Course title

Introduction to functional analysis

Teacher

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Content of the course

This course is a smooth introduction to functional analysis and its applications. The course requires a reasonable background in calculus and linear algebra. The material of the course is intended for students interested in mathematics and their applications to Economics and Data Sciences. The course is organized in two parts. The first part covers fundamental notions of functional analysis while the second covers selected topics in Fourier analysis and its applications as a well as convex analysis and optimization. For additional information on the subject, we highly recommend the reference books by Rudin (1976); Dudley (2004) or Deitmar (2005). More advanced material (which goes beyond the scope of the course) can be found, for instance, in Rudin (1987, 1991) or Brezis (2011). Concerning convex analysis and optimization, we recommend using the recent book by Bubeck (2015).

Part 1 - Fundamental Concepts

1. Normed vector spaces

1.1. Definitions
1.2. Linear operators
1.3. Basic topology
1.4. Denseness, compactness and completeness
1.5. Examples

2. Inner product and Hilbert spaces

2.1. Definitions
2.2. Fundamental identities
2.3. Projection theorem
2.4. Riesz representation theorem
2.5. Examples

3. Integration and approximation

3.1. Riemann vs. Lebesgue integral
3.2. Convergence and differentiation
3.3. Convolution
3.4. Approximation and regularization
3.5. Examples

Part 2 - Selected Topics

4. Fourier series

4.1. Definitions
4.2. Fundamental identities
4.3. Convergence in L^2 sense
4.4. Pointwise convergence
4.5. Applications

5. Fourier transform

5.1. Definitions
5.2. Fundamental identities
5.3. Fourier transform and differentiation
5.4. Fourier transform and convolution
5.5. Applications to partial differential equations

6. Convexity

6.1. Definition and examples
6.2. Convexity and differentiability
6.2. Convexity in probability and statistics
6.4. Gradient algorithms for smooth convex optimization.
6.5. Applications

General instructions

The students are encouraged to spend (on their own or in group discussions) at least twice the time of the lectures to read, understand, learn and challenge the material presented in class. In other words, it is expected that each student will spend at least 2 hours (minimum) reviewing 1 hour of lecture material. It is naturally expected that a serious review of the material studied in class should give rise to many questions. The first 10 to 15 minutes of each lecture will be devoted to the discussion and answering of these questions in the form of a regulated group discussion. The regular homework activities are also intended to answer potential questions of the students and provide additional information.

Evaluation process

The final evaluation grade $\underline{\text{G}}$ of the students will be based on a written midterm test $\underline{\text{T1}}$, a written final test $\underline{\text{T2}}$, group home works $\underline{\text{GHW}}$, individual home works $\underline{\text{IHW}}$ and finally class participation $\underline{\text{CP}}$ according to the formula:

$$\underline{\text{G}} = 0.3\,\underline{\text{T1}} + 0.4\,\underline{\text{T2}} + 0.1\,\underline{\text{GHW}} + 0.1\,\underline{\text{IHW}} + 0.1\,\underline{\text{CP}}.$$

During the course, the students will be given 2 group homeworks and 2 individual homeworks. The group (respectively individual) homework grade $\underline{\text{GHW}}$ (respectively $\underline{\text{IHW}}$) will be computed as the average grade of the 2 assignments.

1. Midterm written test

The midterm written test will consist in a 3 hour test that students will perform individually, at the same time and under the supervision of the teacher. During this test, only a single hand-written A4 sheet of paper (front and back) will be allowed for the students to collect lecture material and results. All other documents or electronic devices will be strictly forbidden. The students are expected to work in total silence and to provide manuscript answers in the order of the given questions, on ruled paper with their name and surname.

2. Final written examination

The final written test will consist in a 3 hour test that students will perform individually, at the same time and under the supervision of the teacher. During this test, only a single hand-written A4 sheet of paper (front and back) will be allowed for the students to collect lecture material and results. All other documents or electronic devices will be strictly forbidden. The students are expected to work in total silence and to provide manuscript answers in the order of the given questions, on ruled paper with their name and surname.

3. Group written reports

The students should work in groups (3 people max) and provide a single document for the group with clear answers to the problem and the names of all group members. The homework should be handed over within 2 weeks, on ruled paper or typed on computer. Reports are expected to be well presented.

4. Individual written reports

The student should work individually and express his ideas and solutions clearly making precise references to the results (seen in class or in the literature) invoked to solve the problem. The homework should be handed over within 2 weeks, on ruled paper or typed on computer, with name and surname. Reports are expected to be well presented.

5. Classroom intellectual engagement and participation

Students are expected to be present at all lectures, arrive in time and pay close attention to lectures. Students are also expected to ask questions and try to answer those given in class by the teacher. At the beginning of each lecture, 10 to 15 minutes will be specifically devoted to the discussion and answering of questions concerning the previous lectures (or homework) in the form of a regulated group discussion.

Make-up, absence and howework policy

• If the final evaluation grade $\underline{G}$ does not meet the minimal requirements, a final written make-up test (3 hours) will be given in september to the concerned students as a final chance to validate the course. In case of success at the make-up test, the students will only be given the minimal grade authorizing to validate the course.

• Potential absence at the written tests should be justified by exceptional medical reasons. In such a situation, official medical certificates are expected to be handed-out as soon as possible. Only for those students able to justify officially their absence, an additional (mid-term or final test) will be prepared. Students unable to justify their absence will unfortunately be given the grade 0 to the related test. An absence at regular lectures should be properly notified in advance by an email to the teacher.

• Howe work assignments will be given in advance with a large preparation time. As a result, home work assignments are expected to be handed out on time and no additional delay will be allowed. Students unable to hand-out their work in time will unfortunately be given the grade 0 to the related assignment.

References

H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2011.

S. Bubeck. Convex optimization: Algorithms and complexity. *Foundations and Trends in Machine Learning*, 8(3-4):231–357, 2015.

A. Deitmar. *A First Course in Harmonic Analysis.* Springer, 2nd edition, 2005.

R. M. Dudley. *Real Analysis and Probability.* Cambridge University Press, 2004.

W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 3rd edition, 1976.

W. Rudin. *Real and Complex Analysis.* McGraw-Hill, 3rd edition, 1987.

W. Rudin. *Functional Analysis.* McGraw-Hill, 2nd edition, 1991.