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Alexander Chigodaev

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Recursive Method for Guaranteed Valuation of Options in Deterministic Game Theoretic Approach

Alexander Chigodaev[1]

Abstract

We adapt a deterministic game theoretic framework in discrete time to super-hedge pricing contingent claims (CCs). The key aspect of this framework is that the worst-case scenario dictates the super-hedging price which protects counter-parties in financial contracts from insolvencies. A general application algorithm for super-hedge pricing of European CC portfolios with piecewise linear payoffs, based on linear programming, is offered for practical usage. Examples of path-dependent European CCs and portfolios of vanilla European CCs are presented to highlight important features of this pricing framework.

JEL Classification: C61, C63, G11.

Key words: worst case scenario, super-hedge, guaranteed price, recursive Bellman equation, martingale measure, European option, Asian option, Lookback option, portfolio of contingent claims.

[1] National Research University (NRU) Higher School of Economics (HSE). Laboratory for Financial Engineering and Risk Management. Research Fellow; E-mail: achigodaev@hse.ru. The research leading to these results has received funding from the Centre for Advanced Studies at NRU HSE. The author would like to thank his colleagues at the laboratory for the helpful insight and assistance: Marat Kurbangaleev, Nikolay Andreev, Victor Lapshin and Sergey Smirnov.

1 Introduction

In complete markets, every contingent claim can be replicated by a combination of other available instruments. It can be said with a high degree of confidence - it is a good approximation to how markets operate, at least. Theoretically, it is possible to make models which maintain the property of completeness and hence lead to the so-called fair price; this type of models are used by the overwhelming majority in the financial industry to price financial instruments.

In incomplete markets, on the other hand, it is impossible to perfectly replicate contingent claims. Consequently, there is no unique risk-neutral measure and hence no unique price to contingent claims. The up-side of incomplete market models is that they yield so-called super-hedging prices which guarantee that contingent claims would be paid off with complete certainty. From the practical standpoint, this might be useful in illiquid and developing markets where contingent claim pricing is otherwise impossible. Alternatively, super-hedging methodology could be implemented by central-counter parties to calculate marginal requirements in derivatives and structured product deals.

The first work on the subject of super-hedging pricing was carried out by Karoui and Quenez [1995]. They proved the optional decomposition theorem in the case of diffusion dynamics of asset prices. The famous work by Kramkov [1996] extended the optional decomposition theorem in continuous time to the case of semi-martingales and suggested how it can be applied to calculate minimal super-hedging prices and strategies. Föllmer and Kabanov [1997] offered a proof of the same theorem under weaker conditions both in discrete and continuous time.

The work on super-hedge pricing had continued to involve and variations on the theme were suggested. Föllmer and Leukert [1999; 2000] proposed the notion of quantile hedging and efficient hedging where they considered maximizing the probability of a successful hedge given a constraint on the cost. Along the same lines is the work by Xu [2006] where a market participant is able to choose the level of risk exposure and set the riskiness of the hedge according to his preference. Thus, we can then view Kramkov's super-hedging prices as a special case when the participant rejects bearing any risk at all. Kolokoltsov [1998] proposed a semi-formal version of the guaranteed pricing approach for one and multi- risky asset contingent claims. He examined it in the case of convex payoff functions but, in our opinion, his treatment of the multi-asset contingent claims is only partially satisfactory because it is certainly more natural for the jointly allowed prices changes, and even more general, to be in the shape of ellipsoids and not rectangles. Furthermore, Rüschendorf [2001] studied the upper (super-hedge) and lower prices of contingent claims with convex payoff functions on measures with bounded support. For that class of functions, he showed that the upper price is given by the Cox-Ross-Rubinstein model (CRR); more on the CRR model below. None of the works mentioned in this paragraph suggested how to calculate super-hedge prices for more general payoffs.

The theoretical foundation for a practical method to calculate super-hedge prices had been established but due to absence of necessity it remained dormant. Nowadays, it has regained attention in the aftermath of the 2008 financial crisis and the ongoing financial instability world-wide. One may find examples in the already mentioned articles here and there, but they are typically supplements to the preceding rigorous proofs and as such obfuscated for non-experts. There are few works related to general methodologies of practical super-hedge price calculation in discrete time especially for more general non-convex payoff functions. Carassus et al. [2006] reported on a discrete time model where the underlying assets may evolve in time unrestrictedly. They summarized a handful of trivial calculations on common contingent claims; trivial in the sense that they correspond to the naive "buy and hold" super-hedge strategies. Our work takes it further and considers, presumably less expensive and more effective, dynamic super-hedging, i.e. where the super-hedging portfolio gets restructured on every time step. Consequently, Carassus and Vargiolu [2010] improved the results of Carassus et al. [2006] by imposing bounds on the changes of asset prices. From the basic principles differing ours, they derived key expressions identical to those in Section 3 of this article; but they did not provide any practical calculations. Braouezec and Grunspan [2015] studied bounds of contingent claims prices in a geometric approach based on determining convex hulls. Unfortunately, their work is impeded by numerical computation and practical calculations are absent.

The purpose of this work is to offer a super-hedge pricing framework for practical calculations based on the deterministic game theoretic approach developed by Smirnov [2016] and which, hereafter, we refer to as the guaranteed approach. It is to be contrasted with the traditional/probabilistic theory for fair and super-hedge pricing in discrete time as, for example, in the textbook by Föllmer and Schied [2004] and the aforementioned articles. For the inquisitive reader, we refer to Smirnov's original work for detailed and rigorous formulation of the framework.

From the onset, we adapt Smirnov's guaranteed approach to the case of one risky asset since we limit ourselves to the study of contingent claims of this type. The asset prices are given by the vector $\mathbf{X}_t = (X_t^0, X_t^1)$ where the zeroth component is the riskless asset. The quantities of each asset in a hedging portfolio form a strategy vector $\mathbf{H}_t = (H_t^0, H_t^1)$. They are decided upon before $\mathbf{X}_{t+1}$ gets revealed. Recall that the riskless asset X_t^0 evolves in time deterministically according to the risk-free interest rate. Therefore, we conveniently set the zeroth asset as the numeraire, i.e. $X_t^0 = 1$ and $\Delta X_t^0 = X_t^0 - X_{t-1}^0 = 0$ for all $1 \le t \le N$, and forget about its propagation in the remainder of the article. Consequently, we redenote the single risky asset $X_t^1 \equiv X_t$ to ease notation. Furthermore, we assume no transactional costs and no trade constraints such that the set of all admissible strategies $\mathbf{H}_t$ is $\mathbb{R}^n$.

The principle feature of Smirnov's framework is that price changes in the risky asset ΔX_t belong to an apriori specified non-empty compact set $K_t(\cdot) \equiv K_t(X_0, ..., X_{t-1})$. No other information about the dynamics of X_t needs to be specified. We consider a particularly simple model where the allowed price changes are independent of price history, i.e $K_t(\cdot) = K_t = X_{t-1}[a, b]$, $X_0 > 0$ with $a < 1 < b$ to ensure absence of arbitrage opportunities[2]. Note that it is instructive to exhibit the price change both as additive and multiplicative processes:

$$X_{t+1} = \phi_{t+1} X_t = X_t + \Delta X_{t+1}. \tag{1.1}$$

We refer to both random ΔX_{t+1} and $\phi_{t+1} \in [a, b]$ as the additive and multiplicative changes respectively; and they may be used interchangeably.

In guaranteed pricing, the task is to assign the smallest price $v_t(\cdot)$ to the contingent claim at hand such that, even in the worst case scenario of the market behaviour, there is sufficient funds to make the full payment on the contingent claim. We refer to it as the super-hedging price. The subscript variable t ($s = N - t$) is the time elapsed from the issuance (remaining time) of the contract.

Smirnov proposed to perform guaranteed pricing in the context of game theory with pure hedger strategies and mixed market strategies. It was rigorously proved that in absence of trade constraints and arbitrage opportunities a game equilibrium exists, i.e. $v_t(\cdot)$ remains unaltered irrespective of who moves first (either the hedger or the market) in the hedging process. In real life though, the hedger would never be able to act retroactively. Further, in equilibrium, for European contingent claims with one underlying asset, the super-hedging

[2] There would be obvious arbitrage opportunities if both a and b were on the same side from unity.

pricing expression is

$$v_t(\cdot) = \sup_{\mathbb{Q}\in P_t(\cdot)} \int d\mathbb{Q}(\Delta X_{t+1})\, v_{t+1}(\cdot, \Delta X_{t+1}) \tag{1.2}$$

where the class of measures representing mixed market strategies $P_t(\cdot)$ is martingale, concentrated in at most 2 points ($n+1$ points in the case of n risky assets) and the topological support of $P_t(\cdot)$ is contained in $K_t(\cdot)$. Equation (1.2) is recursively driven backward in discrete time. It bears the name of its inventor Bellman; hence $v_t(\cdot)$ may be referred to as a Bellman function. The supremum operator instructs to seek a measure $\mathbb{Q} \in P_t(\cdot)$ that attains the maximum of integration. It may be shown that this maximum is attainable for upper semicontinuous payoff function (sufficient conditions are provided in Smirnov [2016]) which we study in this article. Whenever payoff functions are either concave or convex such measure is easily obtained. In the case of piecewise linear payoff functions of mixed concavity such measure can be determined from considerations based on the convex measure. For this class of functions, as we will show below, the solution of equation (1.2) can be reduced to the linear programming problem. Henceforth, and if there is no ambiguity, we refer to the super-hedging price simply as the price to avoid repetition.

In the examples throughout, it was found useful to factor out some reference scale Λ that exists in the calculation and work with dimensionless variables $z_t = \dfrac{Y_t}{\Lambda}$, with Y_t being a state variable. Note that Λ does not have to be a constant nor unique; it may as well be the price of the underlying asset or a state variable related to the price. Effectively, we change appropriately the numeraire to simplify the calculation process and bring the final answer to a simpler form. This is particularly useful, as we show below, for the Asian and Lookback call options where there are two state variables to begin with. In fact, as a result of the numeraire change, their solutions are partially analytic and reduce the computational load on the recursive part. Moreover, it is the author's opinion that working with dimensionless variables is aesthetically more appealing.

2 Convex Payoff

Single asset contingent claims with convex payoff functions are the most widely-spread derivative instruments. For such payoff functions, the supremum is attained when the integration measure Q is concentrated at two points in $K_t = X_{t-1}[a, b]$, $t = 1, \ldots, N$ maximally distant from each other, i.e. the extreme points of K_t (see Dana and Jeanblanc-Picqué [1998]). A characteristic feature of Q here is that it remains the same for all values of the underlying and for all times of existence of such contracts. This makes recursive calculations particularly easy. To find it, we solve for the following linear system which corresponds to the martingality property and normalization of $\mathbb{Q}$:

$$\begin{bmatrix} \Delta X_t^{\text{down}} & \Delta X_t^{\text{up}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.1}$$

where α is the down-move probability and β is the up-move probability and the quantities $\Delta X_t^{\text{down}} = X_{t-1}(a-1)$ and $\Delta X_t^{\text{up}} = X_{t-1}(b-1)$ are the corresponding down-move and up-move price changes. The solution for the probabilities α and β is

$$\alpha = \frac{b-1}{b-a}, \qquad \beta = \frac{1-a}{b-a}. \tag{2.2}$$

It is worthwhile to point out that these equations cannot be inverted to retrieve a and b. Also, note a useful trivial property which comes in handy for the derivations below

$$\alpha \cdot a + \beta \cdot b = 1. \tag{2.3}$$

Altogether for convex functions, the recursive expression that follows from equation (1.2) is

$$v_{t-1}(X_{t-1}) = \alpha\, v_t(X_{t-1}a) + \beta\, v_t(X_{t-1}b), \quad t = 1, \ldots, N. \tag{2.4}$$

One does not have to be an astute reader to quickly realize that equation (2.4), in its appearance, is exactly the binomial tree model due to Cox et al. [1979] and Rubinstein [1994]. However, the difference in the interpretation is crucial between our and the Cox-Ross-Rubinstein (CRR) models. In the CRR model, evolving from $t-1$ to t, the price X_t may end up being only either $X_{t-1}a$ or $X_{t-1}b$, i.e. no other values are permitted. In the guaranteed approach framework, there is an uncountable infinity of values that X_t can take on in the interval $X_{t-1}[a,b]$. In the CRR model, the resulting answer is the fair (risk-neutral) contingent claim price in the sense as we described it for the complete market framework above. There, the parameters a and b, in principle, could be inferred by fitting them to the market observed prices. In the guaranteed approach framework, the answer is the super-hedging price, i.e. due to the worst case scenario played out by the market. It is not meaningful to fit the parameters a and b to the market prices of contingent claims. In fact, these parameters have to be estimated from, say, the historical time series, i.e. they could represent a high percentage confidence band of the returns on the underlying asset. Alternatively, in marginal requirements calculations, a and b would represent typical price limits in a specified time interval by which markets operate, i.e. when these limits are breached, there is a temporary trading halt.

Before proceeding on to the convex examples, we briefly mention the case of the concave payoff functions. The supremum of integral is attained when the measure Q is concentrated at the single point X_{t-1}, corresponding to $\Delta X_t = 0$; this is a consequence of Jensen's inequality. The recursive Bellman equation leads to the trivial result

$$v_{t-1}(X_{t-1}) = v_t(X_{t-1}), \quad t = 1, \dots, N. \tag{2.5}$$

2.1 European Call

Any article that studies the pricing of a selected set of contingent claims would most likely begin with the European option; we are not an exception. Here and everywhere below, without any prejudice though, we calculate the prices of the call type options. Analogously, calculations may be retraced to obtain the prices of the put type options. The payoff of a European call option with the underlying asset price X_N, the strike K and at maturity time N is

$$v_N = \max(0, X_N - K) = K \max(0, z_N - 1) \tag{2.6}$$

where the dimensionless variable $z_N = \dfrac{X_N}{K}$ is introduced for the reasons discussed above. We redefine the payoff function $\tilde{v}_N = \dfrac{v_N}{K}$ and drop the tilde symbol everywhere below. According to the definition of the maximum function[3], we may write the payoff function as

$$v_N(z_N) = \frac{1}{2}(z_N - 1) + \lambda_N(z_N), \quad \lambda_N = \frac{1}{2}|z_N - 1|. \tag{2.7}$$

The advantage of casting it in this form is that the first term stays invariant upon integration. In fact, this trick is possible and suitable for any vanilla and exotic derivatives, with their corresponding λ functions, as we shall see below. By the virtue of equation (2.4)

[3] The maximum of two real numbers is defined by $\max(a, b) = \dfrac{1}{2}\big(a + b + |a - b|\big)$.

and making use of identify (2.3), we quickly find the price at time t given that the price at time $t+1$ is available:

$$\begin{aligned} v_t(z_t) &= \frac{1}{2}(z_t - 1) + \lambda_t(z_t), \quad t = 0, \dots, N-1 \\ \lambda_t(z_t) &= \alpha\, \lambda_{t+1}(z_t a) + \beta\, \lambda_{t+1}(z_t b). \end{aligned} \tag{2.8}$$

This expressions constitute a recursive solution for the price of a European call option at any time prior to maturity N. For path-dependent (and still convex payoff) contingent claims, as we show below, such recursive solutions cannot be simplified any further. The computational time grows as 2^s where s is the number of time steps to maturity. On the desktop computer available to us, we run into the problem of excessive and practically unreasonable computational times around $s \sim 20$. We are not aware how to circumvent this even with parallelization.

Fortunately, in the case of the European option, and for that matter even any portfolio of contingent claims with a convex payoff function, it is possible to express the solution as a computation on $s+1$ points of the payoff function due to the recombining nature of the binomial tree by invoking induction. Once again, this is in agreement with the CRR binomial model and, in fact, has become standard textbook material by now. For instance, the book by van der Hoek and Elliot [2006] treats many common contingent claims with a generic method in the context of the binomial tree model. However, we emphasize that in our approach, seemingly having at lot of resemblance at first sight, the mathematical expression bears a new interpretation, i.e. super-hedge price due to the worst case scenario.

Let us assume that at arbitrary time t we may express the solution in terms of the convex payoff v_N as follows

$$v_t(z_t) = \sum_{i=1}^{s+1} \gamma_i \alpha^{s+1-i} \beta^{i-1} v_N(z_t a^{s+1-i} b^{i-1}), \quad t = 0, \dots, N-1. \tag{2.9}$$

By equation (2.4), which also happens to be the particular case when $s = 1$, we calculate the solution at the earlier time step $t-1$. After a little bit of manipulation and a change of variable on the summation index, we readily find

$$\begin{aligned} v_{t-1}(z_{t-1}) &= \sum_{i=1}^{\widetilde{s}+1} \widetilde{\gamma}_i \alpha^{\widetilde{s}+1-i} \beta^{i-1} v_N(z_t a^{\widetilde{s}+1-i} b^{i-1}), \quad t = 0, \dots, N-1 \\ \widetilde{\gamma}_i &= \gamma_i + \gamma_{i-1}, \quad \widetilde{\gamma}_1 = \gamma_1, \quad \widetilde{\gamma}_{\widetilde{s}+1} = \widetilde{\gamma}_{s+1} \end{aligned} \tag{2.10}$$

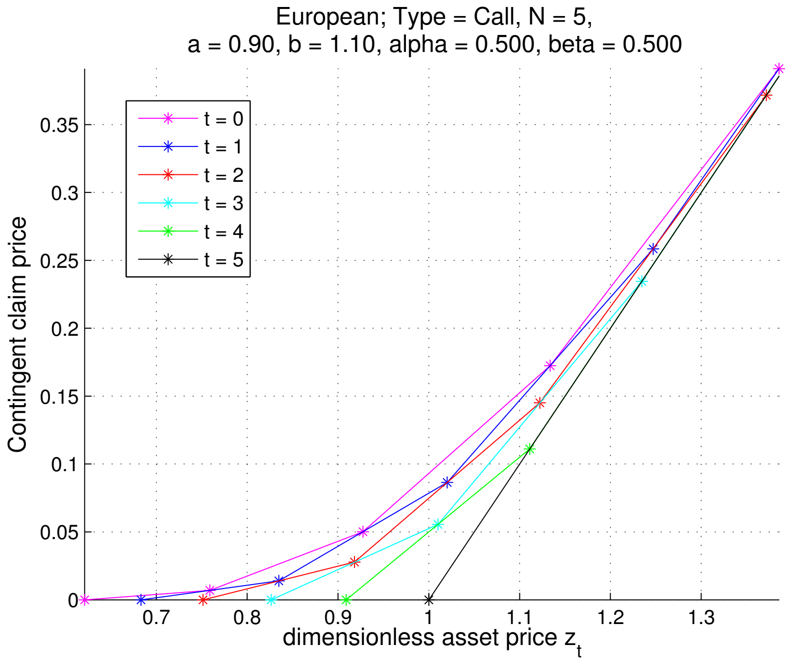

Figure 1: An example of a European call option price expressed in terms of the strike K.

where $\widetilde{s} = s + 1$ and γ_i turns out to be just the binomial coefficient. Most mathematical libraries have the binomial coefficient precomputed which means that it is computationally efficient to use equation (2.10) for large s.

Figure 1 illustrates the solution for a European call option with maturity $N = 5$ at all times during its existence. The values of the parameters a, b and others used in the figures below are selected arbitrarily here and throughout the article. We observe that the curves representing the prices are piecewise linear. The location of the kinks, i.e. points of discontinuity, is straightforward to determine using a simple recursion. Beginning with the kinks at time $t + 1$, we find kinks at time t using

$$z_t^0 = z_{t+1}^0 \, b^{-1}, \quad z_t^i = z_{t+1}^{i-1} \, a^{-1}, \quad \forall i \geq 1 \tag{2.11}$$

where the kinks at $t + 1$ are placed in increasing order and the superscript i indicates numeration. On each subsequent time step the number of kinks increments by one. At maturity N, we begin with a single kink at the strike 1, i.e. $z_N^0 = 1$.

2.2 Asian Arithmetic Call with Floating Strike

Asian options are similar to their European counter-parts except that the strike price is the arithmetic average of the underlying price. Their payoff is path-dependent and becomes known only at the time of maturity. They are usually cheaper due to the reduced volatility of the underlying average price. In addition, the averaging feature makes Asian options' payoffs secure to any drastic changes in the price of the underlying just before maturity.

The payoff of an Asian call option written on the underlying asset with the price X_N and the arithmetic mean S_N at maturity time N is

$$v_N = \max(0, X_N - S_N) = X_N \max(0, 1 - z_N), \tag{2.12}$$

where S_t is defined by and obeys the following identity

$$S_t = \frac{1}{t+1}\sum_{i=0}^{t} X_i = \frac{t}{t+1}S_{t-1} + \frac{1}{t+1}X_t, \tag{2.13}$$

and the dimensionless variable $z_N = \dfrac{S_N}{X_N}$ is the arithmetic average expressed in terms of the underlying final price. In the case of the Asian option, we set the numeraire by factoring out X_t which reduces the number of the state variables to one, i.e. $z_t = \dfrac{S_t}{X_t}$. We parametrize the payoff function in a similar manner as the European call option and omit propagating the scale from line to line

$$v_N(z_N) = \frac{1}{2}\big(1 - z_N + \lambda_N(z_N)\big), \quad \lambda_N(z_N) = |z_N - 1|. \tag{2.14}$$

It is easy to see that this function is convex. The solution at arbitrary time t is given by

$$\begin{aligned} v_t(z_t) &= \frac{1}{2}\left(\frac{t+1}{N+1}(z_t - 1) + \lambda_t(z_t)\right), \quad t = 1, \ldots, N \\ \lambda_t(z_t) &= \alpha \cdot a \cdot \lambda_{t+1}(z^a_{t+1}) + \beta \cdot\ b \cdot \lambda_{t+1}(z^b_{t+1}), \\ z^a_t &= \frac{1}{t+1}\Big(t z_{t-1} a^{-1} + 1\Big), \\ z^b_t &= \frac{1}{t+1}\Big(t z_{t-1} b^{-1} + 1\Big). \end{aligned} \tag{2.15}$$

Here, the first term is not invariant but at least simplifies to a trivial form and the second term is calculated recursively. Note that we have used the identity in equation (2.13) to arrive at the above result.

The domain of z_t is determined by the extreme behaviour of X_t. In one case, the asset price drops by a factor of a on every time step. In the other case, it rises by a factor of b on every step. Then, we express the average of each extreme behaviour in terms of the asset's

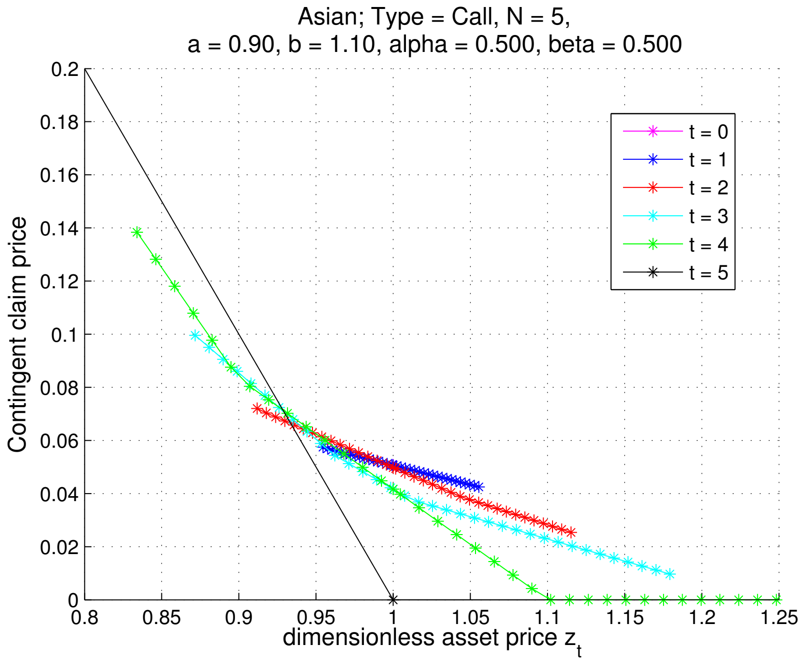

Figure 2: An example of an Asian call option price expressed in terms of the current asset price X_t.

current price. Thus, we deduce that the possible values of z_t on time step t lie between

$$\begin{aligned}\mathrm{Min}_t &= \frac{1}{t+1}\sum_{i=0}^{t} b^{-i}, \\ \mathrm{Max}_t &= \frac{1}{t+1}\sum_{i=0}^{t} a^{-i}.\end{aligned} \tag{2.16}$$

Figure (2) shows that a typical solution to the recursive equations for the price of the Asian option is piecewise linear. The averaging feature of the strike makes it possible for the price at some earlier time t to be lower than at $t+1$ when expressed in terms of the dimensionless variable z_t. Note that the slopes of the price curves are negative which is a consequence of expressing the solution in terms of the current asset price and not mistakenly represent a put type solution. The $t = 0$ curve consists of only one point at $z_0 = 1$, is hidden behind other curves and lies above them.

It is feasible to find the kink points separating the neighbouring linear regions. We can do this recursively by solving systems of inequalities on each time step. The trick is to keep applying the identity in equation (2.13) from step to step. However, the restriction on the allowed values of z_t provided by (2.16) exclude many of the outer kink points, i.e. z_t can not reach the outer linear regions defined by those kinks. Thus, there is no much benefit for calculating them.

Furthermore, due to a resemblance with the European options, it might seem to be possible to reduce the solution from calculating on 2^s to $s+1$ points. Unfortunately, such recombination is impossible because of the path-dependence as seen in equation (2.13).

2.3 Lookback Option with Floating Strike

Lookback options are purchased by market players if they feel confident that the price of an underlying will change mostly in one direction, i.e. either maximally increase or decrease, by the end of the contract. These options are rare and mostly traded on the OTC markets. We demonstrate how to calculate the price of a Lookback call option with a floating strike. The price of this option is a function of the underlying asset X_t and the floating strike $\hat{X}_N$ which is the minimum of X_t for all t and becomes known only at maturity. The extreme values of the payoff are zero on the down side and approach X_N on the up-side if the price of the asset at inception is very small. The option is never out-of-the-money and will always be exercised by the holder unless $X_N = \hat{X}_N$. The payoff function is

$$v_N = X_N - \hat{X}_N. \tag{2.17}$$

The minimum price $\hat{X}_N$ function is defined recursively and requires comparison between neighbouring time steps

$$\hat{X}_t = \min(X_t, \hat{X}_{t-1}), \quad \hat{X}_0 = X_0. \tag{2.18}$$

As before we will express the result in terms of a dimensionless quantity. In this example, it is convenient to choose the underlying X_t itself as the numeraire; we remove it from the equations but keep it in mind that X_t is still there. Using the definition of the $\hat{X}_t$ function and the minimum function, we parametrize the payoff as follows

$$v_N(z_N) = \big(1 - \min(1, \hat{z}_N)\big) = \frac{1}{2}\big(1 - \hat{z}_N + \lambda_N(\hat{z}_N)\big), \quad \lambda_N(z_N) = |\hat{z}_N - 1| \tag{2.19}$$

where the new variable $\hat{z}_N = \dfrac{\hat{X}_{N-1}}{X_N}$ is the minimum of all the previous time steps expressed in units of the terminal price. This parametrization is neat and the calculation turns out to be very orderly. Integration is performed under the same measure as that for the European

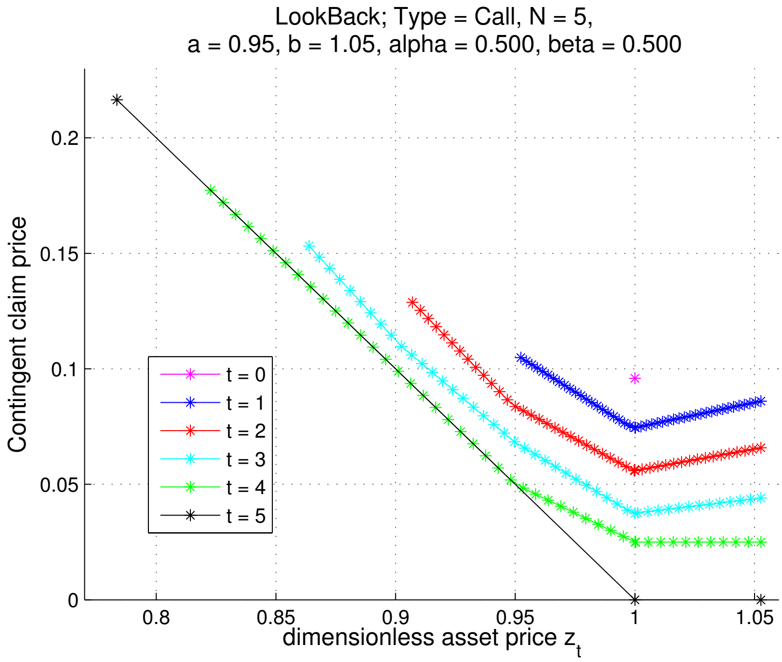

Figure 3: An example of a Lookback call option price expressed in terms of the current asset price X_t.

option[4] but the price function will be given in terms of the variable $\hat{z}_t$. Just as for the Asian option, the payoff function in the variable $\hat{z}_N$ has the slope of negative one. The recursive solution at arbitrary time t is

$$\begin{aligned} v_t(z_t) &= \frac{1}{2}\left(\frac{1-\hat{z}_t}{2^s} + \lambda_t(\hat{z}_t)\right), \quad t = 0, \ldots, N \\ \lambda_t(\hat{z}_t) &= \frac{1}{2}|\hat{z}_t - 1| + \alpha \cdot a \cdot \lambda_{t+1}\Big(a^{-1}\min(1,\hat{z}_t)\Big) + \beta \cdot b \cdot \lambda_{t+1}\Big(b^{-1}\min(1,\hat{z}_t)\Big). \end{aligned} \tag{2.20}$$

Recall that s is the remaining time to maturity. The domain of the variable $\hat{z}_t$ is a little intricate to figure out. We begin at $t = 0$ where it obviously takes on only one value $\hat{z}_0 = 1$. For time steps where $t < N$, we have to solve two inequalities corresponding to the extreme behaviour of X_t. For $\hat{z}_1$ we readily find the domain to be (b^{-1}, a^{-1}). For $\hat{z}_2$ the domain is (b^{-2}, a^{-1}); the right end expressed in $\hat{z}_t$ does not charge. In general, at the time step t, the allowed values of $\hat{z}_t$ lie in (b^{-t}, a^{-1}).

Figure 3 illustrates what a typical solution to equation (2.20) behaves like. If we cascade down along the vertical line $\hat{z}_t = 1$ the asset price does not change and gradually the option value vanishes. The most left (right) point on each curve corresponds to the most increase (decrease), i.e. $X_t = X_{t+1}b^{-1}$ ($X_t = X_{t+1}a^{-1}$) if expressed in absolute units. On any given

[4]The payoff and price functions on every time step t are convex. This fact is better seen in the variable X_t though.

curve we observe that the option price is higher if the asset price goes down rather than remains unchanged. This makes sense and agrees with intuition because if the asset price goes down and re-bounces back on consecutive steps than the potential payoff could be more than if no asset price change occurred.

3 Piecewise Linear Payoff

In financial engineering, especially in the structured products domain, it is commonly practised to construct portfolios from combinations of European calls, puts and/or other common vanilla instruments. The payoff functions of the resulting instruments are typically piecewise-linear but neither fully convex nor concave; they are mixed in concavity and may even have jumps. For such contingent claims, in the context of guaranteed approach, there is no unique measure that could be repeatedly used on the allowed integration domain $K_t(\cdot)$ of the underlying and/or on every time step. Hence, this presents a challenge. In addition and more importantly, the sub-additivity property of the class of measures $P_t(\cdot)$ implies that the calculated super-hedging price of the combined payoff profile is always not greater than the sum of the super-hedging prices of individual contingent claims. This does not occur when one calculates prices in the traditional/probabilistic pricing frameworks where the prices add up linearly. Therefore, we can quickly imagine that it is of a particular craving for portfolio managers and central-counter parties to be able to calculate guaranteed prices of portfolios of contingent claims.

From the technical point of view, it is impossible to determine the integration measure analytically for instruments with piecewise linear payoffs of mixed concavity. At least we do not know how it could be done succinctly. Fortunately, we have been able to devise a numerical algorithm that it is capable of managing it for us. It solves both the task of identifying the appropriate measure $\mathbb{Q}$ and performing integration simultaneously for any piecewise linear European type payoff profile. In fact, we must not worry about the particularities of $\mathbb{Q}$ as they are solely decided by the payoff function; we will see this shortly. The accuracy of this algorithm is a controllable parameter which makes it very robust.

The extension of the pricing framework to piecewise linear payoff functions of mixed concavity is based on the idea of fragmenting the support of the measure K_t, i.e. the integration interval, into several disjoint regions on which the integrand remains either concave or convex. To this end, symbolically, we can represent the measure in the following fashion:

$$\mathbb{Q}(\Delta X_{t+1} \in K_t) = \sum_{i=1}^{m} \mathbb{Q}_i(\Delta X_{t+1} \in K_i)\, \mathbb{Q}(\Delta X_{t+1} \,|\, \Delta X_{t+1} \in K_i) \tag{3.1}$$

where m is the number of disjoint regions, $K_t = \bigcup_{i=1}^{m} K_i$ with the property $K_i \bigcap K_{i+1} = X_t c_{t+1}^{i+1}$ for $i \neq m$ in one dimension[5]. The boundary points of neighbouring regions $X_t c_{t+1}^{i+1} \in X_t[a, b]$ are in written in this product form for convenience. We return to a subtle discussion on how to determine boundary points and correspondingly c_{t+1}^{i+1} below.

Upon fragmentation, we have effectively introduced conditional probabilities and the master equation (1.2) now translates into the following form:

$$v_t(X_t) = \sup_{\mathbb{P}} \sup_{\mathbb{Q}} \sum_{i}^{m} p_i \int d\mathbb{Q}(\Delta X_{t+1} \,|\, \Delta X_{t+1} \in K_i)\; v_{t+1}(X_t, \Delta X_{t+1}), \quad t = 0, \ldots, N-1 \tag{3.2}$$

where $\mathbb{P}$ is the set of probabilities p_i $(i = 1, ..., m)$ of ΔX_{t+1} falling into the corresponding K_i. We insert X_t instead of $(\cdot)$ as the arguments of v_t and v_{t+1} because, in this section, we deal with contingent claims whose payoffs are independent of price history.

If the integrand v_{t+1} is strictly convex on each K_i, then we are led to the quadratic form equation

$$v_t(X_t) = \sup_{\mathbb{P}} \sup_{\mathbb{Q}} \sum_{i}^{m} p_i \left(q_i v_{t+1}(X_t l_i) + (1 - q_i) v_{t+1}(X_t r_i)\right), \quad t = 0, \ldots, N-1 \tag{3.3}$$

with q_i and $1 - q_i$ being the conditional probabilities of X_{t+1} falling exactly at the left $X_t l_i$ and the right $X_t r_i$ end points of the ith fragmentation region[6]. Obviously, we always have $a \leq l_i < r_i \leq b$ for all i. Identically, the requirement of martingality for P_t becomes the quadratic form restriction

$$\sum_{i}^{m} p_i \left(q_i X_t l_i + (1 - q_i) X_t r_i\right) = X_t, \tag{3.4}$$

which casted in this form reflects fragmentation of K_t. In general, we have the following

[5] We have dropped the obvious subscript t on K_i

[6] Again, we have dropped the t subscript from l_i and r_i.

important property and definition

$$X_t r_i = X_t l_{i+1} \equiv X_t c_{t+1}^{i+1}, \quad i \neq m \tag{3.5}$$

which emerges from the fact that the regions K_i and K_{i+1} neighbour each other. With this property, the objective function (3.3) and the martingality restriction (3.4), after some manipulation of the summation index, simplifies to a linear form expression

$$\begin{gathered} v_t(X_t) = \sup_{k_i} \sum_{i=1}^{m+1} k_i \, v_{t+1}(X_t c_{t+1}^i), \quad t = 0, \ldots, N-1 \\ \sum_{i=1}^{m+1} k_i X_t c_{t+1}^i = X_t, \end{gathered} \tag{3.6}$$

where the leftmost $X_t c_{t+1}^1$ and the rightmost $X_t c_{t+1}^{m+1}$ are the end points of the overall integration region K_t and the remaining c_{t+1}^i for $i = 2 \ldots m$ are defined as the ratio of critical points[7] $\widehat{X}_{t+1}^j$ for some j enumerating their quantity and the asset price X_t. The critical points $\widehat{X}_{t+1}^j$ consist of all their predecessors $\widehat{X}_{t+2}^{j'}$ plus new points generated recursively from multiplication of $\widehat{X}_{t+2}^{j'}$ by b^{-1} and a^{-1} where j' enumerates critical points at time $t+2$. Notice that, in order to avoid multiplicity, the generation of additional points is actually equivalent to $\widehat{X}_{t+2}^1 b^{-1}$ and $\widehat{X}_{t+2}^{j'} a^{-1}$ for $j' \neq 1$. Obviously, the index j takes on $2l+1$ values where l is the number of values of the index j'. The quantity of critical points $\widehat{X}_N^j$ at maturity N corresponds solely to $m-1$ where m is the number of fragmentation regions as already stated above. In short summary, before $\widehat{X}_{t+1}^j$ get sorted in increasing order, we have

$$\begin{gathered} c_{t+1}^1 = l_1 = a, \quad c_{t+1}^{m+1} = r_m = b \\ c_{t+1}^i = \frac{\widehat{X}_{t+1}^j}{X_t}, \quad i = 2, \ldots, m \\ \widehat{X}_{t+1}^j = \widehat{X}_{t+2}^{j'}, \quad j = 1, \ldots, l \\ \widehat{X}_{t+1}^{l+1} = \widehat{X}_{t+2}^1 b^{-1}, \quad \widehat{X}_{t+1}^j = \widehat{X}_{t+2}^{j'} a^{-1}, \quad j = (l+2), \ldots, (2l+1). \end{gathered} \tag{3.7}$$

It is crucial to realize that, in calculation, we do not require all critical points to be invoked at once but only those that happen to be contained in the integration interval, i.e. for those i and j which produce $\widehat{X}_{t+1}^j = X_t c_{t+1}^i \in X_t[a, b]$. Finally, the new unconditional probabilities

[7] In the sense that the first derivative of v_{t+1} is not defined at these points.

k_i in (3.6) are defined by

$$
\begin{aligned}
k_1 = p_1 q_1, \quad k_{m+1} = p_m(1-q_m),\\
k_i = p_i q_i + p_{i-1}(1-q_{i-1}) \quad i = 2,\dots,m.
\end{aligned}
\tag{3.8}
$$

In standard literature on linear programming, see for example Vanderbei [2001], equation (3.6) together with the normalization and non-negativity conditions on k_i is known as the primal linear problem. By standard methods, it can be reformulated to its dual form, which results in the following minimization problem

$$
\begin{aligned}
v_t(X_t) = \inf_{\widetilde{k}_i} \sum_{i=1}^{2} \widetilde{k}_i\, \xi_i = \inf_{\widetilde{k}_1,\widetilde{k}_2} \left(\widetilde{k}_1 + X_t \widetilde{k}_2\right), \quad t = 0\dots N\\
\sum_{j=1}^{2(m+1)} A_{ij}\widetilde{k}_j = v_{t+1}(X_t c^i_{t+1}),
\end{aligned}
\tag{3.9}
$$

where the quantities $\widetilde{k}_i$ are the dual variables to k_i, of which there are only two due to the martingality and normalization conditions. Note that $\widetilde{k}_i$ may take on any values in $(-\infty, +\infty)$ unlike its counterparts k_i. The diagonal square matrix A contains -1 in the first $m+1$ entries on the diagonal and $+1$ in the remaining $m+1$ entries on the diagonal. The familiar form of the solution suggests that $\widetilde{k}_1$ is the intercept and $\widetilde{k}_2$ is the slope of a line. Not accidentally then, $\widetilde{k}_1$ and $\widetilde{k}_2$ represents the super-hedging strategy, i.e. the quantities of the riskless and the risky asset, respectively, over a single time period from t to $t+1$. The dual form of linear optimization (3.9) and the geometric interpretation we assign to $\widetilde{k}_i$ agree with the work of Carassus and Vargiolu [2010].

It should be recognized that, say, the price of a European call option from a previous section can also be calculated using the extended theory algorithm with the critical points being absent. It would lead to the same results, although it would be computationally more involved.

3.1 Digital Call

Digital options are considered very risky and difficult to hedge. They come in the European, American or even Asian flavour but have limited usefulness; if any, Digital options are traded over the counter. In this article though, they serve as a vivid example of the piecewise linear payoff function algorithm described above.

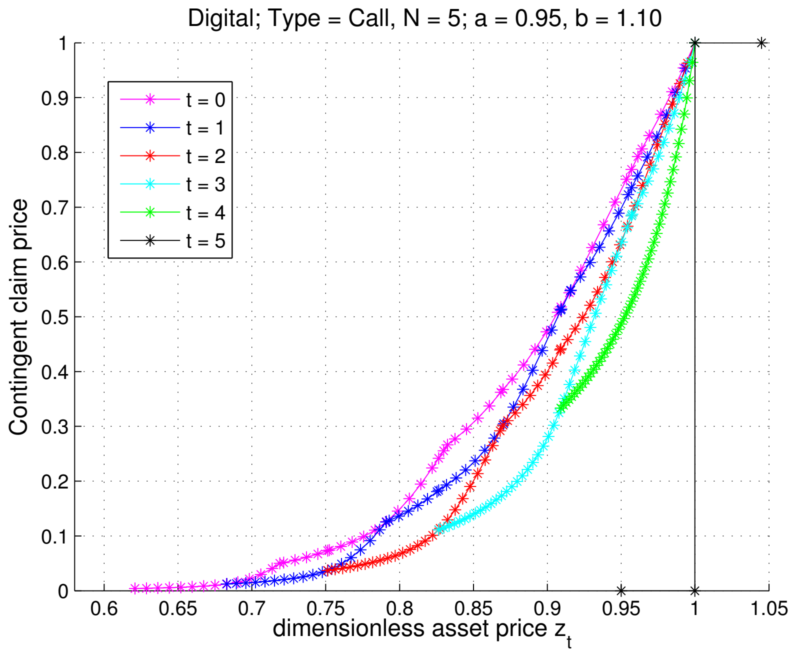

Figure 4: An example of a Digital call option price expressed in terms of the strike K.

The payoff of a Digital call is unity if the asset price X_N ends up above the strike K and zero otherwise. Mathematically, it is represented by a discontinuous step function

$$v_N(X_N) = \begin{cases} 0, & X_N < K \\ 1, & X_N \geq K \end{cases} = K \begin{cases} 0, & z_N < 1 \\ 1, & z_N \geq 1 \end{cases} \tag{3.10}$$

where in the second equality we have introduced a dimensionless variable $z_N = \dfrac{X_N}{K}$. A digital call payoff has a single critical point $\hat{z}_N^1 = 1$ at maturity.

We apply the general payoff algorithm to the digital call payoff in equation (3.10). Figure 4 illustrates the price of a digital call at various times prior to maturity. These prices are depicted as piecewise arc segments. Even though they are obtained purely pointwise, it is instructive to comment on several of its properties. First, the discontinuity gradually disappears with each earlier curve and the trend suggests it would vanish altogether at infinite time from maturity. And second is that the number of kinks, i.e. critical points, on each earlier curve also increases. Thus the curve would not be differentiable at each point below unity at infinite time to maturity. If we now return to the European call option in Figure 1, then we notice that the second property (i.e. infinitely many kinks and non-differentiability) is also present there.

3.2 Call Spread

A call spread combination is popular among portfolio managers who wish to lock in a potential fixed profit for a wide range of the underlying's price in exchange for the risk of gaining nothing. It is constructed by buying a European call option with a strike K_1 and selling a European call option with K_2 both maturing at the same time and $K_2 > K_1$. The payoff of this combined portfolio is

$$v_N(X_N) = \max(0, X_N - K_1) - \max(0, X_N - K_2) = \frac{1}{2}\Big(K_2 - K_1 + |X_N - K_1| - |X_N - K_2|\Big).$$

We express it in terms of the dimensionless variables $z_N = \dfrac{X_N}{K_1}$ and the ratio of the strikes $k = \dfrac{K_2}{K_1}$

$$v_N(z_N) = \frac{K_1}{2}\Big(k - 1 + |z_N - 1| - |z_N - k|\Big). \tag{3.11}$$

Even though there are two kinks in the payoff function there is still a single critical point at $\hat{z}_N^1 = k$; the payoff function maintains its convexness on intervals $(-\infty, k)$ and $(k, +\infty)$. Thus the call spread shares the same computational complexity as the digital call. The shape of the price curves depends on the relative lengths of the integration interval ending at the point $z_N b = k$, i.e. $z_N(b - a)$, and the segment over which the payoff function rises from zero to unity, i.e. $k - 1$. Figure 5 shows the case when $k - 1$ is larger than the integration interval. Here, the curves are almost piecewise linear with the property that the later curve is always less or equal to the earlier one. The other case, when $k - 1$ is less than $z_N(b - a)$, would yield arc curves similar to the digital option except that the discontinuities would be replaced by linear segments with positive slopes.

It is compelling to recognize that the digital option and the European call are both limiting cases of the call spread combination. Consider the payoff of a call spread with an equal number L of longed and shorted contracts and the parameters K_1 and K_2 controlling the strikes

$$\begin{aligned} v_N^N(X_N) &= L\left(\max\big(0, X_N - K_1\big) - \max\big(0, X_N - K_2(1 + L^{-1})\big)\right) \\ &= \frac{L\,K_1}{2}\Big(k(1 + L^{-1}) - 1 + \big|z_N - 1\big| - \big|z_N - k(1 + L^{-1})\big|\Big) \end{aligned} \tag{3.12}$$

where $z_N = \dfrac{X_N}{K_1}$ is the same dimensionless variables introduced for the European call, the

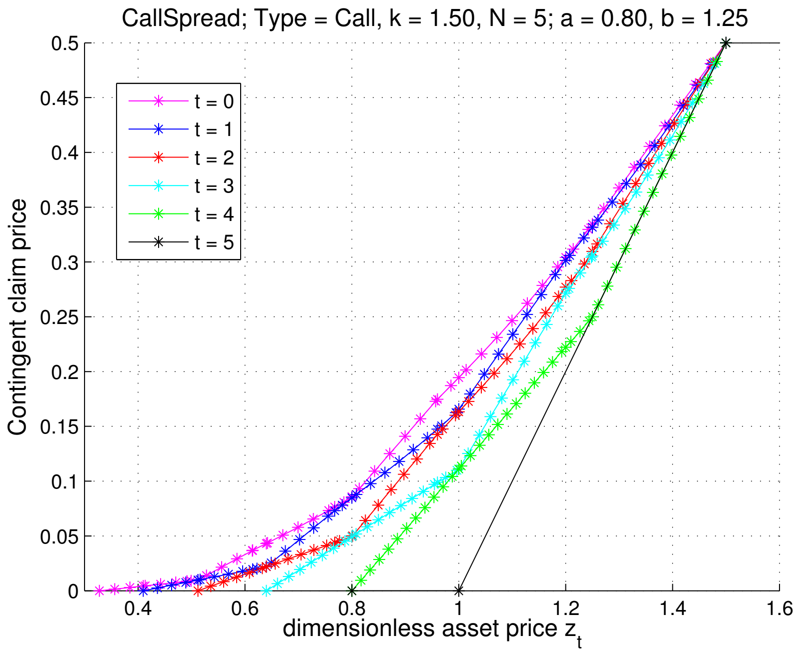

Figure 5: An example of a call spread option price expressed in terms of the strike K_1.

digital option and the call spread option in the examples above. The digital option can be retrieved in the limit where L becomes very large and $K_1 = K_2$. On the other hand, the European option is obtained in the limit when K_2 grows large and $L = 1$. In fact, one may continuously transform a digital call payoff to a European call payoff according to equation (3.12). An identical connection between these three instruments maybe established at times prior to maturity as well.

3.3 European Step Call

The final example is to illustrate the key advantage of pricing in the guaranteed framework, i.e. the price of a portfolio is always less or equal to the prices of its individual constituents. To this end, we consider a portfolio consisting of European calls: one long with the strike K_1, one short with the strike K_2 and one long with the strike K_3; where $K_1 < K_2 < K_3$. We are not aware of a special name for this payoff combination but it makes sense to dub it the Step option. The payoff function may be written as

$$\begin{aligned} v_N(X_N) &= \max\Big(0, X_N - K_1\Big) - \max\Big(0, X_N - K_2\Big) + \max\Big(0, X_N - K_3\Big) \qquad (3.13) \\ &= \frac{K_2}{2}\Big(z_N - \big(k_1 - k_3 - 1\big) + \big|z_N - k_1\big| - \big|z_N - 1\big| + \big|z_N - k_3\big|\Big) \end{aligned}$$

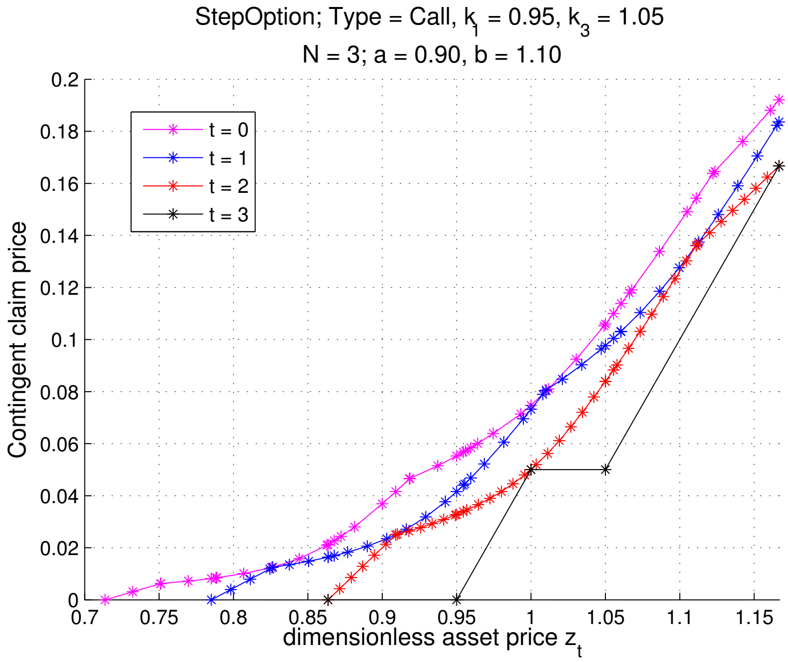

Figure 6: An example of a European call step option price expressed in terms of the strike K_2.

where $z_N = \dfrac{X_N}{K_2}$ is the dimensionless variable in this example. The price curves of the combined contingent claim in Figure 6 are digital-option-like (i.e. arc-like) in the region around the strikes and European-like (i.e. piecewise-linear) on either ends. The right end of the curves is cut off to make the middle region more discernible for viewing. The same goes for the number of time steps; it is sufficient to illustrate the property of sub-additivity with only three time steps. The sum of the individual prices is straightforward to find using equation (2.10) for the long options and (2.5) for the short one. Figure 7 shows the difference between the the sum of the guaranteed constituent prices and the guaranteed price; it shows that the sub-additivity property holds.

4 Conclusion

This article addressed the problem of super-hedging contingent claims in incomplete markets for practical considerations. We applied the results from the deterministic game theoretic (guaranteed) approach due to Smirnov [2016] for the case of contingent claims with one underlying asset. A distinct feature of the guaranteed approach is the fact that the reference probability measure describing dynamics of the asset price is not required to be specified; all that is required are the ranges representing the allowable changes in the price. Other

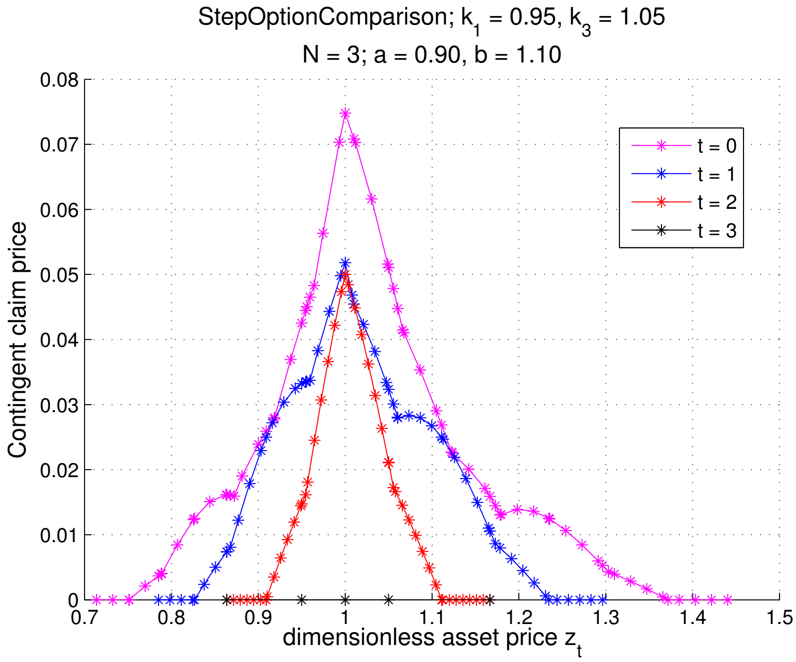

Figure 7: An example of the difference between the sum of the constituent prices and the guaranteed price for a European step combination; expressed in terms of the strike K_2.

guiding principles in Smirnov's derivations are game equilibrium, absence of arbitrage and the martingale property of the asset price.

For convenience, and as a first run, we expressed the asset prices in terms of the riskless asset in order to avoid dealing with the complications arising from the interest rate. Further down, we also found it convenient to introduce new changes of variables specific for each example that we studied. It turned out that to solve the master equation (1.2) implies finding an integration measure which attains supremum, i.e. maximizes the result of integration. In the case of contingent claims with convex payoffs, equation (1.2) is a recursive Bellman equation that has a tractable solution due to the fact that the integration measure is straightforward to identify. Furthermore, in the case of vanilla contingent claims, i.e. such as European call and put or their convex combinations, the recursive solution can be re-summed to a simple one-liner. On the contrary, for portfolios of European contingent claims where the payoff function is piecewise-linear with possible discontinuities, it is required to solve a linear programming problem. The solution may or may not be purely numerical (depending on the method of solution of the linear programming problem) and the computation times depend on the nature of the payoff function, i.e. the critical points.

We have presented a total of six examples of various contingent claims of the European type. The goal was to show the general characteristics of the solutions and exhibit a wider applicability of the proposed methodology. The theoretical framework is highly intuitive even without the understanding of heavy mathematical machinery. It should be easy to grasp for practitioners and non-experts. However, a downside of our methodology, which is typical of anything nowadays, is extensive computation times. There could be yet unexplored ways to decrease computation times with clever tricks involving parallel computation or analytic/combinatoric simplification. They remain to be found.

A natural extension of this research would be to price contingent claims of the American type; they dominate over their European counterparts in organized markets. For American type contracts, it is imperative to incorporate the interest rate into the dynamics of the asset price and to account for a potential premature payoff. In fact, the key equation (1.2) to super-price European contracts was obtained as a limiting case of the expression to super-price American contracts when the potential premature exercise payoffs are set to negative infinity. Thus, the theoretical means of pricing American contract is already available. A practical application and continuation of this research would be to examine portfolios of contingent claims expiring at different times such as the calendar spread. This is readily feasible with European contracts and largely amounts to revamping the code. Likewise, it would be tempting to consider super-hedging of exotic derivatives with only vanilla instruments which requires further theoretical research.

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Contact Author: Alexander Chigodaev (Research Fellow)
Laboratory for Financial Engineering and Risk Management
National Research University Higher School of Economics (Moscow, Russia)
E-mail: achigodaev@hse.ru