0	-	1	0	0
e	0	0	0	-	0	1
f	1	0	0	0	-	0
g	1	0	1	0	1	-

According to the given two-stage choice procedure the alternatives c, g will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{c, g\}$.

Finally, consider the subset $X'' = X \setminus \{a\}$. Then a matrix of majority relation μ looks as

	b	c	d	e	f	g
b	-	0	1	0	1	1
c	0	-	0	0	1	0
d	0	0	-	1	0	0
e	1	0	0	-	0	1
f	0	0	0	0	-	0
g	0	0	1	0	1	-

According to the rule the alternative c will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{c\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

422.5 Monotonicity condition 1

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	1	0	0	0	0
b	0	-	1	0	0	0
c	1	0	-	1	1	0
d	0	0	0	-	0	0
e	0	0	0	0	-	0
f	1	0	0	1	0	-

According to the rule the alternatives a, b, c will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b, c\}$.

Suppose that the relative position of the alternative a is improved such that $a\mu c$ while the relative comparison of any pair of other alternatives remains unchanged. Then a matrix of majority relation looks as

	a	b	c	d	e	f
a	-	1	0	0	0	0
b	0	-	1	0	0	0
c	0	0	-	1	1	0
d	0	0	0	-	0	0
e	0	0	0	0	-	0
f	1	0	0	1	0	-

For this case the alternatives a, b, c, f are considered for the second stage of the choice procedure. For this majority matrix it is possible to construct a profile \vec{P}'_X such that $\{a\} \notin C(\vec{P}'_X, X)$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

440.1 Heredity condition (H)

Let $X = \{a, b, c, d, e, f\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f
a	-	0	1	0	0	0
b	0	-	1	0	0	0
c	0	0	-	1	1	1
d	0	0	0	-	1	0
e	0	0	0	0	-	1
f	0	0	0	1	0	-

According to the two-stage choice procedure the alternatives a, b will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b\}$.

Consider now the subset $X' = X \setminus \{c\}$. A matrix of majority relation μ looks as

	a	b	d	e	f
a	-	0	0	0	0
b	0	-	0	0	0
d	0	0	-	1	0
e	0	0	0	-	1
f	0	0	1	0	-

According to the rule the choice is empty, i.e., $C(\vec{P}_{X'}, X') = \emptyset$.

Then $C(\vec{P}_X, X) \cap X' = \{a, b\} \not\subseteq C(\vec{P}_{X'}, X')$. Thus, the condition **H** is not satisfied.

465.2 Concordance condition (C)

Consider the example from paragraph 468.2 of Appendix 1. Then $C(\vec{P}_X, X) = \{a, b, c, g\}$.

Consider now the subset $X' = X \setminus \{b\}$. A matrix of majority relation μ looks as

	a	b	c	d	e	f
a	-	0	0	0	1	1
b	0	-	0	0	0	0
c	1	0	-	0	0	1
d	0	0	1	-	0	1
e	0	1	1	1	-	0
f	0	0	0	0	1	-

According to the given two-stage choice procedure the alternatives a, b, d will be chosen, i.e.,

$C(\vec{P}_{X'}, X') = \{a, b, d\}$.

Finally, consider the subset $X'' = X \setminus \{a\}$. Then a matrix of majority relation μ looks as

	b	c	d	e	f	g
b	-	0	0	0	0	0
c	0	-	0	0	1	0
d	0	1	-	0	1	0
e	1	1	1	-	0	0
f	0	0	0	1	-	1
g	0	0	1	1	0	-

According to the rule the alternatives b, c, d, g will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{b, c, d, g\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{b, d\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

465.5 Monotonicity condition 1

Let $X = \{a, b, c, d\}$ and a matrix of majority relation μ is the following

	a	b	c	d
a	-	1	0	0
b	0	-	0	1
c	1	0	-	0
d	0	0	1	-

According to the rule the alternatives a, b, c, d will be chosen. Suppose that the relative position of the alternative a is improved ($a\mu d$). Then the alternative d is omitted on the first stage. For this case it is possible to construct a profile \vec{P}'_X such that $\{a\} \notin C(\vec{P}'_X, X)$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

468.1 Heredity condition (H)

Let $X = \{a, b, c, d, e, f, g\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f	g
a	-	0	0	0	1	1	0
b	0	-	0	0	0	0	0
c	1	0	-	0	0	1	0
d	0	0	1	-	0	1	0
e	0	1	1	1	-	0	0
f	0	0	0	0	1	-	1
g	0	0	0	0	1	0	-

According to the given two-stage choice procedure the alternatives a, b, d, g will be chosen, i.e., $C(\vec{P}_X, X) = \{a, b, d, g\}$.

Consider now the subset $X' = X \setminus \{d, e\}$. A matrix of majority relation μ looks as

	a	b	c	f	g
a	-	0	0	1	0
b	0	-	0	0	0
c	1	0	-	1	0
f	0	0	0	-	1
g	0	0	0	0	-

According to the rule the alternatives b, c, g will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{b, c, g\}$.

Then $C(\vec{P}_X, X) \cap X' = \{a, b, g\} \not\subseteq C(\vec{P}_{X'}, X')$. Thus, the condition **H** is not satisfied.

468.2 Concordance condition (C)

Let $X = \{a, b, c, d, e, f, g\}$ and a matrix of majority relation μ is the following

	a	b	c	d	e	f	g
a	-	0	0	0	1	1	0
b	0	-	0	0	0	0	0
c	1	0	-	0	0	1	0
d	0	0	1	-	0	1	0
e	0	1	1	1	-	0	0
f	0	0	0	0	1	-	1
g	0	0	0	1	1	0	-

According to the two-stage choice procedure the alternatives b, g will be chosen, i.e., $C(\vec{P}_X, X) = \{b, g\}$.

Consider now the subset $X' = X \setminus \{d\}$. A matrix of majority relation μ looks as

	a	b	c	e	f	g
a	-	0	0	1	1	0
b	0	-	0	0	0	0
c	1	0	-	0	1	0
e	0	1	1	-	0	0
f	0	0	0	1	-	1
g	0	0	0	1	0	-

According to the given two-stage choice procedure the alternatives b, c, g will be chosen, i.e., $C(\vec{P}_{X'}, X') = \{b, c, g\}$.

Finally, consider the subset $X'' = X \setminus \{a, b, e, f\}$. Then a matrix of majority relation μ looks as

	c	d	g
c	-	0	0
d	1	-	0
g	0	1	-

According to the rule the alternatives c, g will be chosen, i.e., $C(\vec{P}_{X''}, X'') = \{c, g\}$.

Then $C(\vec{P}_{X'}, X') \cap C(\vec{P}_{X''}, X'') = \{c, g\} \not\subseteq C(\vec{P}_X, X)$. Thus, the condition **C** is not satisfied.

468.5 Monotonicity condition 1

Let $X = \{a, b, c, d, e, f, g\}$ and a matrix of majority relation μ is the following

	a	c	d	e	f	g
a	-	0	0	0	1	0
c	1	-	0	0	1	0
d	0	1	-	0	1	0
e	0	1	1	-	0	0
f	0	0	0	1	-	1
g	0	0	1	1	0	-

According to the rule the alternatives e, f are omitted on the first stage. Suppose that the relative position of the alternative a is improved such that $a\mu g$ while the relative comparison of any pair of other alternatives remains unchanged. Then the alternatives e, f are omitted on the first stage. For this case it is possible to construct a profile \vec{P}_X' such that $\{a\} \notin C(\vec{P}_X', X)$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}_X', X)$. Thus, the Monotonicity condition 1 is not satisfied.

Two-stage choice procedures 478-494, 496-504

Two-stage choice procedures 478-494, 496-504 satisfy the same conditions as two-stage choice procedures which use the Fishburn rule and Uncovered set I on the first stage. To prove it we can use the same examples that are used for these rules [11].

Two-stage choice procedures 534-538, 543, 554, 558

Two-stage choice procedures do not satisfy the same conditions as choice procedures used on the second stage. As for other normative conditions, two-stage choice procedures 534-538, 543, 554, 558 do not satisfy any of them as the subset of alternatives for the second stage can be easily changed when the core is applied on the first stage of the rule. To prove it we can provide the following example.

534.5 Monotonicity condition 1.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

	P₁	P₂	P₃	P₄	P₅	P₆	P₇	P₈
a	<i>a</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>d</i>
c	<i>c</i>	<i>c</i>	<i>c</i>	<i>a</i>	<i>d</i>	<i>d</i>	<i>b</i>	<i>b</i>
b	<i>d</i>	<i>d</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>c</i>	<i>c</i>
d	<i>b</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>a</i>	<i>c</i>	<i>a</i>	<i>a</i>

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Suppose now that the position of the alternative a was improved in P'_6 while the relative comparison of any pair of other alternatives remained unchanged.

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8
a	a	a	c	b	b	d	d
c	c	c	a	d	a	b	b
b	d	d	b	c	d	c	c
d	b	b	d	a	c	a	a

Then the alternative d will be omitted after applying the first choice procedure (the core) and the subset $X' = X \setminus \{d\}$ will be presented for the second choice procedure (the plurality rule). A contraction of a profile \vec{P}'_X onto a set X' , i.e., $\vec{P}'_{X'}$, looks as

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8
a	a	a	c	b	b	b	b
c	c	c	a	c	a	c	c
b	b	b	b	a	c	a	a

According to the rule the alternative b will be chosen, i.e., $C(\vec{P}'_X, X) = C(\vec{P}'_{X'}, X') = \{b\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied for the given two-stage choice procedure.

Two-stage choice procedures 561-588

Two-stage choice procedures 561-588 do not satisfy the same normative conditions as two-stage choice procedures which use minimal weakly stable set on the first stage. To prove it similar examples can be used but with larger number of alternatives (see the properties of k -stable set) [11].

Two-stage choice procedures 589-616

To check the properties of two-stage choice procedures 589-616 it is possible to use the same counter-examples that were used to check the properties of the threshold rule [11]. Thus, it remained to check normative conditions which are satisfied for the threshold rule.

- Two-stage choice procedures 589-616 satisfy the Monotonicity condition 1. The proof follows from properties of the threshold rule (an improved position of any chosen alternative x leads to the choice of only this alternative on the first stage of two-stage procedure).
- Two-stage choice procedures 589-616 do not satisfy the non-compensatory condition. To proof it the following example is provided.

Let $X = \{a, b, c, d\}$ and the profile \vec{P}_X is the following

P_1	P_2	P_3
a	b	c
d	a	b
b	c	a
c	d	d

According to the first stage rule, the alternatives a and b will be chosen on the first stage of the two-stage choice procedure. Consider now the subset $X' = X \setminus \{c, d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3
a	b	b
b	a	a

According to the second stage rule, the alternative b will be chosen on the second stage of the two-stage choice procedure. Thus, $C(\vec{P}_X, X) = \{b\}$.

Now let us write the profile \vec{P}_X in the following form

X	φ_1	φ_2	φ_3
a	4	3	2
b	2	4	3
c	1	2	4
d	3	1	1

According to the non-compensatory condition the alternatives a, b (the alternatives a and b are equal) are better than the alternative c and the alternative c is better than the alternative d . Thus, the non-compensatory condition is not satisfied as $\{a, b\} \neq C(\vec{P}_X, X)$.

Two-stage choice procedures 617-700

Two-stage choice procedures do not satisfy the same conditions as Copeland rules 1-3 and choice procedures used on the second stage. To prove it similar examples can be used but with larger number of alternatives [11]. Thus, it remained to check normative conditions which are satisfied for Copeland rules 1-3 and choice procedures used on the second stage.

- Two-stage choice procedures 617-620, 623-624, 628-648, 651-652, 656-676, 679-680, 684-700 satisfy the Monotonicity condition 1 (the proof follows from properties of Copeland rules 1-3 and choice procedures used on the second stage).
- Two-stage choice procedures 621-622, 625-627, 649-650, 653-655, 677-678, 681-683 do not satisfy the Monotonicity condition 1 (the same example that was used to check the properties of the second stage choice procedures can be provided).
- Two-stage choice procedures 617-700 do not satisfy the non-compensatory condition (the proof follows from properties of Copeland rules 1-3).

Two-stage choice procedures 701-728

Two-stage choice procedures do not satisfy the same conditions as choice procedures used on the second stage. To prove it similar examples can be used but with larger number of alternatives [11]. Thus, it remained to check normative conditions which are satisfied for choice procedures used on the second stage.

- Two-stage choice procedures 701, 719-720 do not satisfy the condition **H** (the proof follows from properties of super-threshold rule).
- Two-stage choice procedures 712, 717, 719-720 do not satisfy the condition **C** (the proof follows from properties of super-threshold rule).
- Two-stage choice procedures 712, 717 do not satisfy the condition **O** (the proof follows from properties of super-threshold rule).

- Two-stage choice procedures 701, 712-713, 719-720 satisfy the Monotonicity condition 1 (the proof follows from properties of threshold-rule and choice procedures used on the second stage).
- Two-stage choice procedures 702-711, 714-718, 721-728 do not satisfy the Monotonicity condition 1. Since super-threshold rule narrows the subset of alternatives remained after the first stage of the procedure, alternatives dominated by the chosen one might be eliminated on the first stage and, consequently, on the second stage of the rule the chosen alternative will not be included into a new choice.
- Two-stage choice procedures 701-728 do not satisfy the non-compensatory condition (the proof follows from properties of super-threshold rule).

Two-stage choice procedures 729-784

To check the properties of two-stage choice procedures which use minimax and Simpson procedures on the first stage we can use the same counter-examples which were used to check the properties of minimax and Simpson procedures [11]. Thus, it remained to study those normative conditions which are satisfied for choice procedures used on the first stage.

- Two-stage choice procedures 729, 740-741, 747-748, 757, 768-769, 775-776 satisfy the Monotonicity condition 1 (the proof follows from properties of Minimax and Simpson procedures and choice procedures used on the second stage).
- Two-stage choice procedures 730-739, 742-746, 749-756, 758-767, 770-774, 777-784 do not satisfy the Monotonicity condition 1. To prove it the following example for the two-stage choice procedure 731 is given.

Let $X = \{a, b, c\}$ and the profile \vec{P}_X looks as

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
c	c	b	a	a	a	a	a	c	d	d	d	d	b	c
b	b	c	b	c	c	c	c	b	c	b	b	b	a	d
d	d	d	c	b	b	b	b	d	a	a	a	a	d	a
a	a	a	d	d	d	d	d	a	b	c	c	c	c	b

Let us construct a matrix $S^-(\vec{P}_X, X)$ for the profile \vec{P}_X .

	a	b	c	d
a	-	7	9	6
b	8	-	6	10
c	6	9	-	10
d	9	5	5	-

According to the minimax procedure used on the first stage, the alternatives a, b, c will be chosen, i.e., $C_1(\vec{P}_X, X) = \{a, b, c\}$, where C_1 is a Minimax choice procedure.

Consider now the subset $X' = X \setminus \{d\}$. A contraction of a profile \vec{P}_X onto a set X' , i.e., $\vec{P}_{X'}$, looks as

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
c	c	b	a	a	a	a	a	c	c	b	b	b	b	c
b	b	c	b	c	c	c	c	b	a	a	a	a	a	a
a	a	a	c	b	b	b	b	a	b	c	c	c	c	b

According to the rule the alternative a will be chosen, i.e., $C(\vec{P}_X, X) = \{a\}$.

Consider now a profile \vec{P}'_X , which differs from the profile \vec{P}_X only by improved position of the alternative a in P'_{10} .

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}	P'_{11}	P'_{12}	P'_{13}	P'_{14}	P'_{15}
c	c	b	a	a	a	a	a	c	d	d	d	d	b	c
b	b	c	b	c	c	c	c	b	a	b	b	b	a	d
d	d	d	c	b	b	b	b	d	c	a	a	a	d	a
a	a	a	d	d	d	d	d	a	b	c	c	c	c	b

Let us construct a matrix $S^-(\vec{P}'_X, X)$ for the profile \vec{P}'_X .

	a	b	c	d
a	-	7	10	6
b	8	-	6	10
c	5	9	-	10
d	9	5	5	-

According to the minimax procedure used on the first stage, the alternatives a, b will be chosen, i.e., $C_1(\vec{P}'_X, X) = \{a, b\}$, where C_1 is a Minimax choice procedure.

Consider now the subset $X'' = X \setminus \{c, d\}$. A contraction of a profile \vec{P}'_X onto a set X'' , i.e., $\vec{P}'_{X''}$, looks as

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}	P'_{11}	P'_{12}	P'_{13}	P'_{14}	P'_{15}
b	b	b	a	a	a	a	a	b	a	b	b	b	b	a
a	a	a	b	b	b	b	b	a	b	a	a	a	a	b

According to the rule the alternative b will be chosen, i.e., $C(\vec{P}'_X, X) = \{b\}$.

Then $\{a\} \in C(\vec{P}_X, X)$, $\{a\} \notin C(\vec{P}'_X, X)$. Thus, the Monotonicity condition 1 is not satisfied.

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Исследуются двухступенчатые процедуры выбора, которые представляют собой суперпозицию двух процедур выбора. Показано, какие из рассматриваемых процедур выбора удовлетворяют существующим нормативным условиям, описывающим, каким образом изменяется конечный выбор при изменении предъявляемого множества альтернатив и оценок альтернатив по критериям. Особое внимание уделяется двухступенчатым процедурам, в основе которых лежат позиционные правила, а также правила, использующие мажоритарное отношение, вспомогательную числовую шкалу и турнирную матрицу. Приводится теорема о том, какие нормативные условия выполняются для рассматриваемых двухступенчатых процедур выбора. Оценена вычислительная сложность двухступенчатых процедур выбора и время их выполнения на реальных данных.

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