1. Course Description
   
a. Title of a Course
   
   **Non-Euclidean Geometry**
   
   *Kirichenko V.*
   
b. Pre-requisites
   
   Familiarity with high school geometry, basic logical reasoning skills
   
c. Course Type (compulsory, elective, optional)
   
   Elective (offered both to students enrolled into M.Sc. program in Mathematics at the HSE and to students enrolled into Math in Moscow program)
   
d. Abstract
   
   Geometry studies different models of the real world. Classical models in dimension two are given by Euclidean geometry (geometry of an ideal flat plane), spherical geometry (geometry of the surface of a ball), and hyperbolic geometry, also known as Lobachevskian geometry. The famous Pythagorean theorem holds only in a Euclidean world. This is probably the main reason why we prefer to think of the world around us in terms of Euclidean geometry - this makes calculations easier. Locally, every geometry can be approximated by Euclidean geometry. To discover fascinating facts about non-Euclidean geometries we have to study the global picture. Our geometric intuition alone is not enough to predict what the world might look like far away from us. Hyperbolic geometry is especially counterintuitive (for instance, no matter how long the sides of a triangle are its area cannot exceed a universal constant). In this course, we study non-Euclidean geometries (with main focus on hyperbolic geometry) using first the axiomatic approach of Euclid and Hilbert. The goal is to learn main concepts of hyperbolic geometry in the same manner as we learn high school geometry (in an ideal high school) and to give a rigorous foundation for these concepts (not as in high school, even in an ideal one). After becoming familiar with basic properties of the hyperbolic world we turn to more sophisticated results (such as classification of discrete groups of isometries) using various tools from linear algebra, complex analysis and differential geometry. We do not assume any prior knowledge of these subjects. Everything we need will be explained along the way while non-Euclidean geometries will serve as motivating examples.

2. Learning Objectives
   
   We aim to develop a coherent picture of hyperbolic geometry using axiomatic approach (similarly to the way hyperbolic geometry was discovered by Gauss, Bolyai and Lobachevsky but on a more rigorous foundation developed by Hilbert). Necessary tools
include the list of axioms of Hilbert plane and problems that clarify the significance of axioms. In particular, we postpone definition of Klein and Poincaré models and the formula for the hyperbolic distance until we prove basic theorems of hyperbolic geometry such as additivity of the defect of a triangle.

3. Learning Outcomes

Students develop problem solving skills in linear algebra, complex analysis and differential geometry by using hyperbolic geometry as motivating example and playground. They also develop advanced logical reasoning skills by studying foundations of geometry in the spirit of Hilbert.

4. Course Plan

- Week 1: Euclid's "Elements" and their influence on two millennia of mathematics. Hilbert's "Foundations of Geometry" as a rigorous base for geometry. Comparison of Euclid's postulates and Hilbert's axioms. Finite geometries. Why axioms of order are important: a fake "proof" that all triangles are isosceles.
- Week 3: Alternative formulations of Euclid's fifth postulate (Playfair--Proclus axiom, the angle sum of a triangle, no line lies completely inside an angle, Pythagorean theorem, existence of similar non-equal triangles, existence of rectangles) and counterintuitive theorems in hyperbolic geometry. Hyperbolic geometry in pop culture (common nonsense "In Lobachevskian geometry, parallel lines intersect").
- Week 4: History of hyperbolic geometry: Saccheri, Bolyai, Gauss, Lobachevsky, Beltrami. Gauss's axiom. Axiomatic approach of Bolyai and Lobachevsky, hyperbolic Hilbert plane. Non-Euclidean area, relation between the area and defect of a triangle, a universal constant. When three points lie neither on the same curve nor on the same circle, horocycles and equidistant curves.
• Week 8: Midterm
• Week 9: Reminder on complex numbers. Complex plane and Riemann sphere. Conformal maps. Linear fractional transformations. Conformal automorphisms of Riemann sphere, unit disk and upper half-plane. Poincaré half-plane model.
• Week 11: Modular group. Farey tessellation and continued fractions.
• Week 13: Metrics whose geodesics are segments of straight lines. Beltrami's theorem.

5. Reading List

a. Required

• Euclid, Elements, any edition
• David Hilbert, Foundations of Geometry, any edition
• Robin Hartshorne, Geometry: Euclid and Beyond, Undergraduate Texts in Mathematics, Springer Science & Business Media, 2005

b. Optional

• Robin Hartshorne, Teaching Geometry According to Euclid, Notices of the AMS, 47 (2000), no.4, 460--465
6. Grading System

There will be weekly written homeworks, a midterm and a final exam. The course grade will be determined as follows: 50% Homeworks+20% Midterm+30% Final.

7. Guidelines for Knowledge Assessment

8. Methods of Instruction

The course lasts 14 weeks and consists of 14 lectures and 14 problem solving sessions (one 80-minute long lecture followed by one 80-minute long problem solving session every week).

9. Special Equipment and Software Support (if required)

Not required

- A.B. Sossinsky, Geometries, IUM, 2008