SYLLABUS

Course Description

a `Title of a Course` Introduction to Iwasawa Theory

b `Pre-requisites` Basic knowledge of Algebraic Number Theory

c `Course Type (compulsory, elective, optional)`

d `Abstract`

Let p be an odd prime. In the course of his work on Fermat's Last Theorem, Kummer discovered a connection between arithmetic of fields generated over $\mathbb{Q}$ by p-th roots of unity, and the values of Riemann zeta function at odd negative integers. Besides that, Kummer established congruences between these values that lead to a definition of a p-adic analogue of the Riemann zeta function by interpolation. Almost a century later Iwasawa made equally important discovery that p-adic analogue of the Riemann zeta function is related to arithmetic of fields generated over $\mathbb{Q}$ by *all* p-primary roots of unity.

Iwasawa theory comes out of the following idea of him: in some infinite towers of fields it is often easier to describe all Galois modules simultaneously, not just one of them.

Iwasawa studied the example of p-Sylow subgroups I_n of the ideal class group of the field K_n in the tower of fields $K_1 \subset K_2 \subset K_3 \subset \ldots \subset$ such that $\mathrm{Gal}(K_n|K_1) \cong \mathbb{Z}/p^n\mathbb{Z}$ for a prime p. (In the case of fields $K_n = \mathbb{Q}(\mu_{p^n})$ generated over $\mathbb{Q}$ by p^n-th roots of unity, the group I_1 was known already to Kummer as a principal obstruction to a direct proof of Fermat's Last Theorem.) In a natural way, I_n's are $\mathbb{Z}_p$-modules, and also modules over $G_n = \mathrm{Gal}(K_n|K_1)$. It is not, however, convenient to work with the group ring $\mathbb{Z}_p[G_n]$ since, e.g., it is not a domain.

There are norm maps of $\mathbb{Z}_p[G_m]$-modules $I_m \to I_n$ for $m > n$, so that the inverse limit $\varprojlim I_n$ is a module over the inverse limit $\Lambda := \varprojlim \mathbb{Z}_p[G_n]$. It is much easier to understand the structure of the ring Λ: it is a complete 2-dimensional regular local ring, (non-canonically) isomorphic to the ring of power series $\mathbb{Z}_p[[T]]$. There is a theorem describing the structure of modules over Λ. Here is a simplest example of consequence of this theorem: p divides the class number of one of K_n's of and only if it divides the class number of all K_n's.

There is a profound connection of this theory with special values of L-functions. Iwasawa's idea, called Iwasawa's Main Conjecture, states that the "characteristic ideal" of the module I_n in the ring Λ admits a generator, which is, in a sense, p-adic L-function. This is shown, in particular, for all totally real number fields K_0.

Learning Objectives

The goals are proofs of Iwasawa's Main Conjecture in some cases and, in particular, the following formula:

$$\zeta(1-2k) = (-1)^k \prod_p \frac{|H^1_{\text{ét}}(\mathbb{Z}[1/p], \mu_{p^N}^{\otimes 2k})|}{|H^0_{\text{ét}}(\mathbb{Z}[1/p], \mu_{p^N}^{\otimes 2k})|} \quad \text{for all integer } k \geq 1 \text{ and sufficiently big } N.$$

Course Plan

- Review of class field theory
- Iwasawa Algebras and p-adic (pseudo)-measures

- The algebra $\mathbb{Z}_p[[T]]$, structure of $\mathbb{Z}_p[[T]]$-modules, isomorphism onto Iwasawa algebra
- Iwasawa's theory of $\mathbb{Z}_p$-extensions
- Complex L-Functions, functional equations, values at negative integers
- p-adic L-Functions
- Euler Systems
- Main Conjecture

Reading List

References

[1] J. Coates, R. Sujatha, **Cyclotomic Fields and Zeta Values.** Springer Monographs in Mathematics, Springer-Verlag, 2006.

[2] P. Deligne, K. Ribet, *Values of abelian L-functions at negative integers over totally real fields,* Invent. Math. **59** (1980), no. 3, 227–286.

[3] R. Greenberg, *Introduction to Iwasawa Theory for Elliptic Curves,* `http://www.math.washington.edu/greenber/Park.ps`

[4] S. Lang, **Cyclotomic fields I and II.** Graduate Texts in Mathematics 121, With an appendix by Karl Rubin (Combined 2nd ed.), Berlin, New York: Springer-Verlag, 1990.

[5] J.T. Tate, Fourier Analysis in Number Fields and Hecke's Zeta-Functions, doctoral thesis (Princeton, May 1950).

[6] Lawrence C. Washington, **Introduction to cyclotomic fields.** Graduate Texts in Math. 83 (2nd ed.), Berlin, New York: Springer-Verlag, 1997.

[7] Andrew Wiles, *The Iwasawa conjecture for totally real fields*, Ann. Math. **131** (1990), no.3, 493–540.

Grading System

There will be (i) a problem list, based on [6], (ii) a list of available talks. The maximal grade will correspond to general understanding of the subject and to the most of the problems solved. A part of the required problems can be replaced by a talk.

Methods of Instruction

Lectures, problem sessions and self-study.