Fair Division and Counterfactual-proofness

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joint work with

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- moneyless division problems: family heirlooms; substitutable shifts, seats in overbooked classes to students, or computing resources in peer-to-peer platforms

- goal 1: exploit differences in preferences to reach an efficient division

- equal division is fair but inefficient; equalizing market value of shares is not an option

- goal 2: define a concept of fairness compatible with unequal shares (and efficiency) implementing concrete division methods: several websites Adjusted Winner, Spliddit, $\cdots$

simplifying assumptions:

- additive utilities, ruling out complementarities: a feasible report
- several *indivisible* objects but we can split *a few* using lotteries, time-sharing, $\cdots$
- efficiency $\Longrightarrow$ need to split *at most* $n-1$ among n participants

the two contenders

- **Relative Egalitarian** (REG) solution (aka *Egalitarian Equivalent*, *Adjusted Winner* for $n = 2$)

- **Nash Max Product** (NMP) solution (aka *Competitive Equilibrium with Equal Incomes*)

the punchline

- the NMP solution has better normative and incentives properties than the REG solution
- the NMP solution is *CounterFactual-Proof* and this property is characteristic

Ann, Bob, and six splittable objects

give 100 points to Ann and to Bob

	bicycle	I-phone	clock	tent	violin	parrot
Ann	18	20	16	15	15	16
Bob	20	0	30	13	20	17

note: utilities are *normalized* $\longrightarrow$ individual scales do not matter

Efficiency: order objects by decreasing ratios $\frac{u_x^{Ann}}{u_x^{Bob}}$

	I-phone	tent	parrot	bicycle	violin	clock
Ann	20	15	16	18	15	16
Bob	0	13	17	20	20	30

Ann gets a left tail, Bob a right tail

$\longrightarrow$ split at most one object the classic “utilitarian” and "egalitarian" benchmarks

here utilities are normalized

$\Longrightarrow$ *relative utilitarian* versus *relative egalitarian* solutions

Relative Utilitarian solution

	I-phone	tent	parrot	bicycle	violin	clock
Ann	20	15	(16)	(18)	(15)	(16)
Bob	(0)	(13)	17	20	20	30

	I-phone	tent	parrot	bicycle	violin	clock
Ann	1	1	0	0	0	0
Bob	0	0	1	1	1	1

Relative Egalitarian solution

	I-phone	tent	parrot	bicycle	violin	clock
Ann	20	$+\ 15$	$+\ 16$	$+\ \frac{1}{2}\times 18$	(15)	(16)
Bob	(0)	(13)	(17)	$=\ \frac{1}{2}\times 20$	$+\ 20$	$+\ 30$

	I-phone	tent	parrot	bicycle	violin	clock
Ann	1	1	1	$\frac{1}{2}$	0	0
Bob	0	0	0	$\frac{1}{2}$	1	1

- the utilitarian solution is *integral*: does not split objects (except those with equal scores)

- but it is sharply discontinuous, and can treat one participant very poorly

	bicycle	I-phone	clock	tent	violin	parrot
Ann	13	(18)	(17)	(24)	(18)	(10)
Bob	(5)	19	18	26	20	12

$\longrightarrow$ Ann ends up with a relative value of 13%

- **Fair Share Guarantee** with n participants**:** each share is worth at least $\frac{100}{n}$ points to its recipient

$\longrightarrow$ Relative Egalitarianism guarantees at least 50 points to Ann and to Bob

$\longrightarrow$ we dismiss the Relative Utilitarian rule the *Nash MaxProduct* solution

→ maximize the product $u^1 \times u^2$ of individual utilities (aka *proportional fairness*: Kelly 1998)

→ a well known compromise between utilitarianism and egalitarianism

	I-phone	tent	parrot	bicycle	violin	clock
Ann	20	15	16	(18)	(15)	(16)
Bob	(0)	(13)	(17)	20	20	30

	I-phone	tent	parrot	bicycle	violin	clock
Ann	1	1	1	0	0	0
Bob	0	0	0	1	1	1

- the NMP solution meets **FSG** (any n)

- NMP sits between the RUT and REG solutions ($n = 2$ only)

- the NMP, unlike the REG solution, is *often* integral

objects	x_1	$\cdots$	x_k	x_{k+1}	$\cdots$	x_K
Ann	u_1^A	$\cdots$	u_k^A	u_{k+1}^A	$\cdots$	u_p^A
Bob	u_1^B	$\cdots$	u_k^B	u_{k+1}^B	$\cdots$	u_K^B

objects are arranged "efficiently"

$S_k^A = \sum_{l=1}^{k} u_l^A$; $S_k^B = \sum_{l=k}^{K} u_l^A$

the sequence $\frac{u_l^A}{u_l^B}$ decreases (weakly) in l while $\frac{S_l^A}{S_l^B}$ increases (weakly)

NMP divides objects where they cross (loosely)

NMP is integral **iff** we cannot find k such that $|\frac{S_k^A}{u_k^A} - \frac{S_k^B}{u_k^B}| < 1$

this happens frequently! (work in progress) for two-agent problems we do not have normative arguments to draw a wedge between NMP and REG

we can even choose many compromises between the two rules

$\longrightarrow$ for any $q, 0 \leq q \leq +\infty$ the rule F^q minimizing $\frac{1}{(u^A)^q} + \frac{1}{(u^B)^q}$ satisfies **FSG**

NMP is the limit of F^q when $q \to 0$

REG is the limit of F^q when $q \to +\infty$

.

with three or more agents NMP is much more attractive than REG a **division problem**

$N \ni i$ the agents, $A \ni a$ the objects

$u^i \in \mathbb{R}_+^A$ the utilities (no longer normalized)

feasible allocations: $z = (z^i)_{i \in N}$, $z^i \in [0,1]^A$, $\sum_N z^i = e^A$

corresponding utilities: $U_i = u^i \cdot z^i$

a **division rule**

picks a unique feasible utility vector $(U_i, i \in N)$ and **all** corresponding allocations

Relative Egalitarian allocation z: efficient and for all $i, j \in N$

$$\frac{u^i \cdot z^i}{u^i \cdot e^A} = \frac{u^j \cdot z^j}{u^j \cdot e^A}$$

note: this simple definition requires $u^i \gg 0$ for all i; in the general case we need the leximin refinement

Nash MaxProduct allocation z: maximizes $\Pi_N u^i \cdot z^i$ in the feasible set

define: *Competitive Equilibrium with Equal Incomes* (CEEI) allocation z:
there is a price $p \in \mathbb{R}_+^A$ such that $p \cdot e^A = n$ and

$$z^i \in \arg\max_{y^i \in \mathbb{R}_+^A} \{u^i \cdot y^i | p \cdot y^i \leq 1\} \text{ for all } i$$

Theorem (Gale 1960): NMP $\Longleftrightarrow$ CEEI consequence: at the NMP allocation participants have *equal opportunities*

- **No Envy**: $u^i \cdot z^i \geq u^i \cdot z^j$ for all i, j

compare: the REG allocation may generate envy

	a	b
u^1	1	1
u^2	2	1
u^3	3	1

$$\text{efficiency} + \text{FSG} \Rightarrow z = \begin{array}{cc} a & b \\ 0 & 1-\beta \\ \alpha & \beta \\ 1-\alpha & 0 \end{array}$$

$$z^{REG} = \begin{array}{cc} a & b \\ 0 & 18/23 \\ 11/23 & 5/23 \\ 12/23 & 0 \end{array} \qquad \text{where 3 envies 2}$$

a second defect of the REG rule: *more objects can be bad news*

$$\begin{array}{cccc} & a & b & c \\ u^1 & 3 & 1 & 1 \\ u^2 & 1 & 3 & 1 \\ u^3 & 1 & 1 & 3 \end{array} \hookrightarrow \begin{array}{ccc} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \hookrightarrow U^1 = U^2 = U^3 = 3$$

$$\begin{array}{ccccc} & a & b & c & d \\ u^1 & 3 & 1 & 1 & 0 \\ u^2 & 1 & 3 & 1 & 4 \\ u^3 & 1 & 1 & 3 & 4 \end{array} \hookrightarrow \begin{array}{cccc} a & b & c & d \\ 55/59 & 0 & 0 & 0 \\ 2/59 & 1 & 0 & 1/2 \\ 2/59 & 0 & 1 & 1/2 \end{array} \hookrightarrow U^1 < 3$$

$\Longrightarrow$ incentives to sabotage

- **Resource Monotonicity**: $A \subset B \Longrightarrow U^i(N, A, u_{[A]}) \leq U^i(N, B, u_{[B]})$

- **Population Monotonicity**: $N \subset M \Longrightarrow U^i(N, A, u^{[N]}) \geq U^i(M, A, u^{[M]})$

	FSG	*NoEnvy*	*ResceMon*	*PopulMon*
RelativeEgalit	*YES*	*NO*	*NO*	*YES*
NashMaxProdct	*YES*	*YES*	*YES*	*YES*

recall: FSG + ResceMony + Efficiency = $\varnothing$ in the general Arrow-Debreu domain the key incentive property: CounterFactual-Proofness

can an agent benefit by misreporting

only about objects she does not get?

$\longrightarrow$ YES under the REG rule

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	a	b	c	d
Ann	10	6	3	1
Bob	0	4	8	8

$\longrightarrow$ REG

	a	b	c	d
Ann	1	1	0	0
Bob	0	0	1	1

	a	b	c	d
Ann	10	6	3	1
Bob	4	(0.8)4	(0.8)8	(0.8)8

$\longrightarrow$ REG

	a	b	c	d
Ann	1	0.35	0	0
Bob	0	0.65	1	1

.

$\{v^i$ is truthful to u^i on $Y \subseteq A\} \overset{def}{\Leftrightarrow} \{\frac{v_a^i}{v_b^i} = \frac{u_a^i}{u_b^i}$ for all $a, b \in Y\}$

notation: $[x^i]$ is the support of agent i's share

Define: the division rule f is *CounterFactual-Proof* if for all $i \in N, u \in \mathbb{R}_+^{A \times N}$ and $v^i \in \mathbb{R}_+^A$

$$v^i \text{ is truthful to } u^i \text{ on } [f^i(v^i, u^{-i})] \Longrightarrow u^i \cdot f^i(u) \geq u^i \cdot f^i(v^i, u^{-i})$$

Fact: *the NMP rule is (Group) CounterFactual-Proof*

proof: assume $f^i(u) = X^i \subset A$ and $f^i(v^i, u^{-i}) = Y^i$ (no shared items)

$i \hookrightarrow j \overset{def}{\Longleftrightarrow} \{u^i \cdot Y^i > u^i \cdot X^i$ and $Y^i \cap X^j \neq \varnothing\}$

apply the FOC for $a \in Y^i \cap X^j$, first at u then at (v^i, u^{-i}):

$$\frac{u^i \cdot X^i}{u^j \cdot X^j} \geq \frac{u^i_a}{u^j_a} \text{ and } \frac{v^i_a}{u^j_a} \geq \frac{v^i \cdot Y^i}{u^j \cdot Y^j} \Longrightarrow \frac{u^i \cdot X^i}{u^j \cdot X^j} \geq \frac{u^i \cdot Y^i}{u^j \cdot Y^j}$$

because v^i is truthful to u^i on Y^i

if $i \hookrightarrow j$ we get $u^j \cdot Y^j > u^j \cdot X^j$ hence $j \hookrightarrow k$ for some k

start from i who benefits from misreporting v^i

let B be the set of agents reachable from i by a path $i \hookrightarrow j \hookrightarrow k \hookrightarrow \cdots$

we see $\cup_B Y^j = \cup_B X^j$ and $u^j \cdot Y^j > u^j \cdot X^j$ for all $j \in B$

contradicting the efficiency of $f(u)$

Axiomatic characterization of NMP

- **Efficiency** the rule F picks an efficient outcome

- **Scale Invariance**: changing u^i to λu^i for any positive λ, is irrelevant

- **Symmetry:** with respect to agents and with respect to objects

- **Equivalent Objects:** we can merge two objects a, b such that $\frac{u^i_a}{u^i_b} = \frac{u^j_a}{u^j_b}$ for all $i, j \in N$

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Theorem: the NMP rule is characterized by these four axioms and Group-CounterFactual-Proofness *note*: the only fairness axiom is Symmetry

many rules meet Eff, SI, SYM and EO

for instance REG, RUT

and the rules minimizing $sign(q)\sum_N(\frac{u^i}{u^i \cdot e^A})^q$ for any $q, -\infty \leq q \leq +\infty$

Conclusion

- the Nash MaxProduct rule is a very compelling Fair Division rule under additive utilities

- next step: testing this statement in an online experimental setting

Thank You