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Jacques-François Thisse, Philip Ushchev

WHEN CAN A DEMAND SYSTEM BE DESCRIBED BY A MULTINOMIAL LOGIT WITH INCOME EFFECT?

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When can a demand system be described by a multinomial logit with income effect?*

Jacques-François Thisse† Philip Ushchev‡

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Abstract

We show that a wide class of demand systems for differentiated products, such as those generated by additive preferences, indirectly additive preferences, and Kimball-like homothetic preferences, can be given a multinomial logit foundation provided that the conditional indirect utility is nonlinear and varies with the whole price array.

Keywords: discrete choice, multinomial logit, demand systems, additive preferences, homothetic preferences

JEL classification: D43, L11, L13.

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†National Reseach University Higher School of Economics, CORE-UCLouvain and CEPR. Email: jacques.thisse@uclouvain.be

‡National Reseach University Higher School of Economics. Email: fuschev@hse.ru

1 Introduction

Models of imperfect competition used in trade, growth and macroeconomics typically rely on a representative consumer (RC) whose preferences play two distinct roles: they generate the market demands for goods and are used to assess the social desirability of policies. For this approach to be justified, RC demands must stem from the aggregation of individual decisions made by heterogeneous consumers. Otherwise, it is hard to figure out what the implications of policies are, as the welfare gains or losses of a RC have no meaning per se. Indeed, a RC can be "unrepresentative" in that she can be better off under some change in the economy, whereas most or all members of the underlying population are worse off (Kirman, 1992). In other words, working only with a RC can lead to misleading results. To assess accurately the impact of various policies, especially their efficiency and redistributive effects, we must know who the heterogeneous consumers whose aggregate demands are those of the RC are.

One natural strategy to micro-found such demand systems is to appeal to discrete choice theory (Berry *et al.*, 1995; Handbury and Weinstein, 2015). Recently, Armstrong and Vickers (2015) have identified a necessary and sufficient condition for a demand system to be consistent with a population of heterogeneous consumers who make mutually exclusive and indivisible choices among differentiated products and an outside good. This approach is in line with the Hotelling tradition where individuals consume one unit of the good (Hotelling, 1929; Perloff and Salop, 1985; Anderson *et al.,* 1995). Adding an outside good through a given reservation price makes the set-up mainly IO-oriented. Our purpose, instead, is to capture income effects that are needed to micro-found several demand systems used in applications of models with imperfect competition.

In this paper, we assume that individuals make *mutually exclusive choices* but *purchase in volume*. This allows us to capture the situation typical in trade, growth and macroeconomic models in which a RC's demands depend on her income due to the budget constraint. Moreover, since preferences are generally not homothetic, the patterns of consumer choices may significantly vary with income. Our first result is in line with recent research which highlights the principal role of non-homotheticity (Fajgelbaum *et al.*, 2011; Handbury, 2013): we show that in the absence of an outside good the market demand system can be described by the standard multinomial logit with income effects *if and only if the representative consumer is endowed with indirectly additive preferences.* Apart from the CES, the classes of additive, indirectly additive and homothetic preferences are disjoint (Samuelson, 1965). Therefore, to cope with RCs who have non-CES additive or homothetic preferences, we must get rid of the independence of irrelevant alternatives (IIA) property.

Rather than appealing to the gallery of discrete choice models developed to obviate the shortcomings of the IIA, we build on the idea that the utility associated with the consumption of a variety depends on the whole range of varieties from which the purchasing decision is made.

Why is such an assumption plausible? Perhaps because the individual ascribes an *intrinsic* value to the freedom of choice (Sen, 1988) and/or is influenced by the menu from which the choice is made (Sen, 1997). In both cases, the individual indirect utility associated with a particular variety depends on its price and a *market aggregate* that reflects the value of the whole set of varieties. This gives rise to choice probabilities which keep the flexibility of the multinomial logit set-up, but where the IIA property no longer holds (Billot and Thisse, 1999). Examples of preferences that can be rationalized along these lines include CARA (Behrens and Murata, 2007), addilog (Bertoletti *et al.*, 2016), linear expenditure system (Simonovska, 2015) and, more generally, additive and indirectly additive preferences (Zhelobodko *et al.*, 2012; Bertoletti and Etro, 2016) and Kimball-like homothetic preferences (Itskhoki and Gopinath, 2010). More generally, our approach can cope with the demand system developed by Arkolakis *et al.* (2015), which encompasses both additive and Kimball-like preferences.

Our results should help to provide a reconciliation between the different modeling strategies used in industrial organization and the contemporary trade literature. They also suggest new perspectives to model discrete choices and underscore the versatility of the $(us)/(us+them)$ principle, which is also confirmed by McFadden and Train (2000) who show that the mixed logit can approximate general random utility models.

The remainder of the paper is organized as follows. Given the pervasiveness of the CES in many economic fields, we discuss in the next section two alternative logit models consistent with this demand system, which both serve as the starting point of our analysis. Section 3 provides the logit foundation of RCs endowed with indirectly additive preferences and shows that these preferences are the only ones which are consistent with the standard multinomial logit. Section 4 presents a random utility model that lies behind the ACDR demand system proposed by Arkolakis *et al.* (2015). To highlight the relevance of our approach, we deal with RCs whose preferences are additive or Kimball-like homothetic, which are increasingly employed in the trade and macroeconomic literature. The merit of our approach is also highlighted by the fact that there is no random utility model that can rationalize a simple additive utility function such as CARA. The concluding section provides a summary and discusses possible extensions.

2 The CES or two discrete choice models

It is well known that the CES demand system

$$X_i(\mathbf{p}, y) = \frac{p_i^{-\sigma}}{\sum_{k=1}^n p_k^{1-\sigma}} y, \qquad i = 1, ..., n,$$

where $\sigma > 1$ is the elasticity of substitution across varieties, can be micro-founded by a logit model with a unit mass of heterogeneous consumers whose individual indirect utility for variety

i is given by

$$V_i(y, p_i) = \ln y - \ln p_i + \varepsilon_i, \qquad i = 1, ..., n, \tag{1}$$

where y is the individual income or expenditure, while the ε_i are Gumbel i.i.d. random variables with standard deviation $\mu\pi/\sqrt{6}$. The corresponding multinomial logit yields the CES in which the elasticity of substitution among varieties is $\sigma = (1 + \mu)/\mu > 1$ (Anderson *et al.*, 1992). Note that here a consumer who chooses variety i buys y/p_i units of this variety, whereas she consumes 1 or 0 unit in Armstrong and Vickers (2015).

Alternatively, it is readily verified that the CES demand system can be generated by the logit model in which the indirect utility of variety i is

$$V_i(\mathbf{p}) = -\ln p_i + \ln P + \varepsilon_i, \qquad i = 1, ..., n, \tag{2}$$

where P is the CES price index given by

$$P \equiv \left(\sum_{k=1}^{n} p_k^{1-\sigma} \right)^{1/(1-\sigma)}. \tag{3}$$

In (2), the utility of variety i is pinned down by its "real price" p_i/P within the range of varieties, which reflects the idea that the consumer is better off when her most-preferred variety is also a good deal:

$$\mathbb{P}_i = \frac{p_i^{1-\sigma}}{\sum_{k=1}^{n} p_k^{1-\sigma}} = \left(\frac{p_i}{P}\right)^{1-\sigma}, \qquad i = 1, ..., n. \tag{4}$$

If the price p_i remains unchanged, the probability $\mathbb{P}_i$ rises with the price index P because variety i becomes relatively more attractive.

Although the two models are observationally equivalent in the sense that they yield the same market demand system, (1) and (2) describe two different facets of the individual choice process. In the former, the consumer focuses on her budget constraint while, in the latter, she cares about real prices. Moreover, in (1) the indirect utility of variety i depends on the price of this variety only, whereas in (3) it depends on the entire price schedule through the price index P. In what follows, we extend (1) to micro-found *indirectly additive preferences* and (3) to provide discrete choice foundations to demand systems stemming from *additive* or *Kimball homothetic* preferences.

3 Indirectly additive preferences

Consider a RC who has indirectly additive preferences:

$$\mathcal{V}(\mathbf{p}, y) = \sum_{k=1}^{n} v\left(\frac{y}{p_k}\right), \tag{5}$$

where $v(z)$ is differentiable. For $\mathcal{V}$ to satisfy the properties of an indirect utility, we also assume that v is strictly increasing and strictly concave. By Roy's identity, the RC's demands are given by

$$X_i(\mathbf{p}, y) = \frac{y}{p_i} \frac{v'(y/p_i) \cdot y/p_i}{\sum_{k=1}^{n} v'(y/p_k) \cdot y/p_k}, \qquad i = 1, ..., n. \tag{6}$$

If the elasticity of $v'(z)$ is larger than -1, the demand for variety i decreases with its own price and increases with income (Bertoletti and Etro, 2016).

To show how the demand system (6) can be micro-founded, consider the following random utility model:

$$V_i = \ln \psi\left(\frac{y}{p_i}\right) + \varepsilon_i, \qquad i = 1, ..., n, \tag{7}$$

where ε_i are Gumbel i.i.d. random variables with standard deviation equal to $\sqrt{6}\mu/\pi$, while

$$\psi(z) = \left[zv'(z)\right]^{\mu}.$$

The indirect utility (7) boils down to (1) when $\psi(z) = z$ and $\mu = 1/(\sigma - 1)$. Hence, indirectly additive preferences may be viewed as a parsimonious extension of the CES.

Furthermore, we have

$$\frac{\partial \mathbb{E}(V_i - V_j)}{\partial p_k} = 0,$$

which means that (7) satisfies the IIA.

The corresponding multinomial logit probabilities are given by

$$\mathbb{P}_i = \frac{\psi^{1/\mu}(y/p_i)}{\sum_{k=1}^{n} \psi^{1/\mu}(y/p_k)}, \qquad i = 1, ..., n. \tag{8}$$

In this expression, μ measures the degree of heterogeneity across consumers. When μ goes to 0, $\mathbb{P}_i = 1$ if i is the cheapest variety, and 0 otherwise; when μ goes to ∞, $\mathbb{P}_i = 1/n$ because consumers do not care anymore about price differences.

The probability of choosing i strictly decreases in p_i and strictly increases in p_k for all $k \neq i$. Furthermore, unlike in the CES, an income hike affects choice probabilities when $\psi(z)$ is not a power function. To be precise, assume that the elasticity of $\psi(z)$ is decreasing.[1] In this event, the probability of buying expensive varieties increases with income because their marginal

[1] This assumption renders demand functions "increasingly elastic." This property is appealing because it gives rise to a pro-competitive income effect under monopolistic competition (Bertoletti and Etro, 2016).

utility is higher, whereas the probability of choosing cheap varieties decreases for the opposite reason (Appendix A). In other words, *as individuals get richer, they shift their consumption from cheaper to more expensive varieties.*

The following proposition provides the discrete choice foundation of preferences (5).

Proposition 1. *Assume* $\psi(z) = [zv'(z)]^{\mu}$. *Then, the multinomial logit model* (8), *in which each consumer buys a single variety* i *in volume* y/p_i, *micro-founds the demand systems* (6) *generated by indirectly additive preferences* (5).

Indeed, the expected demand for variety i being given by

$$X_i \equiv \frac{y}{p_i}\mathbb{P}_i, \qquad i = 1, ..., n,$$

setting $\psi(z) \equiv [zv'(z)]^{\mu}$ in (7) yields (6).

Conversely, for any multinomial logit model (7), there exists a RC endowed with indirectly additive preferences (5) where

$$v(x) = \int_0^x \frac{\psi^{1/\mu}(z)}{z}\mathrm{d}z$$

if the elasticity of ψ is smaller than μ, which is needed for v to be concave.

To sum up, there is a one-to-one correspondence between a RC having indirectly additive preferences and a population of heterogeneous consumers having logit probabilities in which the conditional indirect utility of variety i depends only upon the ratio y/p_i.

Remark 1. Ever since Eaton and Kortum (2002), it has become increasingly popular in the trade and urban economics literature to use the Fréchet distribution rather than the Gumbel distribution (Bryan and Morten, 2015; Redding, 2015; Tombe and Zhu, 2015). Therefore, we find it appropriate to show here that our approach can be equivalently restated as a multiplicative random utility model in which the disturbance terms follow the Fréchet distribution.

The c.d.f. of a Gumbel distribution whose mean is $m + \gamma\mu$ (γ being the Euler-Mascheroni constant, $\gamma \approx 0.577216$) and standard-deviation $\mu\pi/\sqrt{6} > 0$, is given by

$$G(x) = \exp\left[-\exp\left(-\frac{x - m}{\mu}\right)\right],$$

where $x \in \mathbb{R}$. Given the scale parameter $T > 0$ and the shape parameter $\theta > 0$, the c.d.f. of a Fréchet distribution is given by

$$F(x) = \begin{cases} \exp\left(-Tx^{-\theta}\right), & x > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

where $z > 0$. Its mean is given by $\exp(\gamma/\theta)T^{-1/\theta}$ while its log has a standard deviation $\pi/(\theta\sqrt{6})$.

The function $G(x)$ may be rewritten as follows:

$$G(x) = \exp\left(-Tz^{-1/\mu}\right)$$

which is equal to (9) if we set

$$T = \exp\left(\frac{m}{\mu}\right) \qquad x = \ln z \qquad \text{and} \qquad \theta = 1/\mu.$$

Therefore, (7) can be recast in terms of the Fréchet distribution as follows:

$$V_i = \psi\left(\frac{y}{p_i}\right)\epsilon_i, \qquad i = 1, ..., n, \tag{10}$$

where the ϵ_i are Fréchet i.i.d. random variables. It is then readily verified that the choice probabilities are still given by (8), where $\theta = 1/\mu$ keeps measuring the dispersion of consumers' tastes. The same applies to the results obtained in the sections below.

Remark 2. Competition among firms may be viewed as a large number of *contests* in which the contender/firm i receives a prize equal to its per-variety profit $y(p_i - c)/p_i$ when it wins/supplies a particular consumer, c being the marginal production cost. The share of contests firm i wins is equal to (8) (Jia *et al.*, 2013). Note that the income y affects here the structure of contests because richer consumers purchase expensive varieties more frequently than cheap varieties. All of this suggests the possibility of bridging these two strands of literature.

4 Multinomial logit with cross-effects

As shown by Proposition 1, working with a multinomial logit in which conditional indirect utilities vary only with own prices makes it impossible to cope with demand systems more general than (6). A seemingly natural way out is to consider a non-linear and non-additive random utility model:

$$V_i = \Psi\left(\frac{y}{p_i}, \varepsilon_i\right) \tag{11}$$

where Ψ is increasing while the ε_i obey any multivariate distribution. Surprisingly, despite its generality (11) does not allow the micro-founding of additive preferences such as CARA (Appendix B).[2] This impels us to choose a different modeling strategy.

[2]It is readily verified that the demand system generated by CARA with an outside good does not satisfy the necessary (and sufficient) condition obtained by Armstrong and Vickers (2015) for a demand system to be consistent with a random utility model in which consumers buy one unit of a single variety, or not at all.

4.1 The ACDR demand system

Consider the ACDR demand system proposed by Arkolakis *et al.* (2015):

$$X_i(\mathbf{p}, y) = AD\left(\Lambda p_i\right), \qquad i = 1, ..., n, \tag{12}$$

where $A(\mathbf{p}, y)$ and $\Lambda(\mathbf{p}, y)$ are two *market aggregates* such that the budget constraint holds, while $D(z)$ is a strictly decreasing function whose elasticity is larger than 1. Note that Arkolakis *et al.* (2015) do not impose any functional restriction on D. Since D is independent of the variety i, the demand system (12) is derived from symmetric preferences. In addition, $D\left(\Lambda p_i\right)$ does not depend directly on the RC's income. However, we will show that the market aggregate Λ may depend on y.

The budget constraint implies that the demand functions may be rewritten as follows:

$$X_i(\mathbf{p}, y) = \frac{y}{p_i} \frac{p_i D\left(\Lambda p_i\right)}{\sum_{k=1}^n p_k D\left(\Lambda p_k\right)}, \qquad i = 1, ..., n. \tag{13}$$

Observe that the demand system (13) now depends only on *one* aggregate Λ. The intuition behind this aggregate will be given below in two special, but important, cases.

Variety i's market share $\mathbb{P}_i \equiv p_i X_i / y$ is as follows:

$$\mathbb{P}_i = \frac{\Lambda p_i D\left(\Lambda p_i\right)}{\sum_{k=1}^n \Lambda p_k D\left(\Lambda p_k\right)}.$$

This expression has a *logit* structure which displays two appealing features. First, the choice probabilities do not satisfy the IIA property because Λ is a market aggregate. Second, the demand system (12) is consistent with the following random utility model:

$$V_i = -\ln \varphi(\Lambda p_i) + \varepsilon_i, \qquad i = 1, ..., n, \tag{14}$$

where the ε_i are Gumbel i.i.d. random variables with standard deviation $\mu\sqrt{6}/\pi$, while $\varphi(\cdot)$ is the following strictly increasing function:

$$\varphi(z) \equiv \left[zD(z)\right]^{-\mu}. \tag{15}$$

Hence, the functional form of φ is determined by that of D, and vice versa. The corresponding choice probabilities are given by

$$\mathbb{P}_i = \frac{\varphi^{-1/\mu}(\Lambda p_i)}{\sum_{k=1}^n \varphi^{-1/\mu}(\Lambda p_k)}, \qquad i = 1, ..., n. \tag{16}$$

and thus the expected market demands $X_i \equiv \mathbb{P}_i y / p_i$ are equal to (13). In particular, when $\Lambda = 1/P$, where P is the CES price index, and $\varphi(\cdot)$ a power function, (16) boils down to (4).

In (16), Λ may be interpreted as a demand shifter that affects the ratio between any two probabilities. To illustrate, assume $n = 2$ and $p_1 > p_2$. Then,

$$\mathbb{P}_1 = \frac{1}{1 + [\varphi(\Lambda p_2)/\varphi(\Lambda p_1)]^{-1/\mu}}.$$

This probability decreases with Λ if and only if the elasticity of $\varphi^{-1/\mu}$ with respect to Λ is increasing. More generally, we show in Appendix C that for $n > 2$ a hike in Λ lowers the probability of purchasing more expensive varieties and raises the probability of choosing cheaper ones if and only if the elasticity of φ with respect to Λ is decreasing. In other words, a higher Λ generates effects similar to those triggered by a lower income in the above section. This is in accordance with the predictions of Proposition 1. Indeed, setting $\Lambda = 1/y$ we fall back on (7).

Proposition 2. *Assume $\varphi(z) \equiv [zD(z)]^{-\mu}$. Then, the ACDR demand system is micro-founded by the multinomial logit model* (16).

Despite some resemblance, Propositions 1 and 2 are very different. In Proposition 1, the conditional indirect utility $\psi(y/p_i)$ of variety i depends on its *own* price only. In Proposition 2, the conditional indirect utility $\varphi(\Lambda p_i)$ depends on the *whole* price vector $\mathbf{p}$ via the aggregate $\Lambda(\mathbf{p}, y)$. It is legitimate to ask why we need such cross-effects. The reason has been given above: we cannot micro-found non-CES additive preferences by discrete choice models in which the deterministic part of the conditional indirect utility associated with a variety depends on its price alone. In contrast, when cross-effects are taken into account through the aggregate Λ, the multinomial logit is sufficiently general to generate a very broad family of demand systems. We illustrate the relevance of this claim in two special, but widely-used, cases in which the RC is endowed with additive or Kimball-like homothetic preferences, and determine the aggregate for each type of preferences.

4.2 Additive preferences

Consider now a RC who has additive preferences:

$$U(\mathbf{X}) = \sum_{k=1}^{n} u(X_k). \tag{17}$$

where $u(z)$ is differentiable, strictly increasing and strictly concave. In this case, the RC's demand functions are given by

$$X_i(\mathbf{p}, y) = \frac{y}{p_i} \frac{\lambda p_i \left(u'\right)^{-1}(\lambda p_i)}{\sum_{k=1}^{n} \lambda p_k \left(u'\right)^{-1}(\lambda p_k)}, \qquad i = 1, ..., n, \tag{18}$$

where $z \cdot \left(u'\right)^{-1}(z)$ is strictly decreasing in z, while $\lambda(y, \mathbf{p})$ is the RC's marginal utility of income determined by

$$\frac{1}{y}\sum_{k=1}^{n} p_k \left(u'\right)^{-1}(\lambda p_k) = 1. \tag{19}$$

Since $u(z)$ is strictly increasing and strictly concave, for any given $\mathbf{p}$ and y (19) has a unique positive solution $\lambda(\mathbf{p}, y)$.

The demand system (18) can be obtained as a special case of (13) by setting $D(z) = \left(u'\right)^{-1}(z)$ and $\Lambda = \lambda$, as defined by (19). For example, under CARA we have $D(z) = (\ln\alpha - \ln z)/\alpha$, while $D(z) = (1-\beta z)/z$ gives rise to the linear expenditure system. Plugging $D(z) = \left(u'\right)^{-1}(z)$ and $\Lambda = \lambda$ into (15), we obtain the corresponding choice probabilities:

$$\mathbb{P}_i = \frac{\varphi^{-1/\mu}(\lambda p_i)}{\sum_{k=1}^{n}\varphi^{-1/\mu}(\lambda p_k)}, \qquad i = 1, ..., n, \tag{20}$$

where $\varphi(z) = \left[z\cdot\left(u'\right)^{-1}(z)\right]^{-\mu}$. Consequently, the expected market demands $X_i = \mathbb{P}_i y/p_i$ are micro-founded by the following random utility model:

$$V_i = -\ln\left[\lambda p_i\cdot\left(u'\right)^{-1}(\lambda p_i)\right] + \varepsilon_i, \qquad i = 1, ..., n.$$

In sum, we end up with the following proposition.

Proposition 3. *Assume* $\varphi(z) = \left[z\cdot\left(u'\right)^{-1}(z)\right]^{-\mu}$. *Then, the multinomial logit model* (16), *in which each consumer buys a single variety* i *in volume* y/p_i, *while* Λ *is the solution* $\lambda(\mathbf{p}, y)$ *to* (19), *micro-founds the demand systems* (18) *generated by additive preferences* (17).

The meaning of the aggregate Λ in (12) should now be clear: Λ is the marginal utility λ of income if and only if the RC's preferences are additive. Since a high (low) value of Λ means that the budget constraint is tight (slack), the probability of buying a cheap (expensive) variety rises at the expense of the expensive (cheap) varieties.

While indirectly additive preferences are associated with the standard logit, additive preferences involve cross-effects among varieties' prices. This concurs with the Generalized Extreme Value approach developed by McFadden (1978) where the deterministic part of V_i depends on the attributes of all alternatives. Comparing (8) and (20) shows that the functional form of the choice probabilities is determined by the marginal indirect utility $v'(z)$ when preferences are indirectly additive, and by the inverse $\left(u'\right)^{-1}(z)$ of the marginal utility in the case of additive preferences.

4.3 Homothetic preferences à la Kimball

Finally, consider a RC whose utility function $U(\mathbf{X})$ is homothetic and described by Kimball's flexible aggregator (Kimball, 1995; Sbordone, 2007). For any strictly increasing and strictly concave function $\mathcal{K}(\cdot)$ such that $\mathcal{K}(0) < 1/n < \mathcal{K}(\infty)$, there exists a homothetic utility $U(\mathbf{X})$

that satisfies:

$$\sum_{k=1}^{n} \mathcal{K}\left(\frac{X_k}{U(\mathbf{X})}\right) = 1 \tag{21}$$

for any consumption vector $\mathbf{X}$. The inverse demand for variety i is given by

$$p_i = \frac{1}{\lambda}\frac{\partial U}{\partial X_i} = \frac{1}{\lambda} \cdot \frac{\mathcal{K}'\left(\frac{X_i}{U(\mathbf{X})}\right)}{\sum_{k=1}^{n} \frac{X_k}{U(\mathbf{X})}\mathcal{K}'\left(\frac{X_k}{U(\mathbf{X})}\right)}, \tag{22}$$

which depends on both the utility level U and the marginal utility of income λ.

For any homothetic preferences, we have:

$$\lambda(\mathbf{p}) = \frac{1}{P(\mathbf{p})} \qquad \mathcal{V}(y, \mathbf{p}) = \frac{y}{P(\mathbf{p})}, \tag{23}$$

where P is the ideal price index, while $\mathcal{V}$ is the indirect utility. Furthermore, Roy's identity implies

$$X_i(\mathbf{p}, y) = \frac{y}{P}\frac{\partial P}{\partial p_i}. \tag{24}$$

Since $U = \mathcal{V}$ at the consumer's optimum, plugging (23)-(24) into (22) gives the expression:

$$\frac{p_i}{\mathcal{P}} = \mathcal{K}'\left(\frac{PX_i}{y}\right). \tag{25}$$

where $\mathcal{P}$ is a price aggregate defined as follows:

$$\mathcal{P} \equiv \frac{P}{\sum_{k=1}^{n} \frac{\partial P}{\partial p_k}\mathcal{K}'\left(\frac{\partial P}{\partial p_k}\right)}.$$

Note that $\mathcal{P}$ differs from the ideal price index P, except in the CES case in which both coincide.

Solving (25) for X_i yields the demand functions:

$$X_i(\mathbf{p}) = \frac{y}{P}(\mathcal{K}')^{-1}\left(\frac{p_i}{\mathcal{P}}\right). \tag{26}$$

Combining (26) with (21), we obtain

$$\sum_{k=1}^{n} \mathcal{K}\left[(\mathcal{K}')^{-1}\left(\frac{p_k}{\mathcal{P}}\right)\right] = 1, \tag{27}$$

which uniquely pins down the price aggregate $\mathcal{P}(\mathbf{p})$. The properties of $\mathcal{K}(z)$ imply that equation (27) has at most one solution, while $\mathcal{K}(0) < 1/n < \mathcal{K}(\infty)$ and the intermediate value theorem yield that (27) has at least one solution. In addition, it follows from (27) that $\mathcal{P}$ is an increasing and homogeneous linear function of the price vector $\mathbf{p}$ and is independent of y. As a consequence, very similarly to the case of additive preferences, the demands given by (26) depend on a single aggregate.

The choice probabilities $\mathbb{P}_i = p_i X_i / y$ are then given by

$$\mathbb{P}_i = \frac{\frac{p_i}{\mathcal{P}} (\mathcal{K}')^{-1} \left(\frac{p_i}{\mathcal{P}}\right)}{\sum_{k=1}^{n} \frac{p_k}{\mathcal{P}} (\mathcal{K}')^{-1} \left(\frac{p_k}{\mathcal{P}}\right)}, \tag{28}$$

which can be obtained directly from (16) by setting

$$\varphi(z) \equiv z \cdot (\mathcal{K}')^{-1}(z) \qquad \Lambda(\mathbf{p}) = \frac{1}{\mathcal{P}(\mathbf{p})},$$

where Λ is no longer the RC's marginal utility of income, unless preferences are CES.

The following proposition provides a summary.

Proposition 4. *Assume* $\varphi(z) = [z \cdot (\mathcal{K}')^{-1}(z)]^{-\mu}$. *Then, the multinomial logit model* (16), *in which each consumer buys a single variety* i *in volume* y/p_i *and* $\Lambda = 1/\mathcal{P}$ *is implicitly defined by* (27), *micro-founds the demand systems generated by Kimball-like preferences* (21).

Since preferences are now homothetic, the income level has no impact on choice probabilities. By contrast, the value of the price aggregate matters. When $\mathcal{P}$ is high (low), varieties are on average expensive (cheap). This increases (decreases) the probability of choosing a cheap (expensive) variety.

Observe that Propositions 3 and 4 are special cases of Proposition 2 when $D(z) = (u')^{-1}(z)$ and $D(z) = (\mathcal{K}')^{-1}(z)$, respectively. Under both additive and Kimball-like preferences, the RC's demands and individual demands depend on a *single* market aggregate. Note, however, the following difference between the two demand systems: under additive preferences, the aggregate λ depends on income, and so do the choice probabilities. In contrast, under Kimball-like preferences $\mathcal{P}(\mathbf{p})$ is truly a *price aggregate*, a feature that stems from homotheticity. Finally, Proposition 1 may also be viewed as a special case of Proposition 2 if we set $D(z) = v'(z)$ and $\Lambda(y, \mathbf{p}) = 1/y$. Thus, the aggregate varies with income but not with prices, which makes the case of a RC with indirectly additive preferences somewhat "polar" to that of Kimball-like preferences.

5 Summary and extensions

The above analysis shows that a wide range of demand systems can be bolstered by the multinomial logit when it is recognized that the conditional indirect utility of a product depends on *the income and the prices of all products*. As a consequence, we are able to maintain the flexibility of the most popular ever discrete-choice model (i.e. the logit) while working with rich patterns of substitution among differentiated products. As suggested by Sattinger (1984), we are able to micro-found a great many demand systems by allowing consumers to purchase in volume. One distinctive feature of our approach is that it generates aggregative oligopoly games which display especially appealing properties (Acemoglu and Jensen, 2013; Anderson *et al.*, 2015). When the number of varieties is uncountable, we fall back on monopolistic competition. In this case, the aggregates entering the demand functions (12) are still endogenous. Although firms treat the aggregates parametrically, the market outcome captures interactions among firms that are absent in the CES case (Bertoletti *et al.*, 2016; Zhelobodko *et al.*, 2012).

Our approach can be extended in several directions. First, the demand system (13) is symmetric because the function $D(\cdot)$ is the same across varieties. Asymmetric patterns of substitution among varieties may be generated by making the taste parameters μ_i variety-specific. A parsimonious way to account for vertical differentiation proposed by Fajgelbaum *et al.* (2011) is to assume that high-quality products are more differentiated than low-quality products. Ranking varieties by decreasing order of quality, we have $\mu_1 > \mu_2 > ... > \mu_n$ in the choice probabilities (8). It should be stressed, however, that quality is an ordinal concept, which makes it difficult to pin down a more precise relationship between the value of μ_i and the quality level of variety i.

Second, building on the mixed logit model developed by McFadden and Train (2000), we may introduce an additional source of heterogeneity across consumers who have different incomes while the ε_i are drawn from Gumbel distributions in which the taste parameter μ varies with the consumer-type. A first step in this direction has been taken by Osharin *et al.* (2014) who combine the CES and the mixed logit to study the joint impact of taste and income heterogeneity on the market outcome.

Last, under which conditions can a demand system generated by a RC endowed with general symmetric preferences be micro-founded by a discrete choice model? In particular, what are the conditions to be imposed on function f for the demand system

$$X_i(\mathbf{p}, y) = \frac{y}{p_i} \frac{p_i f(\Lambda, p_i)}{\sum_{k=1}^{n} p_k f(\Lambda, p_k)}, \qquad i = 1, ..., n,$$

to be micro-founded? We leave these questions for future research.

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Appendix

A. The behavior of choice probabilities (8) with respect to income.

Using (8), it is readily verified that the elasticity of $\mathbb{P}_i$ with respect to y satisfies the following condition:

$$\mathcal{E}_y(\mathbb{P}_i) = \frac{1}{\mu}\left[\mathcal{E}_{y/p_i}(\psi) - \sum_{k=1}^{n} \mathbb{P}_k \mathcal{E}_{y/p_k}(\psi)\right].$$

Hence, $\mathcal{E}_y(\mathbb{P}_i) > 0$ if and only if the following inequality holds:

$$\mathcal{E}_{y/p_i}(\psi) > \sum_{k=1}^{n} \mathbb{P}_k \mathcal{E}_{y/p_k}(\psi).$$

Since the right-hand side of this inequality is independent of i, we have the following result.

Claim A. *If the elasticity $\mathcal{E}_{y/p_i}(\psi)$ is a decreasing function of y/p_i, then there exists a variety $i_0 \in \{1, ..., n\}$, such that*

$$\mathcal{E}_y(\mathbb{P}_i) > 0 \quad \text{iff } p_i > p_{i_0}.$$

Note that the converse of Claim A holds when the elasticity $\mathcal{E}_{y/p_i}(\psi)$ is an increasing function of y/p_i.

B. CARA cannot be micro-founded by (11).

Consider the following random utility model:

$$V_i = \Psi\left(\frac{y}{p_i}, \varepsilon_i\right), \tag{A.1}$$

where the function $\Psi(\cdot,\cdot)$ and the cumulative distribution $G(\cdot)$ of the random vector $\boldsymbol{\varepsilon}$ are unspecified. We only assume that $\Psi(\cdot,\cdot)$ is a strictly increasing and differentiable function in both variables, while $G(\cdot)$ is absolutely continuous.

Claim B. *Assume that $n \geq 3$. Then, the demand system generated by CARA preferences*

$$U(\mathbf{X}) = -\sum_{j=1}^{n} \exp(-\alpha X_j) \tag{A.2}$$

cannot be generated by (A.1).

The argument is by contradiction. The choice probabilities associated with (A.2) must satisfy

$$\mathbb{P}_i = \Pr(V_i = \max V_k) = \frac{p_i}{\alpha y}\left[\frac{\alpha y - \mathcal{H}(\mathbf{p})}{\mathcal{P}(\mathbf{p})} - \ln p_i\right],$$

where $\mathcal{P}(\mathbf{p})$ and $\mathcal{H}(\mathbf{p})$ are, respectively, the *aggregate price index* and the *entropy* of the price vector $\mathbf{p}$:

$$\mathcal{P}(\mathbf{p}) \equiv \sum_{k=1}^{n} p_k, \qquad \mathcal{H}(\mathbf{p}) \equiv \sum_{k=1}^{n} p_k \ln\left(\frac{1}{p_k}\right).$$

Computing the cross-derivative $\partial\mathbb{P}_i/\partial p_j \partial p_k$, where $i \neq j \neq k$, yields

$$\frac{\partial \mathbb{P}_i}{\partial p_j \partial p_k} = \frac{2}{[\mathcal{P}(\mathbf{p})]^2}\left[\frac{\alpha y - \mathcal{H}(\mathbf{p})}{\mathcal{P}(\mathbf{p})} - 1 + \ln\sqrt{p_j p_k}\right].$$

Evaluating this expression at $(y, \mathbf{p}) = \left(\frac{n}{2\alpha}, \mathbf{1}\right)$, we get:

$$\left.\frac{\partial \mathbb{P}_i}{\partial p_j \partial p_k}\right|_{(y,\mathbf{p})=\left(\frac{n}{2\alpha},\mathbf{1}\right)} = -\frac{1}{n^2} < 0. \tag{A.3}$$

It remains to show that (A.1) and (A.3) are incompatible.

Let $G_{-i}(\cdot)$ be the cumulative distribution function of the random vector $\boldsymbol{\varepsilon}_{-i}$. Observe also that the equation $\Psi(x, \varepsilon) = V$ has a unique solution in ε, and denote this solution $\epsilon(x, V)$. Clearly, the function $\epsilon(\cdot, \cdot)$ is decreasing in the first argument and is increasing in the second. Finally, set

$$\epsilon_k^i \equiv \epsilon\left[\frac{y}{p_k}, \Psi\left(\frac{y}{p_i}, \varepsilon_i\right)\right]. \tag{A.4}$$

Note that $p_i = p_k$ implies $\epsilon_k^i = \varepsilon_i$. Furthermore, because $\epsilon(\cdot, \cdot)$ is decreasing in its first argument, we have

$$\frac{\partial \epsilon_k^i}{p_k} > 0. \tag{A.5}$$

Using (A.4), the choice probabilities generated by (A.1) may be expressed as follows:

$$\mathbb{P}_i = \int_{\infty}^{-\infty} G_{-i}\left(\epsilon_1^i, \ldots, \left[\epsilon_i^i\right], \ldots \epsilon_n^i\right) \mathrm{d}G_i(\varepsilon_i),$$

where $G_i(\cdot)$ is the marginal distribution of ε_i.

Evaluating the cross derivative $\partial\mathbb{P}_i/\partial p_j\partial p_k$ at $\mathbf{p}=\mathbf{1}$ yields

$$\left.\frac{\partial\mathbb{P}_i}{\partial p_j\partial p_k}\right|_{\mathbf{p}=\mathbf{1}} = \left.\left(\frac{\partial\epsilon_j^i}{p_j}\cdot\frac{\partial\epsilon_k^i}{p_k}\right)\right|_{p_{j,k}=1}\int_{\infty}^{-\infty}\left.\frac{\partial^2 G_{-i}}{\partial\varepsilon_j\partial\varepsilon_k}\right|_{\boldsymbol{\varepsilon}_{j,k}=\varepsilon_i}\mathrm{d}G_i(\varepsilon_i). \tag{A.6}$$

Using (A.5) and (A.6) together with the fact that for the cross-derivatives of a cumulative distribution function must be non-negative shows that (A.6) is non-negative. Given (A.3), we come to a contradiction.

C. The behavior of choice probabilities (16) with respect to Λ.

Using an argument similar to the one employed in Appendix A where P_i are given by (16), we can show the following result:

Claim C. *If the elasticity $\mathcal{E}_{\Lambda p_i}\left(\varphi^{-\frac{1}{\mu}}\right)$ is a decreasing function of Λp_i, then there exists a variety $i_0 \in \{1,...,n\}$, such that*

$$\mathcal{E}_{\Lambda}(\mathbb{P}_i) < 0 \quad \text{iff } p_i > p_{i_0}.$$

Contact details:
Philip Ushchev
National Research University Higher School of Economics (Moscow, Russia).
Center for Market Studies and Spatial Economics, leading research fellow.
E-mail: fuschev@hse.ru