Discrete Mathematical Models

Faculty of Economic Sciences

Bachelor, 2 course

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Class Times and Locations -

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Office Location

Office Hours

Section 1. General information about the course

In this course the selected topics of modern discrete mathematical models in economics, politics, business and social studies are provided. It includes social choice theory, double-sided matchings, fair division, power distribution and structural balance, and a soft introduction to game theory. Many real world examples are presented like power distribution analysis in Russian Parliament and International Monetary Fund, analysis of structural balance in Shakespeare’s Macbeth, etc. This course will help students internalize practical and effective mathematical methods and tools of discrete mathematical models, and apply them to solve many real world problems.

Section 2. Course goals, learning objectives, expected learning outcomes

Course goals. By the end of this course students:

1. will know and understand basics of graph theory, matching theory, theory of binary relations, social choice theory, fair division theory, theory of power distribution and structural balance, and game theory, main stages of mathematical modeling;
2. will be able to apply methods and algorithms for practical purposes, i.e. to find a stable matching, collective decisions by different rules, fair divisions, Nash equilibria, etc.;
3. will be able to solve problems similar to those presents in lectures and textbooks;
4. will be able to prove formal statements on topics covered and present proofs in compact and complete way;
5. will be able to model and analyze real-life problems from the economics, politics, social issues and everyday life;
6. will be able to read related literature and find needed information for further education.

Prerequisites. Students should have some background in mathematics:

1. know terminology of set theory,
2. be able to make operations over sets,
3. be able to make operations over matrices,
4. know and be able to use logic symbols in order to write down statements,
5. understand the structure of mathematical statements, the difference between necessary and sufficient conditions,
6. understand the mathematical notations and be able to read math textbooks,
7. use Internet searching systems to find literature and data for real world problems analysis.

This course will be helpful in studying:
1. game theory,
2. social choice theory,
3. political analysis,
4. microeconomics.

Section 3. Course Outline

<table>
<thead>
<tr>
<th>№</th>
<th>Topic/Focus/Activity</th>
<th>Week</th>
<th>Course format (in hours)</th>
<th>Readings and assignments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>lectures</td>
<td>seminars</td>
</tr>
<tr>
<td>2</td>
<td>Binary relations, graphs and utility functions</td>
<td>4</td>
<td>4</td>
<td>Binary relations and their properties. Operations on binary relations. Interpretation of binary relations and its properties by graphs. Special types of binary relations: partial order, weak order, linear order. Indifference relation and its properties for special cases of</td>
</tr>
</tbody>
</table>
binary relations. Model of ordinal utility.  

Literature:  
1. [2] (Chapters 2 and 3).  


Literature:  
1. [2]  
2. [15, 22]


Literature:  
1. Basic textbook: [1]  
2. Additional readings: [3,7,10,11,12,13,16]


Literature:  
1. Basic textbook: [2]  
2. Additional readings: [2,6,9,12]

History of fair division problem. «Cut-

**Literature:**
1. Basic textbook: [18] (section 9).
2. Additional readings: [7]

<table>
<thead>
<tr>
<th>7</th>
<th>Game theory</th>
<th>4</th>
<th>4</th>
</tr>
</thead>
</table>


**Literature:**
1. Basic textbook: [19]
2. Additional readings: [21,23]

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**Description of course methodology and forms of assessment to be used:**

<table>
<thead>
<tr>
<th>Type of grading</th>
<th>Type of work</th>
<th>Module</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Running (week)</td>
<td>Homework</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Final</td>
<td>Examination</td>
<td></td>
<td>At the end</td>
</tr>
</tbody>
</table>
Section 4. Texts, readings and other informational resources

Basic textbooks:

Additional readings:
5.1. Examples list of questions for homework and exam

Example of homework

1. Prove that $A \cup B \subseteq C \iff A \subseteq C$ and $B \subseteq C$.

2. Let $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the preferences are:

   \[
P(m_1) = w_3, w_2, w_1, w_4; \quad P(w_1) = m_4, m_3, m_2, m_1; \\
P(m_2) = w_4, w_2, w_3, w_1; \quad P(w_2) = m_3, m_2, m_4, m_1; \\
P(m_3) = w_4, w_3, w_1, w_2; \quad P(w_3) = m_3, m_4, m_1, (w_3), m_2; \\
P(m_4) = w_2, w_4, (m_4), w_1, w_3; \quad P(w_4) = m_2, m_1, m_4, m_3.
\]

Is the matching below stable?

\[
\mu = \begin{pmatrix}
    w_2 & w_4 & w_3 & w_1 \\
    m_1 & m_2 & m_3 & m_4
\end{pmatrix}
\]

3. Let $M = \{m_1, m_2, m_3, m_4, m_5\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the preferences are:

   \[
P(m_1) = w_3, w_1, w_2, w_4; \quad P(w_1) = m_1, m_3, m_2, m_4, m_5; \\
P(m_2) = w_4, w_3, w_1, w_2; \quad P(w_2) = m_3, m_1, m_2, m_5, m_4; \\
P(m_3) = w_4, w_3, w_1, w_2; \quad P(w_3) = m_5, m_4, m_1, m_2, m_3; \\
P(m_4) = w_1, w_4, w_2, w_3; \quad P(w_4) = m_1, m_5, m_4, m_3, m_2; \\
P(m_5) = w_1, w_2, w_4, (m_5), w_3; \\
\]

Construct the matching $\mu_M$ and $\mu_W$.

4. Let $A$ be a nonempty final set and the utility function be $u: A \to R_+$. The binary relation $P$ is defined as follows: $xPy \iff u(x) - u(y) > \varepsilon$, where $\varepsilon$ is a fixed positive number. What properties
has this binary relation \( P \)?

5. Prove that binary relation \( P \) is transitive if and only if \( P^2 \subseteq P \).

6. Construct an example of indifference relation for irreflexive semitransitive relation that is not total.

7. Construct a majority graph based on the preferences of agents \( N = \{1,2,3,4\} \) over the candidates \( A = \{x_1, x_2, x_3, x_4, x_5\} \):

\[
\begin{align*}
P_1 &: x_5 > x_1 > x_4 > x_3 > x_2; \\
P_2 &: x_1 > x_5 > x_3 > x_4 > x_2; \\
P_3 &: x_4 > x_1 > x_2 > x_5 > x_3; \\
P_4 &: x_5 > x_1 > x_3 > x_4 > x_2.
\end{align*}
\]

Is there a Condorcet winner?

8. A company of three friends decides how to spend the evening. There have four alternatives: to go to the disco (D), to go to the cinema (C), to go to the theatre (T) or to go to the photo exhibition (F). The preferences of friends are:

<table>
<thead>
<tr>
<th></th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>C</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>D</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>

What will be a collective decision if the maximin procedure is used? What will be a collective decision if a minimax procedure is used?

Example of final exam

1. Find the maximal internally stable sets for weak order. How can we find its number of internal stability?

2. Find a core of presented graph or show that it does not exist.
3. Find the winning coalitions in voting with quota (5; 2,1,1,1,1,1,1,1,1) and find the Banzhaf index for all agents.

4. Board of Directors consists of five persons P, A, B, C, D. The President of Bank P has 3 votes and the other members of the Board of Directors have one vote for each other. The decision-making rule is “at least 5 vote for”. It is known that P and vice-presidents A and B never vote together for the same decision for some reasons. Find Banzhaf index for all members of the Board of Directors.

5. The firm has to rent office rooms in Moscow. The firm’s representative and the renter discuss the following questions: the price of 1 m², the area of the office, the duration of rent, the cost of the routine maintenance, number of parking places. The importance of these questions for the firm and for the renter is in the Table below.

<table>
<thead>
<tr>
<th>Question</th>
<th>Firm</th>
<th>Renter</th>
</tr>
</thead>
<tbody>
<tr>
<td>the price of 1 m²</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>the area of the office</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>the duration of rent</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>the cost of the routine maintenance</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>number of parking places</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

Construct the fair decision using the “adjusted winner” procedure if all items (questions) are divisible.

6. Two firms produce chocolate. The number of customers of each firm is roughly the same. If they do not advertise their product then their profits are the same and equal to 100,000 rubles. The advertisement costs 20,000 rubles, and if two firms advertise then their income will increase by 10,000 rubles, so the profit is 90,000 rubles. If only one firm advertise, then its profit will be 140,000 rubles and the profit of other firm will be 60,000 rubles. Construct the payoff matrix and find all Nash equilibria in pure strategies. Are they Pareto-optimal?

7. Find Nash equilibria in pure and mixed strategies in game with payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>(3,1)</th>
<th>(0,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>(2,2)</td>
<td></td>
</tr>
</tbody>
</table>

**5.2. Grading system and how both the course and final test will be graded**
The lecturer evaluates students’ work on seminars and this grade $O_{sem}$ will be announced to students before the final exam. There is one homework in the middle of the course after the completion of the 4th topic. Students are encouraged to work together to help each other in understanding the course material and completing the homework problems. However, everybody has to write up his/her own solutions. Late homework will not be accepted. The common mistakes made in the homework will be discussed during the seminars.

The cumulative grade before the final exam includes the grade $O_{sem}$ and the grade for the homework $O_{hw}$:

$$O_{cum} = 0.3* O_{sem} + 0.7* O_{hw} .$$

The final grade for the course is formed

$$O_{final} = 0.3* O_{cum} + 0.7* O_{exam}$$

and will be rounded off by the following rule

<table>
<thead>
<tr>
<th>The final grade before rounding</th>
<th>The final grade after rounding</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1-1.4</td>
<td>1</td>
</tr>
<tr>
<td>1.5-2.4</td>
<td>2</td>
</tr>
<tr>
<td>2.5-3.9</td>
<td>3</td>
</tr>
<tr>
<td>4-4.4</td>
<td>4</td>
</tr>
<tr>
<td>4.5-5.4</td>
<td>5</td>
</tr>
<tr>
<td>5.5-6.4</td>
<td>6</td>
</tr>
<tr>
<td>6.5-7.4</td>
<td>7</td>
</tr>
<tr>
<td>7.5-8.4</td>
<td>8</td>
</tr>
<tr>
<td>8.5-9.4</td>
<td>9</td>
</tr>
<tr>
<td>9.5-10</td>
<td>10</td>
</tr>
</tbody>
</table>

The correspondence between the final grade in 10-point scale and in 5-point scale is as follows:

<table>
<thead>
<tr>
<th>the final grade in 10-point scale</th>
<th>the final grade in 5-point scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Insufficient</td>
</tr>
<tr>
<td>2</td>
<td>Insufficient</td>
</tr>
<tr>
<td>3</td>
<td>Insufficient</td>
</tr>
<tr>
<td>4</td>
<td>Satisfactory</td>
</tr>
<tr>
<td>5</td>
<td>Satisfactory</td>
</tr>
<tr>
<td>6</td>
<td>Good</td>
</tr>
<tr>
<td>7</td>
<td>Good</td>
</tr>
<tr>
<td>8</td>
<td>Excellent</td>
</tr>
<tr>
<td>9</td>
<td>Excellent</td>
</tr>
<tr>
<td>10</td>
<td>Excellent</td>
</tr>
</tbody>
</table>

There is no possibility to get an extra point on the resit to compensate the low cumulative grade.

**Section 6. Academic Integrity**
The Higher School of Economics strictly adheres to the principle of academic integrity and honesty. Accordingly, in this course there will be a zero-tolerance policy toward academic dishonesty. This includes, but is not limited to, cheating, plagiarism (including failure to properly cite sources), fabricating citations or information, tampering with other students’ work, and presenting a part of or the entirety of another person’s work as your own. HSE uses an automated plagiarism-detection system to ensure the originality of students’ work. Students who violate university rules on academic honesty will face disciplinary consequences, which, depending on the severity of the offense, may include having points deducted on a specific assignment, receiving a failing grade for the course, being expelled from the university, or other measures specified in HSE’s Internal Regulations.