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AN EXTENSION OF A CLASS OF COST SHARING METHODS TO THE SOLUTIONS OF THE CLASS OF TWO-PERSON COOPERATIVE GAMES

Two-person games and cost/surplus sharing problems are worth for studying because they are the base for their extending to the classes of such problems with variable population with the help of very powerful consistency properties. In the paper a family of cost-sharing methods for cost sharing problems with two agents [Moulin 2000] is extended to a class of solutions for two-person cooperative games that are larger than both cost-sharing and surplus-sharing problems, since cooperative games have no no restrictions on positivity of costs and surpluses. The tool of the extension is a new invariance axiom – self covariance – that can be applied both to cost-sharing methods and to cooperative game solutions. In particular, this axiom replaces the Lower composition axiom not applicable to methods for profit sharing problems.

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1 Introduction

The solutions for two-person games and for the cost/surplus sharing problems are worth for studying, since they, together with some consistency axiom(s), can define solutions for the classes of game (cost/surplus sharing problems) with variable population. The last class of problems and their solutions have been developed more in detail than those of cooperative games.

The main difference between cost/profit sharing problems and two-person games consists in positivity of all components in the domains of their definitions: in the former ones all their components, including the solutions, are nonnegative, and this restriction is superfluous for two-person cooperative games. Mainly for this reason the most axioms characterizing solutions for both classes of problems differ one from another. For example, the popular axiom translation covariance, meaning independence of solutions w.r.t. arbitrary shifts of individual utilities cannot be applied to cost/profit sharing problems. Only scale covariance (positive homogeneity) is used for both classes. Conversely, axioms Upper and Low compositions for solutions of cost sharing problems can not be extended to cooperative games with arbitrary sets of players.

The number of cost sharing methods is enough big, because for cost sharing problems there are many natural properties whose formalization can be applied for axiomatizations of the methods [Moulin 2002, Thomson 2003]. Two-person cost sharing problems can be considered as subadditive games with nonnegative values of characteristic functions. Efficient solutions for such a subclass of games coincide with methods for cost sharing problems. If the axioms characterizing properties of a cost sharing method can be extended to solutions of subadditive games with arbitrary set of players, then the former, possibly, will be extendable to solutions for the whole class of subadditive games. The same correspondence exists between profit-sharing methods and solutions of superadditive games.

In particular, note that the most known TU game solutions are covariant w.r.t. positive linear transformations of individual utilities. This axiom needs no explanation in the context of problems dealing with transferable utilities including classical cooperative games and cost sharing problems. However, the latters cannot possess this property, since arbitrary shifts (that may be negative) of individual utilities of players can lead a cost problem out the positive domain. Thus, weakenings of covariance, applicable for cost sharing methods, may be useful for their characterizations.

Recently the author [Yanovskaya 2012] proposed a weakening of translation covariance axiom, called by self-covariance. It turned out that the egalitarian solutions for cooperative games with transferable utilities (TU games) satisfy this axioms and can be characterized with its help. Moreover, self-covariance can be applied for cost-sharing methods, since the shifts of individual utilities defining the axioms, do not violate positivity of the costs.

In this paper this axiom is applied to solutions of two-person cooperative games together with efficiency, anonymity, path independence and scale covariance. All such solutions are described and characterized separately for super- and subadditive games. The coincidence of the solutions for subadditive games. On the subclass of subadditive games with nonnegative characteristic functions such solutions coincide with the family of cost-sharing methods defined and characterized in [Moulin 2000]. These methods cannot be extended for profit sharing problems, because one of the axioms characterising it – the Low Composition – cannot be applied to profit sharing method, since transformations in its definition may go out a profit sharing problem from the whole class of such problems. It turned out that the self-covariance axiom can replace it, and with its help we obtain the extension of Moulin's methods to the class of two-person profit sharing problem.

The paper is structured as follows. In section 2 the main definitions of cooperative game solutions and of cost sharing methods are given. Section 3 describes all cooperative game solutions satisfying efficiency, single-valuedness, anonymity, and self covariance. This class turns out very large, and in section 4 one more axiom –Path Independence – is added, and the corresponding family of solutions satisfying all them is described. It is shown that the restriction of this family by nonnegative subadditive games, or, in other words by the set of two-person cost sharing methods, coincides with the family characrized in [Moulin 2000]. Concluding remarks show the simple extension of all results to superadditive two-person games and to profit sharing methods.

2 Correspondences between cost-sharing problems and two-person cooperative games

In this section we give definitions of cost-sharing problems and cooperative games with two agents and show a connection between cost-sharing methods and cooperative game solutions.

A *cost sharing problem* for two agents $\{i,j\}$ is a triple (c_i, c_j, T), where $c_i, c_j > 0$ are claims of the agents, $T > 0$ is the total cost to be allocated between the agents. Thus, $c_i + c_j > T$. Denote $c = (c_i, c_j)$. A *cost sharing method* for a class $\mathcal{C}$ of cost sharing problems is a function $r : \mathcal{C} \to \mathbb{R}^2_+$ such that $r_i(c,T) + r_j(c,T) = T$.

A method r for a class of cost sharing problems with two agents $N = \{i,j\}$ is *anonymous*, if $r_i(c_i, c_j, T) = r_j(c_j, c_i, T)$ for all $c_i, c_j, T : c_i + c_j \geq T > 0$.

The most known cost sharing method is *proportional*: for every problem (c_i, c_j, T) it is defined as $r^{pr}(c_i, c_j, T) = t(c_i, c_j)$, where t satisfies the equality $t(c_i + c_j) = T$.

A cooperative game is a pair (N, v), where N is a finite set of players, $v : 2^N \to (R)$ is a characteristic function with a convention $v(\emptyset) = 0$. In this paper we restrict ourselves by two-player sets. If this set $N = \{i,j\}$ is fixed, then a cooperative game is completely defined by three numbers $v(\{i\}), v(\{j\}), v(\{i,j\})$. A *solution* φ for two-person cooperative game associates with every game $(\{i,j\}, v)$ a payoff vector $\varphi(\{i,j\}, v) \in \mathbb{R}^2$. In the sequel we consider only the classes of two-person TU games with a fixed set of players N. We will denote them as i, j and for simplicity exclude the notation N from all the formulas. Moreover, for brevity and similarity with cost sharing problems in the sequel we use notation $v(\{i\}) = v_i, v(\{j\}) = v_j, v(\{i,j\}) = T$.

Therefore, a subclass of subadditive cooperative games with nonnegative characteristic functions coincides with the two-agent cost-sharing problems. The set of all cost-sharing methods coincides with the subset of solutions for such a subclass of two-person games such that every solution satisfies the defined on the next page properties efficiency and single-valuedness.

First, consider the class of subadditive two-person games $\mathcal{G}_+^2$ with nonnegative values of characteristic functions of the great coalition, i.e. the class $\mathcal{G}_2^+ = \{(N,v)\}$, where N is an arbitrary two-element set, and $v : 2^N \to \mathbb{R}^2$ is a characteristic function satisfying $v_i + v_j \geq v(\{i,j\}) \geq 0, v(\emptyset) = 0$. Thus, the domain of this class of games is larger than that for cost sharing problems: we allow negative values individual values v_i, v_j of characteristic functions.

Recall some well-known properties of cooperative game solutions that will be further applied for their characterizations. Since in the paper we consider only two-person games we formulate them for the class of all two-person games $\mathcal{G}_2$

A solution φ for the class $\mathcal{G}_2$ is

- *non-empty* or satisfies *nonemptiness (NE)*, if $\varphi(\{i,j\},v)) \neq \emptyset$ for every game $(\{i,j\},v) \in \mathcal{G}_2$;
- *efficient (EFF)*, if $\varphi_i(\{i,j\},v) + \varphi_j(\{i,j\},v) = v(\{i,j\}$ for every game $(\{i,j\},v) \in \mathcal{G}_2$;
- *single valued (SV)*, if $|\varphi(\{i,j\},v)| = 1$ for every game$(\{i,j\},v) \in \mathcal{G}_2$;
- *positively homogeneous (PH)*, if for every $\alpha > 0$ and a game $(\{i,j\},v) \in \mathcal{G}_2$ it holds $(\{i,j\},\alpha v) \in \mathcal{G}_2$ and $\varphi(\{i,j\},\alpha v) = \alpha\varphi(\{i,j\},v)$;
- *translation covariant (TCOV)*, if for every game $(\{i,j\},v)) \in \mathcal{G}_2$ and a vector $b \in \mathbb{R}^2$
$$x \in \varphi(N,v) \implies x + b \in \varphi(N, v+b),$$
where $(v+b)_k) = v_k + b_k$ for $k = i, jN$, and $(v+b)(\{i,j\})) = v(\{i,j\} + b_i + b_j$;
- *covariant (COV)*, if it is positively homogeneous and translation covariant;
- *weakly covariant (WCOV)*, if it is positively homogeneous and translation covariant with only respect to shift $b \in \mathbb{R}^N$ with equal coordinates;
- *anonymous (ANO),* if for every game $(\{i,j\},v)) \in \mathcal{G}_2$ and the unique permutation $\pi(\{i,j\}) = (\{j,i\})$ the following equality holds: $\varphi(\pi(\{i,j\}),\pi v) = \pi(\varphi(\{i,j\},v))$. Here the function πv is defined by $\pi v_i = v_j, \pi v_j = v_i, \pi v(\{i,j\}) = v(\{i,j\})$,
- *self-covariant (self-COV)*, if it is positively homogeneous and for every number $A \geq -1$ the equalities
$$\varphi(\{i,j\}, v + A\varphi(\{i,j\},v)) = (A+1)\varphi(N,v) \tag{1}$$

hold for all games $(\{i,j\},v) \in \mathcal{G}_2^+$.

Compare these properties with those applied for characterizations of cost-sharing methods. First, efficiency and single-valuedness are contained in the definition of cost-sharing methods themselves. Positive homogeneity and Anonymity are applied in such characterizations without any modifications.

The translation covariance property, as well as weak covariance, cannot be applied for such a purpose, because the shifts of individual utilities applied in their definitions may violate positivity of components defining the cost sharing problems. On the contrary, self-covariance can be considered as a property of cost-sharing methods.

Let us give two more properties of cost-sharing methods that characterize, together with positive homogeneity, a family $\mathcal{H}_2^*$ of cost-sharing methods [Moulin 2000] that will be extended to solutions for two-person sub-additive games in section 4. As the the previous properties of cooperative games we define them only for two-agent cost sharing problems.

A cost sharing method r for two-agent cost sharing problems satisfies

– *Upper Composition (UC)* or *Path Independence*, if for all x and T, T' such that $0 \leq T \leq T' \leq x_i + x_j$ it holds
$$r(x,T) = r(r(x,T'),T);$$

– *Lower Composition (LC)*, if for all x and T, T' such that $0 \leq T' \leq T \leq x_i + x_j$ it holds
$$r(x,T) = r(x,T') + r(r(x, x - r(x,T')), T - T');$$

These are structural-invariance properties allowing to decompose the computation of shares when available resources are estimated from above or from below.

In the sequel we will consider only efficient and single-valued solutions, and will understand under the word "solution"an efficient and single-valued solution. In this definition the set of solutions for subadditive games with nonnegative values of characteristic functions coincides with the set of cost-sharing methods for cost sharing problems. This correspondence is a tool for the following characterizations of cooperative game solutions.

We begin with the definition of a family $\mathcal{H}_2^*$ of cost-sharing methods characterized in [Moulin 2000]. Every method r from this family is defined by an ordered partition of the positive orthant to cones (their number can be infinite) with vertices in zero point. The positive parts of the coordinate axis are the sides of the cones. The orderedness means that the sides of every cone are ordered by a binary index 1 or 2. On the side of cones the method is proportional, i.e., if k is the slope of any side of the cone, and the point $c = (c_i, c_j)$ is placed on this side, then
$$r(c,T) = (t, kt), \text{ where } t = \frac{T}{k+1}. \tag{2}$$

If $c = (c_i, c_j)$ belongs to the interior of a cone, whose sides have the slopes k_1, k_2, the slope k_2 has index 1, k_1 has index 2. then

$$r(c,T) = (t, T-t), \tag{3}$$

$$t = \begin{cases} \dfrac{T}{1+k_1}, & \text{if } T \leq \dfrac{(1+k_1)(c_2 - k_2 v_1)}{k_1 - k_2} \\ \dfrac{T - (c_2 - k_2 c_1)}{1+k_2} & \text{otherwise.} \end{cases} \tag{4}$$

This means that the point $r(c,T)$ either lies on the side with index 2, or the points c and $(t, T-t)$ lie on the direct line, parallel to the side of the cone having index 1, or

Theorem 2.1 (Moulin 2000) *The family $\mathcal{H}_2^*$ is the unique family of cost-sharing methods satisfying PH, UC, and LC.*

Since most cooperative game solutions are anonymous, we consider only the subfamily $\mathcal{H}_2^a \subset \mathcal{H}_2^*$ of anonymous cost-sharing methods. Just cost-sharing methods from this subfamily we will extend to solutions of games from the set $\mathcal{G}_2^+$.

Let us extend the methods from the family $\mathcal{H}_2^a$ to a family of anonymous, positive homogeneous, and self-covariant solutions of games from the class $\mathcal{G}_2^+$. Such a family of two-person subadditive games we denote by Φ.

In view of anonymity of the methods from $\mathcal{H}_2^a$ it suffices to define the extensions only for half a domain

$$L = \{x \mid x_i + x_j \geq 0, x_i \geq x_j\} \tag{5}$$

of characteristic function values (v_i, v_j) for games from the class $\mathcal{G}_2^+$. Divide this domain on cones. In such a partition the positive half-axis $x_j = 0, x_i \geq 0$ may not be a side of some cone.

Note that the slopes of the sides of the cones for methods from the family $\mathcal{H}_2^a$ and cost sharing problems (c,T) with $c_i \geq c_j$ belong to the set $[0,1]$. For partitioning of the set (5) on the corresponding cones the domain of their slopes is increased to $[-1,1]$. Each ordered partition of the set (5) on cones defines the following solution φ :: Let $(v,T) \in \mathcal{G}_2^+$ be an arbitrary game.

If the ray with the slope $k = \frac{v_j}{v_i}$ is one of the side of a cone of the partition, then $\varphi(\{i,j\}, v, T)$ is the proportional solution.

If v belongs to the interior of a cone, whose sides have the slopes k_1, k_2, $k_1 < k_2$, then $\varphi(v,T) = (t, T-t)$, where t is defined by formulas (3),(4), where c is replaced by v.

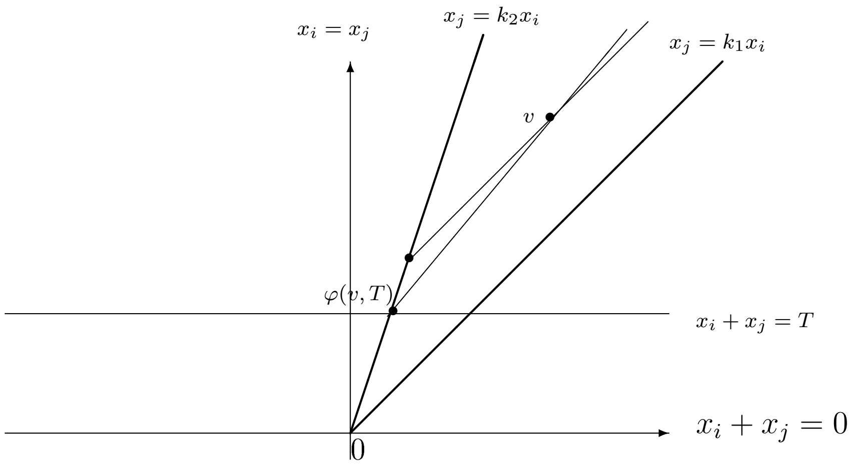

Fig.1

On Fig.1 the piece-wise line between the points v and $\varphi(v, T)$ is the locus of solution points for games (v, H) when H is going from $v_i + v_j$ to T.

From the definition of the family Φ it follows that they satisfy Efficiency, Single-valuedness, Anonymity, and Positive Homogeneity. Since the properties of Upper and lower Composition till now were not defined for cooperative game solutions, we lay aside their discussion till section 4.

Proposition 2.1 *Solutions from the family Φ verify self-covariance.*

Proof . Let $(v, T) \in \mathcal{G}_2^+$ be an arbitrary game, $\varphi \in \Phi$ be an arbitrary solution. The proportional solution satisfies self-covariance, hence, if v is placed on a side of a cone of the partition, then the Proposition has been proved.

Let now v be placed inside the cone with slopes k_1, k_2 (Fig.1) All points $\frac{1}{A+1}(v + A\varphi(v, T))$ for $A \geq -1$ are placed on the ray going out from the point $\varphi(v, T)$ and passing through v. By the definition of the solution φ, the following equality holds

$$\varphi(\frac{1}{A+1}(v + A\varphi(v, T))) = \varphi(v, T)$$

(see also Fig.1) for $A \geq -1$. Now by positive homogeneity of solutions from Φ we obtain the result.

$\square$

However, the properties ANO, PH, and Self-Cov are still insufficient for axiomatic characterizations of solutions for two-person subadditive games from the family Φ. Thus, in the next section we describe a larger family of all solutions satisfying these axioms.

3 The family Ψ

Denote by Ψ a family of all solutions for the class $\mathcal{G}_2^+$ satisfying efficiency, single-valuedness, anonymity, and self-covariance.

A particular subclass of subadditive games is the class of the additive ones. Let us find all solutions for this class satisfying axioms under consideration. Denote by $\mathcal{G}^{ad}$ the set of additive two-person games. Every game from this class is defined by a vector $v = (v_i, v_j)$ of individual values of the characteristic function. Thus, we denote additive two-person games with the player set $\{i, j\}$ by v.

Proposition 3.1 *A solution φ for the class $\mathcal{G}_2^{ad}$ satisfies efficiency, anonymity, positive homogeneity, and self-covariance if and only if it is trivial, i.e., $\varphi(\{i,j\}, v) = v$ for all $v \in \mathcal{G}_2^{add}$, or egalitarian $\varphi^e(\{i,j\}, v) = (\frac{v_i+v_j}{2}, \frac{v_i+v_j}{2})$.*

Proof . Let $v = (v_i, v_j)$ be an arbitrary vector. Assume that $\varphi(v) = x = (x_i, x_j)$, where $x_i + x_j = v_i + v_j, x \neq v$. By Self-covariance and positive homogeneity of φ it holds the equality $\varphi(\alpha v + (1-\alpha)x) = x$ for all $\alpha \geq 0$. If there is $\alpha \geq 0$ such that the equal share vector $(\frac{v_i+v_j}{2}, \frac{v_i+v_j}{2}) = \alpha v + (1-\alpha)x$, then, by anonymity and self-covariance of φ, we obtain $\varphi(v) = (\frac{v_i+v_j}{2}, \frac{v_i+v_j}{2})$ for all v.

In the opposite case we would have the relation

$$v_i \geq v_j \iff \varphi_i(v) \geq \varphi_j(v). \tag{6}$$

By using self covariance of φ, it is not difficult to check that this relation holds only if $v = \varphi(v)$.

□

Let $\varphi \in \Psi$ be an arbitrary solution. We call a *φ^v-path* a locus of solution vectors $\varphi(v, T)$ for game with a fixed $v = (v_i, v_j)$ and variable $T \leq v_i + v_j$.

A multi-valued function $\varphi^{-1} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is the *converse solution function* of φ if

$$v \in \varphi^{-1}(x, T) \iff x = \varphi(v, T).$$

Since for efficient solutions a solution vector $\varphi(v, T) = (x_i, x_j)$ satisfies the equality $x_i + x_j = T$, the converse solution function in fact depends on two variables $x = (x_i, x_j)$.

Lemma 3.1 *If $\varphi \in \Psi$, then the set $\varphi^{-1}(x)$ is a convex cone with the vertex x and sides in positive directions of the sums of coordinates.*

Proof . Let $v \in \varphi^{-1}(x)$. It means that $\varphi(v, T) = x$ and, by self-covariance and positive homogeneity of φ, the ray going out from x and passing through v is contained in the set $\varphi^{-1}(x)$. If this ray coincide with the set $\varphi^{-1}(x)$, then it is a degenerate cone.

Assume that there are 2 points $v, w \in \varphi^{-1}(x)$ not lying on same line with x. Put a point u inside the angle generated by the ray from x and passing through v and w. Suppose that $\varphi(u, T) = z \neq x$. Then $z \in \varphi^{-1}(x)$, and the ray from z through u intersects one side of the angle in a point y. By self covariance of φ it should hold two inconsistent equalities $\varphi(y,T)=z$ and $\varphi(y,T)=x$, that is impossible by single-valuedness of φ.

Therefore, the set $\varphi^{-1}(x)$ is a convex cone with the vertex x. $\square$

Denote by L_1 the half-line

$$L_1=\{x\in\mathbb{R}^2, \,|\, x_i+x_j=1, x_i\geq x_j\}. \tag{7}$$

Give some more properties of solutions from the family Ψ.

Lemma 3.2 *For every $\varphi\in\Psi$ the set $\{\varphi(v,1)\,|v_i+v_j>1, v_i\geq v_j\}$ either coincides with the half-line L_1, or equals an interval $[(1/2,1/2),(A,1-A)]$ for some $A\geq 1/2$.*

Proof .First, we prove that the set of points $x_i\geq 1/2$, for which there is a point v, $v_i+v_j>1$ such that $\varphi(v,1)=(x_i,1-x_i)$, is either an interval $[(1/2,1/2),(A,1-A)]$, or the half-line $\bigcup_{x_i=1/2}^{\infty}(x_i,1-x_i)$.

Suppose that for some $y_i\geq 1/2$ there is a neighborhood $U(y_i)=(a_i,b_i)$ of a point y_i (it may consists of the unique point y_i) such that $(\varphi(v,1))_i\cap U(y_i)=\emptyset$ for all $v: v_i+v_j\geq 1, v_i\geq v_j$, and let this neighborhood be maximal, i.e., either $\varphi^{-1}(\bar{a})\neq\emptyset$, or $\varphi^{-1}(\bar{b})\neq\emptyset$, where $\bar{a}=(a_i,1-a_i), \bar{b}=(b_i,1-b_i)$.

Consider the following cases:

1. $\varphi^{-1}(\bar{a}),\varphi^{-1}(\bar{b})\neq\emptyset$. Without loss of generality we may suppose that $a_i>b_i$. The sets $\varphi^{-1}(\bar{a}),\varphi^{-1}(\bar{b})$ by Lemma 3.1 are the angles with the vertices $\bar{a},\bar{b}$ and slopes of their sides $\alpha_1,\alpha_2,\ \alpha_1\geq\alpha_2,\ \beta_1,\beta_2,\ \beta_1\geq\beta_2$ respectively. Since these rays do not intersect, $\alpha_2>\beta_1$.

Let v_1 and v_2 be arbitrary points on the rays from a with the slope α_2, and from b with the slope β_1 respectively, and let $w=\gamma v_1+(1-\gamma)v_2$ for some $\gamma\in(0,1)$. Then by Lemma 3.1 $\varphi(w,1)=\bar{a}$ or $\bar{b}$. Let $\varphi(w,1)=\bar{a}$. Then the ray from the point a through the point w should be contained in the set $\varphi^{-1}(\bar{a})$ that is impossible. Analogously the case $\varphi(w,1)=\bar{b}$ is impossible as well.

2. $U(y_i)=y_i$. Then for every sequence $y_{ni}\to y_i$ there are v_n such that $\varphi(v_n,1)=(y_{ni},1-y_{ni})$ for sufficiently large v_n. Let $y_n\to y, y_{in}<y_i$, $z_n\to y, z_{in}>y_i$, and $\varphi(w_n)=y_n$. Denote $[\alpha_n^1,\alpha_n^2]$ $[\beta_n^1,\beta_n^1]$ the slopes of the boundary rays of the sets $\varphi^{-1}(y_n)$ and $\varphi^{-1}(z_n)$, respectively. Then $\alpha_n^2\geq\beta_n^1$ for all n . Without loss of generality we can suppose that there exist the limits $\lim_{n\to\infty}\alpha_n^2=\alpha$, $\lim_{n\to\infty}\beta_n^1=\beta$. They satisfy the equality $\alpha\leq\beta$.

If $\alpha=\beta$, then take an arbitrary point v on the ray from y in the direction $\alpha=\beta$. Then $\varphi(v,1)=y$, since in the contrary for sufficiently large n this ray would intersect the rays from y_n in the directions α_n^2, or the rays from z_n in the directions β_n^1, that is impossible.

If $\alpha<\beta$, then take a point v on the ray from y in a direction $\gamma\in(\alpha,\beta)$. Similar to the previous case we obtain that $\varphi(v,1)=y$.

3. The case when one of the sets $\varphi^{-1}(\bar{a})$ or $\varphi^{-1}(\bar{b})$ is empty and another is not, is proved by combinations of proofs for cases 1 and 2.

Thus, we have proved that on the half-line L_1 the range of the function φ^{-1} is either an interval $[(1/2, 1/2)(A, 1-A)]$ or the whole half-line L_1..

Therefore,

$$\varphi^{-1}(x) = \emptyset \implies \varphi^{-1}(y) = \emptyset \text{ for all } y \in L_1, y_i > x_i.$$

Corollary 3.1 *In the conditions of Lemma 3.2 for every $y \in L_1$ from $\varphi^{-1}(y) \neq \emptyset$ it follows that $\varphi^{-1}(z) \neq \emptyset$ for every $z \in L_1$ satisfying $z_i < y_i$.*

Corollary 3.1 shows that the set $\{x | x_i + x_j = 1, x_i \geq x_j, \varphi^{-1}(x) \neq \emptyset\}$ either is an interval $[1/2, a]$, or the half-line $[1/2, \infty)$.

Corollary 3.2 *If the set $\{\varphi(v, 1) \,|v_i + v_j > 1\}$ is bounded, equal $[(-A, A+1), (A, 1-A)]$ for some $A \geq 1/2$, then $\varphi(v, 1) = (A, 1-A)$ for all $\{v|v_i + v_j \geq 1,\, v_j < kv_i + 1 - A(1+k)\}$, where $k \geq -1$ equals the minimal slope of rays of the cone $\varphi^{-1}(A, 1-A)$.*

Proof . The ray $x_j = kx_i + 1 - A(k+1)$ for $x_i \geq A$ belongs to the cone $\varphi^{-1}(A, 1-A)$. Let v be a vector satisfying $v_j < kv_i + 1 - A(1+k)$. Denote $\varphi(v, 1) = y$. Then, by the condition of the Lemma, $y_i \leq A$. Assume that $y_i < A$. By self-covariance of φ the following inclusion holds: $[y, v] \subset \varphi^{-1}(y)$. Let z be the intersection point of the interval $[y, v]$ with the ray $x_j = kx_i + 1 - A(k+1)$. Then, again by self-covariance of φ, it should be $\varphi(z, 1) = y, \varphi(z, 1) = (A, 1-A)$ that is impossible. Therefore, $\varphi(v, 1) = (A, 1-A)$. $\square$

The next Lemma shows continuity of the function φ^{-1}, $\varphi \in \Psi$ in x in domains $x_i + x_j = T$ for arbitrary T. Give some notation.

Let x_n be a sequence of points with a constant sum of coordinates $x_{in} + x_{jn} = T$ satisfying $x_{in} > x_{jn}$ and converging to x. Without loss of generality suppose that the sequence x_n is monotone, i.e., the sequence x_{in} is either increasing, or decre.asing.

Let x_{in} be increasing and suppose that the sets $\varphi^{-1}(x_n)$ are one-dimensional, i.e.,rays. Denote the slopes of the rays $\varphi^{-1}(x_n)$ by k_n. Then $k_n \in [-1, 1]$.

Since the rays $\varphi^{-1}(x_n)$ do not intersect (Lemma 3.1), the sequence k_n is decreasing. Therefore, there is a limit $\lim_{n\to\infty} k_n = k$.

Lemma 3.3 *Let a solution φ satisfy the conditions of Lemma 3.1, and for the sequence $x_n \to x$ defined above all the sets $\varphi^{-1}(x_n)$ contain the rays with slopes $k_n \in [-1, 1]$. If $\lim_{n\to\infty} k_n = k$, then the set $\varphi^{-1}(x)$ contains the ray with the slope k.*

Proof . It suffices to consider only the case $k > 0$.

1. $\varphi^{-1}(x)$ is a ray. Suppose that the direction of this ray does not equal k. Then, by monotonicity of k_n the slope of the ray $\varphi^{-1}(x)$ is greater than k. Consider the angle with the vertex x and sides with the directions k and that of $\varphi^{-1}(x)$. Then for every point v , belonging to this angle, it holds the equality $\varphi(v, T) = x$, since for every point $y : y_i + y_j = T, y \neq x$ the interval connecting v and x, intersects either the ray $\varphi^{-1}(x)$, or the rays $\varphi^{-1}(x_n)$ for sufficiently large n that is impossible by Lemma 3.1. Therefore, $\varphi^{-1}(x)$ cannot be a ray that contradicts the assumption.

2. $\varphi^{-1}(x)$ equals an angle with the vertex x and sides with the slopes $k_1, k_2, k_1 \neq k_2$. Suppose that $k \notin [k_1, k_2]$. Let v be an arbitrary point in the angle with the vertex x and sides with the slopes $k, k_i, i = 1, 2$ if $|k - k_i| < |k - k_j|, i, j = 1, 2$. Then $\varphi(v, T) = x$, since for another point y on the direct line $x_i + x_j = T$ the interval, connecting v and y, intersects either the set $\varphi^{-1}(x)$, or the sets $\varphi^{-1}(x_n)$ for sufficiently large n, that is impossible by Lemma 3.1. In particular, the point v may be placed on the boundary of the corresponding angle, implying that this boundary is contained in the set $\varphi^{-1}(x)$. □

Let us describe solutions for the class $\mathcal{G}_2^+$ from the family Ψ with the help of their converse.

First we define such converse solution functions for a fixed value T, let it be $T = 1$. Then a converse function φ^{-1} depends only on one variable x_i. By anonymity, as earlier, consider only $x_i \geq 1/2$.

For every number $A > 1/2$ define a family $\mathcal{H_A}$ of multi-valued functions $h : [1/2, \infty) \to [-A, A]$, possessing the following properties: $h \in \mathcal{H_A}$, if

1) $1 \in h(1/2)$;

2) $h(t) = [a_t, b_t]$, $a_t \leq b_t]$ for all $t \geq 1/2$.

3)the function h has the closed graphic: if $x_n \to x, y_n \in h(x_n), y_n \to y$, then $y \in h(x)$,

4) h is weakly decreasing:

$$\liminf_{x_n \to x^-} h(x_n) \geq \limsup_{x_m \to x^+} h(x_m).$$

Every function $h \in \mathcal{H_A}$ defines an efficient solution ψ^h for the class $\mathcal{G}_2^+$ by a converse solution function $(\psi^h)^{-1}(x)$ for $x = (x_i, x_j), x_i + x_j = 1, A \geq x_i \geq x_j$ as follows:

$$(\psi^h)^{-1}(x) = \begin{cases} \text{the ray from } x \text{ with the slope } h(x_i), & \text{if } |h(x_i)| = 1, \\ \text{the angle with the vertex } x \text{ whose slopes of sides are } a, b, & \text{if } h(x_i) = [a, b]. \end{cases} \tag{8}$$

The solution ψ^h itself is defined for every game $(v, 1) \in \mathcal{G}_2^+$ by the relation

$$x = \psi^h(v, 1) \iff v \in (\psi^h)^{-1}(x), \;\; x_i + x_j = 1. \tag{9}$$

Equality (8) shows that for all $v \in L_1$ the inequality $(\phi^h(v, 1))_i \leq A$ holds.

For arbitrary T

$$\psi^h(v_i, v_j, T) = T\psi^h\left(\left(\frac{v_i}{T}, \frac{v_j}{T}\right), 1\right). \tag{10}$$

To games (v, T) from $\mathcal{G}_2^+$ with $v_j > v_i$ we extend the solution ψ^h by the anonymity property:

$$\psi_i^h(v_i, v_j, T) = \psi_j^h(v_j, v_i, T). \tag{11}$$

Now, given a number A and a function $h \in \mathcal{H}_\mathcal{A}$, the solution ψ^h has been defined for the whole class $\mathcal{G}_2^+$.

Theorem 3.1 *A solution φ for the class $\mathcal{G}_2^+$ belongs to the family Ψ if and only if there is a number $A > 1/2$ and a function $h \in \mathcal{H}_\mathcal{A}$ such that $\varphi = \psi^h$.*

Proof . Let $h \in \mathcal{H}_A$ be an arbitrary function. Let us show that $\psi^h \in \Psi$.

Properties 3) and 4) of h imply that for every vector $v = (v_i, v_j)$, $v_j \leq v_i, v_i + v_j > 1$ there is a point x such that $v \in \psi^{-h}(x)$. Hence, the solution ψ^h is non-empty for games $(v, 1) \in \mathcal{G}_2^+$ and, by using extension (10), it is non-empty for all games from $\mathcal{G}_2^+$.

Single-valuedness of $\psi^h(t)$ for every t follows from nonintersectability of the values $(\psi^h)^{-1}$: $(\psi^h)^{-1}(x) \cap (\psi^h)^{-1}(y) = \emptyset$ for every $x \neq y, x_i + x_j = y_i = y_j$ that is provided by properties 2) and 4) of h and the definition of the converse solution function (8).

Anonymity of ψ^h follows from (11).

Property 4) of the functions h provides non-intersectability of the sets $\psi^{-h}(x), \psi^{-h}(y)$ for $x \neq y$ implying single-valuedness of ψ^h. At last, self-covariance follows from the definition of the converse solution functions whose values are convex cones.

Let now $\varphi \in \Psi$ be an arbitrary solution, and let $A = \max x$ such that there exists $v, v_i \geq v_j$ satisfying $\varphi(v, 1) = (x, 1 - x)$. We should prove that $\varphi = \psi^{h^\varphi}$ for some $h^\varphi \in \mathcal{H}_A$. For simplicity of notations without loss of generality suppose that $A = \infty$ and denote $\mathcal{H}_A = \mathcal{H}$.

Let us construct the function $h^\varphi \in \mathcal{H}$, implying the equality $\varphi = \psi^{h^\varphi}$.

Evidently, $\varphi((1/2, 1/2), 1) = (1/2, 1/2)$, so, we put $(1/2, 1/2) \in h^\varphi(1/2)$.

For arbitrary $t \geq 1/2$ we put

$$h^\varphi(t) = [\alpha_1^t, \alpha_2^t], \tag{12}$$

where $\varphi^{-1}(t, 1 - t)$ equals the angle with the vertex $(t, 1 - t)$ and the sides having slopes α_1^t, α_2^t. Hence, the function h^φ satisfies property 2).

Since the sets $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$, for $x_i + x_j = y_i + y_j = 1$ are not empty by the assumption $A = \infty$, closed, and do not intersect, we obtain that the function $h^\varphi(t)$ (12) is not increasing in t, i.e., satisfies property 4). By Lemma 3.3 the function h^φ satisfies property 3).

Therefore, $h^\varphi \in \mathcal{H}$, and $\psi^{h^\varphi} \in \Psi$. It remains to show that $\varphi = \psi^{h^\varphi}$. By equality (12) for every $t \geq 1/2$ the set $h^\varphi(t) = [k_1^t, k_2^t]$ defines uniquely the set $\varphi^{-1}(t, 1-t)$ such that k_1, k_2 are the slopes of the sides of the angle $\varphi^{-1}(t, 1-t)$. Hence, by (8) $\varphi^{-1} = (\psi^{h^{\varphi}})^{-1}$, from what by (9) it follows that $\varphi = \psi^{h^{\varphi}}$.

This proof was given for the subclass of $\mathcal{G}_2^+$ of games (v, T), satisfying $T = 1$, $v_i \geq v_j$. The extension of the result to the whole class $\mathcal{G}_2^+$ is hold with the help of anonymity (11) and positive homogeneity (10) of solutions from the family Ψ. $\square$

Examples .

1.If $h(t) = 1$ for all $t \in [1/2, \infty)$, then the solution ψ^h is standard;

2. $h((\frac{1}{2}) = [0, 1]$, $h(t) \equiv 0$ for all $t > \frac{1}{2}$, then the solution ψ^h is the extension of the constrained egalitarian solution defined for superadditive games [Dutta 1990] to subadditive two-person games [Arin, Iñarra 2002];

3. If $h(\frac{1}{2}) = [-1, 1]$ and $h(t) = -1$ for all $t > \frac{1}{2}$, , then the solution ψ^h is egalitarian: $\psi^h(v, 1) = (1/2, 1/2)$ for all $v, v_i + v_j \geq 1$.

4. If $h(t) = \frac{1-t}{t}$, , then the solution ψ^h is proportional.

All solutions 1-4 belong to the class Φ. An example of a function $h \in \mathcal{H}$, defining a solution $\psi^h \in \Psi$, but not belonging to the family Φ, is defined by the function $h(t) = \text{tg}\, \frac{1-t}{1+t}$. In fact, for every $t \geq 1/2$ the set $(\psi^h)^{-1}(t, 1-t)$ is a ray, and the function $h(t)$ has no intervals with constant values. Therefore, the solution ψ^h is not defined by a partition of the half space $\{(x_i, x_j) \mid x_i + x_j \geq 0\}$ on the cones inside which the function h is constant, and on whose sides the solution is proportional.

4 An axiomatic characterization of the family Φ

The family Ψ, characterized in Theorem 4 is very large and contains the family Φ. In this section we characterize the family Φ by adding one more axiom. Recall that the restriction of the family Φ to the class of two-person cost-sharing problems coincides with the family $\mathcal{H}_2^a$. The last one is characterized by Anonymity, upper and lower composition, and positive homogeneity (Moulin 2000). The Upper and Lower composition properties were not applied to the characterization of the family $\Psi \supset \Phi$ (Theorem). Therefore, it could be natural to choose one of these axioms. However, the lower composition property can not be applied to a characterization of methods for profit-sharing problems because given two profit-sharing problems $(c, t), (c, T-t), t < T$ the problem $(c, T-t)$, participating in the definition of this property, may turn out to be a cost sharing problem. Since the ultimate goal of the paper is an extension and a characterization of the family Φ to the whole class of two-person games, this axiom does not match to fulfilling the goal.

On the contrary, Upper composition, or the Path Independence property matches completely for both classes of sharing problems.

It can be applied for single-valued solutions of two-person both sub- and superadditive games without any modification.

A solution φ for the class of all class $\mathcal{G}_2$ of two-person TU games is *path independent (PI)*, if for any games $(v,t),(v,T),t<T)$

$$\varphi(v,T)=\varphi(\varphi(v,t),T).$$

In the sequel we use the following notation; k_1xk_2, axb denote the angles with the vertex x and the sides with the slopes k_1,k_2, $k_1<k_2$ or passing through the points a,b respectively.

Evidently, if a solution φ satisfies self-covariance and path independence, and for a fixed $x\in\mathbb{R}^2$ the set $\varphi^{-1}(x)$ is a ray, then the path φ^v is an interval on the ray $\varphi^{-1}(x)$.

To begin with, we will consider the subclass $\mathcal{G}_+^2\subset\mathcal{G}_2$ and solutions for it.

Lemma 4.1 *Let φ be a solution for the class $\mathcal{G}_+^2$, satisfying the axioms NE, EFF, SV. ANO, self-C0V, and PI. Let k_10k_2 be an a maximal angle such that for every interior point x the solution φ on the ray $0x$ is not proportional. Then for any point z inside the angle the set $\varphi^{-1}(z)$ is a ray having slope k_1 or k_2.*

Proof . By the condition of the Lemma the path $\varphi^z(v,t)$ for every interior point z and $t\in[0,v_i+v_j]$intersects one of the sides of the angle in a non-zero point. Let it intersect the side $x_j=k_2x_i$ in a point y.

Let us show that the the interval $[y,z]$ has the slope k_1. Denote the slope of the interval $[y,z]$ by k and assume that $k\neq k_1,k_2$. Consider the following cases:

1. $k<k_1$. Then the ray with this slope from the the set $\varphi^{-1}(z)$ intersects the ray $x_j=k_1x_i$ that is impossible.
2. $k_1<k<k_2$. Then by positive homogeneity of φ the solution is proportional on the ray inside the angle with the slope k that is impossible by the assumption.
3. $k>k_2$. This case is analogous to the case 1.

Since the point z was chosen arbitrarily in the interior of the angle, then their converse $\varphi^{-1}(z)$ consists of the unique ray with the same slope k_1 or k_2.

For every point $w:w_j=k_2w_i$ the value of the converse function $\varphi^{-1}(w)$ is the angle k_1ak_2. □

Lemma 4.2 *Let φ be a solution for the class $\mathcal{G}_+^2$, satisfying the conditions of Lemma 4.1.*

If in a point $a=(a_i,a_j),a_i+a_j=1,a_i>1$ the set $\varphi^{-1}(a)$ is an angle k_1ak_2, $k_1<k_2$, then
1) the ray $0a$ passes through this angle and
2) on the rays $0a$ and those from 0 with the slopes k_1,k_2 the solution φ is proportional.

Proof . Denote by k_a the slope of the ray $0a$ to the line $x_i+x_j=1$.

Consider the following cases:

1. $k_a<k_1<k_2$ (Fig.2).

By positive homogeneity of φ $\varphi^{-1}(\alpha a) = \alpha\varphi^{-1}(a)$ for every $\alpha > 0$. Let $v \in \varphi^{-1}(\alpha a) \cap (\varphi^{-1}(a)$ for some $\alpha < 1$. Then $\varphi(v, 1) = a, \varphi(v, \alpha) = \alpha a$, By path independence of φ $\alpha a = \varphi(v, \alpha) = \varphi(\varphi(v, 1), \alpha)$. Since $\varphi(v, 1) = a$, from the last equality we obtain $\varphi(a, \alpha a) = \alpha$, that means proportionality of the solution φ on the ray $0a$. Therefore, the part of the ray $0a$ going out from a belongs to the set $\varphi^{-1}(a)$, that is possible only if $k_a = k_1$. This equality implies that on the ray $0a$, coinciding with the ray having the slope k_1, the solution φ is proportional.

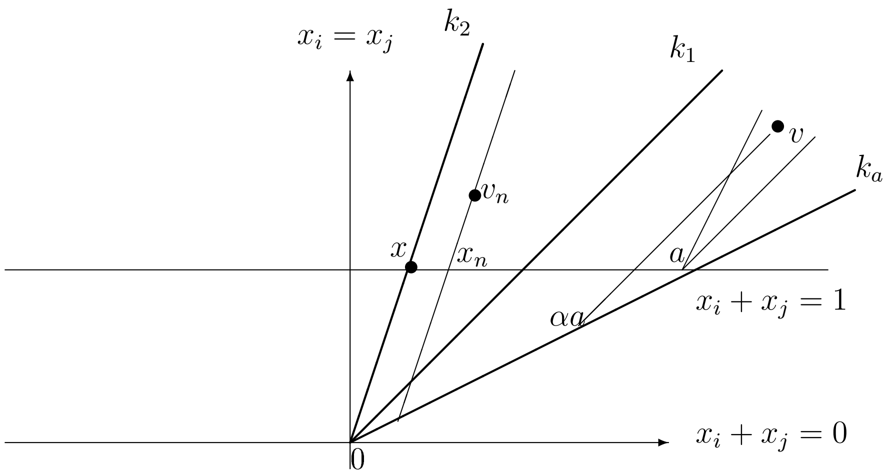

Fig.2

It remains to show that on the ray from 0 with the slope k_2 the solution φ is proportional. Let w be an arbitrary point satisfying $w_i + w_j = 1$ and lying inside the angle $k_1 0 k_2$ (Fig.3).

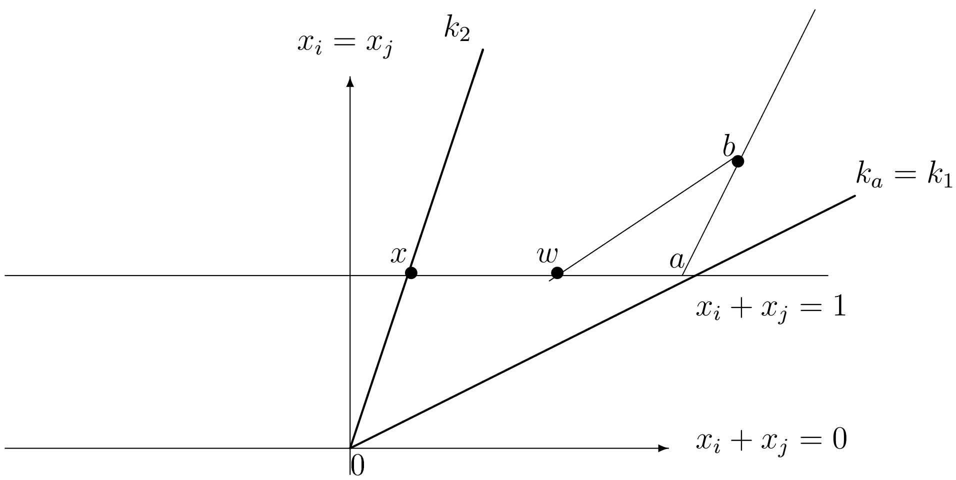

Fig.3

The slope of any ray from the set $\varphi^{-1}(w)$ is less than or equal k_2. If it does not equal k_2, then its continuation will intersect the ray from a with the slope k_2. Then in the intersection point b we should have $\varphi(b, 1) = w$, and $\varphi(b, 1) = a$ that is impossible. Hence, for every point w inside the angle $k_1 0 k_2$ $\varphi^{-1}(w)$ is the ray from w with the slope k_2.

Let a sequence $x_n \to x$, $x_{ni} > k_2 x_{nj}$ be a sequence of points on the line $x_i + x_j = 1$. Then the sets $\varphi^{-1}(x_n)$, are the rays with the direction k_2 By Lemma 3.3 the ray from x with the slope k_2 belongs to the set $\varphi^{-1}(x)$, providing proportionality of the solution φ on the ray $0k_2$.

2. $k_1 < k_2 \leq k_a$. Similarly to the proof of case 1) we obtain that this assumption leads to proportionality of φ on the ray $0a$ and $k_2 = k_a$.

3. $k_1 < k_a < k_2$ (Fig.3) . First, we show proportionality of φ on the ray $0a$. Take an arbitrary point w in the angle $k_1 a k_2$. Then $\varphi(w, 1) = a$. Positive homogeneity of φ implies $\varphi(w, \alpha) = \alpha a$ for every $0 < \alpha < 1$, and by path independence of φ the last two equalities imply $\varphi(a, \alpha) = \alpha a$, that means proportionality of φ on the ray $0a$.

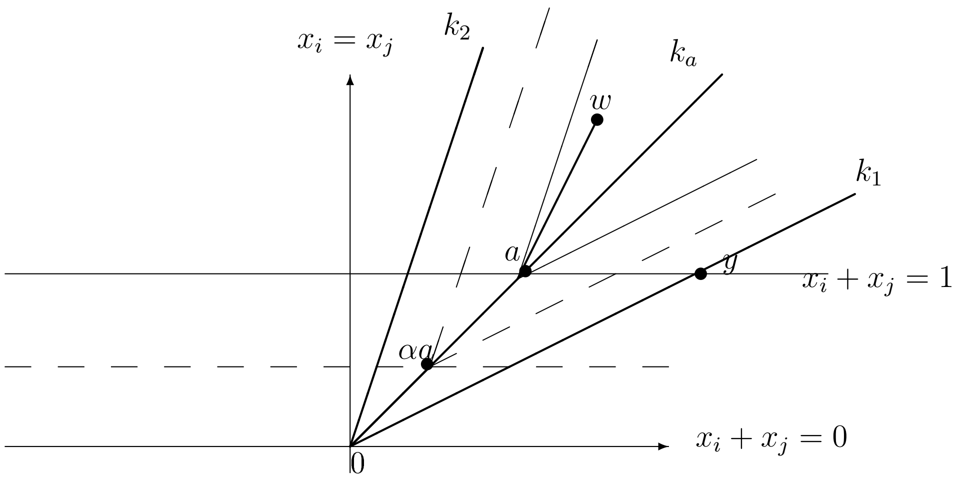

Fig.4

Now let us show proportionality of the solution φ on the ray from 0 with the direction k_1. Consider the point $y = \left(\frac{1}{1+k_1}, \frac{k_1}{1+k_1}\right)$ on the ray (Fig.4). By positive homogeneity of the solution φ and by Lemma 3.3 we have $\varphi^{-1}(y) \neq \emptyset$.

Let $v \in \varphi^{-1}(y)$ be an arbitrary point. Then $\varphi(v, 1) = y$. Consider the following cases:

i) The point v is placed on the ray from 0 with the slope k_1. Then the claim has been proved.

ii) The point v is placed out of the angle $k_1 0 k_a$ (Fig 4).

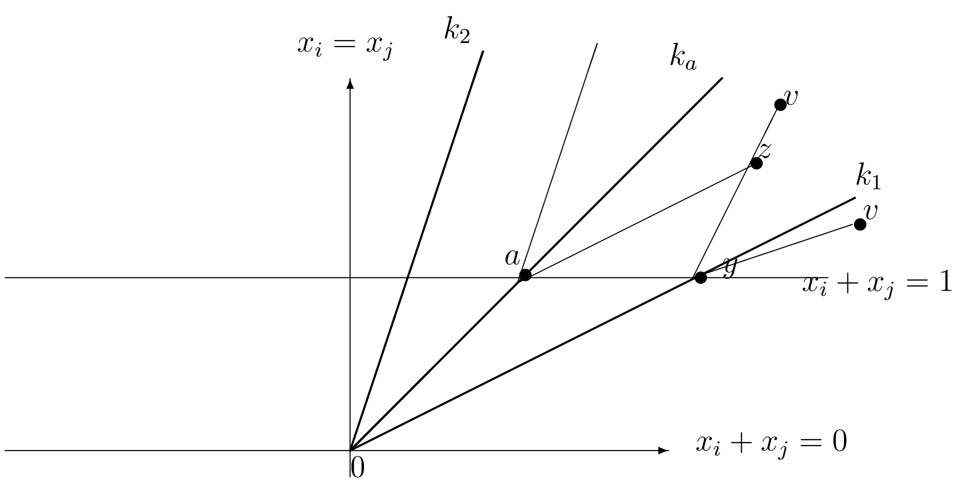

Fig.5

Then by Lemma 3.1 the angle with the vertex y whose one side has the slope k_1 and the other lies on the ray $0v$, is contained in the set $\varphi^{-1}(y)$.

iii) The point v is placed inside the angle k_a0k_1. (Fig.5) Then the slope of the interval $[v, y]$ should be between k_1 and k_a. However, if it greater than k_1, then the ray from y through the point v will intersect the ray from a in the direction k_1. In the intersection point z we should have $\varphi(z, 1) = a, \varphi(z, 1) = y$ that is impossible.

Hence, from pp. ii) and iii) it follows that the ray from y with the direction k_1 is contained in the set $\varphi^{-1}(y)$, implying proportionality of the solution φ on the ray $0k_1$.

Similarly proportionality of φ on the ray $0k_2$ is proved. □

Lemma 4.3 *If in the conditions of Lemma 4.1 there exists an angle k_10k_2, $k_1 < k_2 \le 1$ such that for every interior point u of the angle the set $\varphi^{-1}(u)$ is a ray, lying inside the sector as well, and for every boundary point z of the angle the set $\varphi^{-1}(z)$ does not intersect with the interior of the sector, then the solution φ is proportional in all the sector.*

Proof . If the set of points $X = \{x : x_i + x_j = 1, x_i \ge x_j\}$, placed inside the angle k_10k_2 and such that on the rays $0x$ the solution φ is proportional, is dense in the interval $[\bar{a}, \bar{b}]$, where $\bar{a} = (a_i, 1 - a_i), \bar{b} = (b_i, 1 - b_i)$, $k_1 = \frac{1-b_i}{b_i}, k_2 = \frac{1-a_i}{a_i}$, then then by Lemma 3.3 it is proportional in all the angle k_10k_2.

Consider the interval $[\bar{a}, \bar{b}]$ of the points of the sector, satisfying $a_i + a_j = b_i + b_j = 1$. Then the rays from $\bar{a}, \bar{b}$ in the directions $0\bar{a}, 0\bar{b}$ respectively, belong to the sets $\varphi^{-1}(\bar{a}), \varphi^{-1}(\bar{b})$. The set of the slopes of the rays $\varphi^{-1}(u)$ for $u \in (\bar{a}, \bar{b})$ should fill the interval (k_a, k_b), where k_a, k_b are the slopes of the rays $0\bar{a}, 0\bar{b}$ respectively (Lemma 3.1). Therefore, there exists a point $\bar{c} \in (\bar{a}, \bar{b})$ such that the ray $\varphi^{-1}(\bar{c})$ lies on the ray $0\bar{c}$, and on the ray $0\bar{c}$ the solution φ is proportional.

Now the angle $\bar{a}0\bar{b}$ is partitioned into two angles $\bar{a}0\bar{c}$ and $\bar{c}0\bar{b}$ satisfying the conditions of the Lemma. Such a procedure of dividing every sector into two ones can be continued, and in the limit, by Lemma 3.3 we obtain that for every $u \in (\bar{a}, \bar{b})$ the ray $\varphi^{-1}(u)$ lies on the ray $0u$, that means that on this ray the solution φ is proportional. □

Theorem 4.1 *A solution φ for the class $\mathcal{G}_2^+$ two-person games satisfies axioms EFF, ANO, PH,PI and Self-Cov if and only if $\varphi \in \Phi$.*

Proof . The *'if part'*. Let $\varphi \in \Phi$ be an arbitrary solution for the class $\mathcal{G}_2^+$. The first three properties follow from the definition of the family Φ given in Section 2. Self-covariance of these solutions has been checked in Proposition 2.1. Path independence is one of the properties of the family $\mathcal{H}_2^*$ two-person cost sharing methods (Moulin 2000). It is evidently saved when extending their anonymous subfamily $\mathcal{H}_+^a$ to the class $\mathcal{G}_2^+$ of subadditive two-person games with nonnegative characteristic function values of two-person coalitions.

The *'only if'* part. Let φ be an arbitrary solution for the class $\mathcal{G}_2^+$, verifying all the axioms stated in the Theorem. By Theorem 3.1 $\varphi \in \Psi$. To prove that $\varphi \in \Phi$ it suffices to show this fact for the subclass $\mathcal{G}_2^1 \subset \mathcal{G}_2^+$ of games such that $(\{i,j\}, v) \in \mathcal{G}_2^1$ if and only if $v(\{i,j\}) = 1, v_i + v_j \geq 1$, and $v_i \geq v_j$. Then by applying anonymity and positive homogeneity we prove the result for the whole class $\mathcal{G}_2^+$.

We begin to prove this part of the Theorem with finding the values of the converse solution function $\varphi^{-1}(x)$ for all points of the half-line L_1, and show that every such a function generates a partition of the half-line into the cones defining the solution φ for the class $\mathcal{G}_2^+$ by formulas being rewriting of (2),(3) for solutions for the class of subadditive two-person games $\mathcal{G}_2^+$.

First, denote by $L_2 \subset L_1$ the maximal set such that that for every $\bar{a} = (a, 1-a) \in L_2$ the set $\varphi^{-1}(\bar{a}) \neq \emptyset$. This set is non-empty, since $(1/2, 1/2) \in L_2$ by anonymity. By Lemma 3.2 $L_2 = [(1/2, 1/2), (a, 1-a)]$ for some $a \geq 1/2$.

Denote $L^p \subset L_2$ the set such that for every point $x \in L^p$ the solution φ on the ray $0x$ is proportional. Such a set is not empty, since $(1/2, 1/2) \in L_p$. By Lemmas 3.3 and 4.1 the set L^p consists of isolated points and of intervals.

Let us find the values of the converse solution function $\varphi^{-1}(\bar{x})$ for $\bar{x} \in (\bar{a}, \bar{b})$, where $(\bar{a}, \bar{b}) \in L_2 \setminus L^p$, is a maximal open interval in which $\varphi^1(\bar{x})$ is single-valued.

By Lemma 4.1 the rays $\varphi^{-1}(\bar{x})$ for $\bar{x} \in (\bar{a}, \bar{b})$, $\bar{x} = (x, 1-x), \bar{b} = (b, 1-b)$, have the same slope k_{ab} coinciding with the minimal slope of the rays from $\varphi^{-1}(\bar{a})$, or with the maximal slope of the rays from $\varphi^{-1}(\bar{b})$. From the same Lemma it follows that the sets $\varphi^{-1}(\bar{x})$ are not degenerate cones only if on the ray $0\bar{x}$ the solution φ is proportional implying that the solution φ ismproprtional on the rays $0\bar{a}, 0\bar{b}$.

Evidently, if on the ray $0\bar{x}$ the solution φ is proportional, then for $\bar{x} \in L_2$ the ray $\varphi^{-1}(\bar{x})$ has the slope $\frac{1-x_i}{x_i}$.

Thus, we have obtained that for solutions φ satisfying all the properties given in the Theorem, the function h^φ, defined in the previous section for solutions from the family Ψ, satisfies two more properties:

5) If $|h^\varphi(t)| > 1$ for some $t > 1/2$, then $\frac{1-t}{t} \in h^\varphi(t)$,
6) If for all $t \in (a_i, b_i)$ $|h^\varphi(t)| = 1$, then either $h^\varphi(t)$ is constant , for all $t \in (a_i, b_i)$, or $h^\varphi(t) = \frac{1-t}{t}$.

Let a function $h : [1/2, \infty) \to [-1, 1]$ satisfy the 1)–6). Consider the solution ψ^h, defined in (8) on the domain $L_1 = \{(x_i, x_j) \,|\, x_i + x_j = 1, x_i \geq x_j\}$.

We prove that $\psi^h \in \Phi$. Define a solution ψ^h

Let $v = (v_j, v_j) \in L$, $T < v_i + v_j$. If v belongs to the interior of the angle defined by the rays

$0\bar{a}, 0\bar{b})$. Then $\psi^h(v,1) = (t, 1-t)$, where either

$$t = \frac{v_i}{v_i + v_j}, \text{ if } h(t) = \frac{1-t}{t} \text{ for all } t \in (a,b), \tag{13}$$

or

$$t = \begin{cases} \dfrac{1}{1+k_a}, & \text{if } 1 \leq \dfrac{(1+k_a)(v_j - k_b v_i)}{k_a - k_b} \\ \dfrac{1-(v_j - k_b v_i)}{1+k_b}, & \text{otherwise.} \end{cases} \tag{14}$$

where k_a, k_b are the slopes of the rays $0\bar{a}, 0\bar{b}$, or else

$$t = \begin{cases} \dfrac{1}{1+k_b}, & \text{if } 1 \leq \dfrac{(1+k_b)(v_j - k_a v_i)}{k_b - k_a} \\ \dfrac{1-(v_j - k_a v_i)}{1+k_a}, & \text{otherwise.} \end{cases} \tag{15}$$

In the first case the solution ψ^h is proportional, in the last two cases it is defined by the paths $(\psi^h)^v$ when T is varied from $v_i + v_j$ to 1. These paths are piece-wise lines consisting of two intervals. The choice of the two possibilities (14) and (15) for the solution $\psi^h(v,1)$ in every interval of the partition of the half-line L_1 in which the solution ψ^h is not proportional, can be done uniquely, if this partition is ordered. Then we obtain definition ψ^h by (14), if the interval $(\bar{a}, \bar{b})$ is indexed as $(\bar{a}^1, \bar{b}^2))$, and definition ψ^h by (15), if the ordered interval is $(\bar{a}^2, \bar{b}^1)$.

Then, definitions (13)–(15) added by the set L^p and by an ordering of the partition of the half-line L_1 defined by the function h, uniquely determine the solution $\psi^h \in \Psi$, and they coincide with those for solutions from the family Φ given in section 3 implying $\psi^h \in \Phi$.

In particular, we can put $h = h^\varphi$, so, $\psi^{h^\varphi} \in \Phi$ from what it follows that $\varphi = \psi^{h^\varphi}$ for every $\varphi \in \Phi$.

Hence, we have obtained that a solution φ satisfying the conditions of the Theorem, belongs to the family Φ for subadditive games $(v,T) \in \mathcal{G}_2^+$ with $T = 1$ and $v_i \geq v_j$. By the properties PH and ANO this result holds for all games from $\mathcal{G}_2^+$ as follows:

Let $(v,T) \in \mathcal{G}_2^+$ be an arbitrary game. If $v_i \geq v_j$, then for arbitrary $T > 0$

$$\varphi(v,T) = T\varphi\left((\frac{v_i}{T}, \frac{v_j}{T}), 1\right). \tag{16}$$

If $v_j > v_i$, then

$$\varphi(v,T) = T\varphi\left((\frac{v_j}{T}, \frac{v_i}{T}), 1\right). \tag{17}$$

Note that the games in the right-hand part of equalities (16), (17), satisfy the subclass of $\mathcal{G}_2^+$ for which the proof of the Theorem has been already obtained.

Evidently, the games in left-hand parts of equalities (16), (17) belong to the family Φ as well.

$\square$

5 An extension of solutions from the family Φ to the class of all two-person games

In this section we extend the family Φ to the whole class of subadditive two-person games such that the extensions save all the properties given in Theorem 4.1.

Theorem 5.1 *Let $\bar{\varphi}$ be an extension of a solution $\varphi \in \Phi$ to games (v, T) where v, T satisfy the inequalities $v_i \geq v_j, v_i + v_j > T, T < 0$ the extension verifies axioms NE, SV, ANO, Self-COV, and PI. Denote $t_0 = \lim_{t\to\infty} h^{\varphi}(t)$. Then the extension $\bar{\varphi}$ is defined by the following equalities:*
If $t_0 > -1$, then

$$\varphi(v,T) = \begin{cases} \bar{\varphi}(v,T) = \left(\dfrac{T}{2}, \dfrac{T}{2}\right), & \text{if } v_i = v_j \text{ or} \\ & T \leq v_j - \dfrac{1-t_0}{t_0} v_i), \\ t_0 \left(T - (v_j - \dfrac{1-t_0}{t_0} v_i)\right) & \text{otherwise .} \end{cases} \tag{18}$$

If $t_0 = -1$, , then $\varphi(v, T) = \left(\frac{T}{2}, \frac{T}{2}\right)$.

Proof . Equalities (18) (and its symmetric part for $vj > v_i$) determine a nonempty single-valued and anonymous solution $\bar{\varphi}$ for subadditive games (v, T) with $T \leq 0$. It is not difficult to check that the solution $\bar{\varphi}$ satisfies the properties Self-Cov and PI. Path independence of solution $\bar{\varphi}$ provided by equalities (18) shows that it is, in fact, an extension of the solution φ to negative values T.

Assume that there is an extension $\bar{\varphi}$ to the whole class of subadditive two-person games, satisfying axioms NE, SV, ANO, Self-COV, and PI. sider two cases.

1. There exists a point $(t_0, 1 - t_0), t_0 > 1/2$ such that $h^{\varphi}(t) = \frac{t_0}{1-t_0}$ for all $t \geq t_0$ (Fig. 6)

Let v be an arbitrary point satisfying $v_i > \frac{1-t_0}{t_0} v_j$, $v_i + v_i > 1, T < 0$. Denote $\bar{\varphi}(v, T) = y$. Let us show that the interval $[y, v]$ has the slope $\frac{t_0}{1-t_0}$. In fact, the less slope is impossible by the definition of t_0. If it is greater than $\frac{t_0}{1-t_0}$, then the ray from y with this slope would intersect the ray from zero with the slope $\frac{t_0}{1-t_0}$, and in the intersection point z by self covariance and the definition of t_0 we would receive two paths: zy by self-covariance, and $z0$ by the definition of the solution φ, that is impossible.

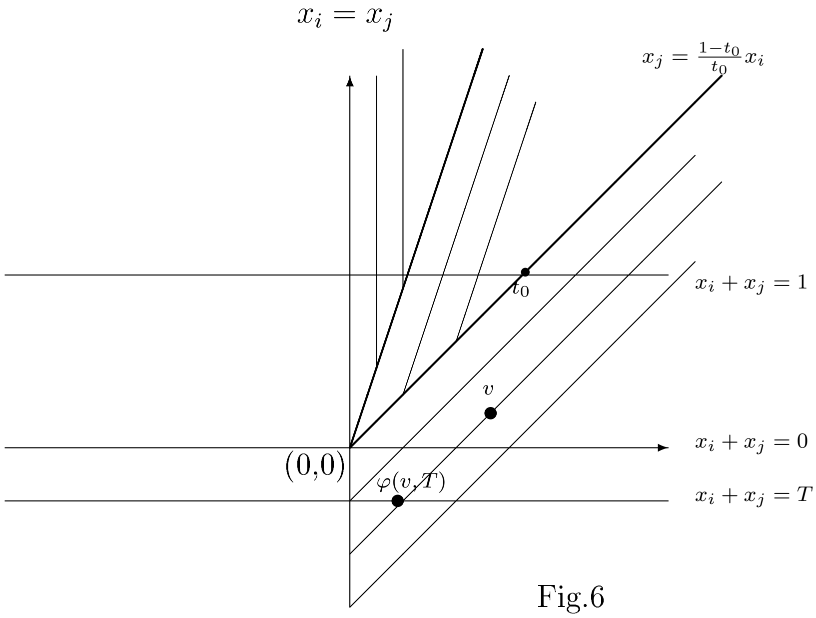

Fig.6

2. $\lim_{t\to\infty} h^{\varphi}(t) = 0$. In this case $\varphi(v,T) = (\frac{T}{2}, \frac{T}{2})$ for all $T \leq 0$ and $v : v_i + v_i > T, v_i \geq v_j$. Indeed, if $\bar{\varphi}(v,T) = y \neq (\frac{T}{2}, \frac{T}{2})$, then the ray from y passing through the point v would intersect the line $x_i + x_j = 0$ and the rays $\varphi^{-1}(t,-t)$ for sufficiently large t, and we would return to case 1.

Therefore, we have proved that the for every $\bar{y} : y_i \geq y_j, y_i + y_j < 0 y_i \neq y)j$ the set $\bar{\varphi}^{-1}(\bar{y})$ is the ray parallel to the axis x_i. If $y_i = y_j$, then the set $\bar{\varphi}^{-1}(\bar{y})$ is the angle whose sides are parallel to the coordinate axis.

□

6 Concluding remarks

In all the text new solutions were investigated only for subadditive two-person games. The natural question arises whether it is possible to define and to characterize the solutions satisfying the same axioms for superadditive two-person games as well.

It turns out that there is no problem to represent such a solution for this class of games. Let (v,T) be a sub-additive two-person game. Then the game $(-v,-T)$ is superadditive, and vise versa. We can extend a solution $\bar{\varphi}$ for the class of subadditive two-person games, considered in the previous section, to the class of superadditive two-person games as follows: let (v,T) be a superadditive game. Put

$$\bar{\varphi}(v,T) = -\bar{\varphi}(-v,-T) \tag{19}$$

It is clear that this extension verifies all the axioms stated in Theorem 5.1, though, possibly, self-covariance is not so evident. Let us show that this property. Let (v,T) be a superadditive game. Then

$$\varphi(v+A\varphi(v,T))=-\varphi(-v-A\varphi(-v,-T))=-(A+1)\varphi(-v,-T)=(A+1)\varphi(v,T). \qquad (20)$$

Coming back, to profit-sharing methods, we can note that for this class of problems the family Φ applied to the positive domain, is not so rich as the family $\mathcal{H}_2^a$. In fact, equality (20) shows that such methods are the reversed ones of family Φ for negative values T. They coincide with the class of cooperative game solutions described in [Yanovskaya, 2014] for superadditive games as the solutions satisfying the axioms ANO, Self-Cov, SI, and Weak Covariance that here is replaced by the Path Independence axiom.

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