Markov Chain Models in Economics, Management and Finance

Intensive Lecture Course

in High Economic School, Moscow Russia

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This course introduces a newly developed optimization technique for a wide class of discrete and continuous-time finite Markov chains models.

Along with a coherent introduction to the Markov models description (controllable Markov chains classifications, ergodicity property, rate of convergence to a stationary distribution) some optimization methods (such as Lagrange multipliers, Penalty functions and Extra-proximal scheme) are discussed.

Based on the these models and numerical methods Marketing, Portfolio Optimization, Transfer Pricing as well as Stackleberg-Nash Games, Bargaining and Other Conflict Situations are profoundly considered.

While all required statements are proved systematically, the emphasis is on understanding and applying the considered theory to real-world situations.
Structure of the course

1-st Lecture Day:
Basic Notions on Controllable Markov Chains Models, Decision Making and Production Optimization Problem.

2-nd Lecture Day:
The Mean-Variance Customer Portfolio Problem: Bank Credit Policy Optimization.

3-rd Lecture Day:
Conflict Situation Resolution: Multi-Participants Problems, Pareto and Nash Concepts, Stackelberg equilibrium.

4-th Lecture Day:
Bargaining (Negotiation).

5-th Lecture Day:
Partially Observable Markov Chain Models and Traffic Optimization Problem.


Recommended Bibliography

Papers recently published (2)


Basic Notions on Controllable Markov Chains Models, Decision Making and Production Optimization Problem
PART 1: MARKOV CHAINS AND DECISION MAKING

Definition

A stochastic dynamic system satisfies the Markov property, as it is accepted to say (this definition was introduced by A.A. Markov in 1906), "if the probable (future) state of the system at any time $t > s$ is independent of the (past) behavior of the system at times $t < s$, given the present state at time $s$".

This property can be nicely illustrated by considering a classical movement of a particle which trajectory after time $s$ depends only on its coordinates (position) and velocity at time $s$, so that its behavior before time $s$ has no absolutely any affect to its dynamic after time $s$. 
Definition

$x(t, \omega) \in \mathbb{R}^n$ is said to be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ with state space $\mathbb{R}^n$ and the index time-set $J := [t_0, T] \subseteq [0, \infty)$. Here

- $\omega \in \Omega$ is an individual trajectory of the process, $\Omega$ a set of elementary events;
- $\mathcal{F}$ is the collection ($\sigma$-algebra) of all possible events arising from $\Omega$;
- $P$ is a probabilistic measure (probability) defined for any event $A \in \mathcal{F}$.

The time set $J$ may be
- *discrete*, i.e., $J = [t_0, t_1, \ldots, t_n, \ldots)$ - then we talk on a **discrete-time stochastic process** $x(t_n, \omega)$;
- *continuous*, i.e., $J = [t_0, T)$ - then we talk on a **continuous-time stochastic process** $x(t, \omega)$. 
Markov Processes

Stochastic process: illustrative figures

Discrete time process.

Continuous time process.
Definition

\( \{x(t, \omega)\}_{t \in J} \) is called a Markov process (MP), if the following Markov property holds: for any \( t_0 \leq \tau \leq t \leq T \) an all \( A \in \mathcal{B}^n \)

\[
P \left\{ x(t, \omega) \in A \mid \mathcal{F}_{[t_0, \tau]} \right\} \overset{a.s.}{=} P \{ x(t, \omega) \in A \mid x(\tau, \omega) \}
\]
Finite Markov Chains

Main definition

Let the *phase space* of a Markov process \( \{x(t, \omega)\}_{t \in \mathcal{T}} \) be *discrete*, that is,

\[
x(t, \omega) \in X := \{(1, 2, \ldots, N) \text{ or } \mathbb{N} \cup \{0\}\}
\]

\( \mathbb{N} = 1, 2, \ldots \) is a countable set, or finite

**Definition**

A Markov process \( \{x(t, \omega)\}_{t \in \mathcal{T}} \) with a discrete phase space \( X \) is said to be a **Markov chain** (or **Finite Markov Chain** if \( \mathbb{N} \) is finite)

a) in continuous time if

\[
\mathcal{T} := [t_0, T), \ T \text{ is admitted to be } \infty
\]

b) in discrete time if

\[
\mathcal{T} := \{t_0, t_1, \ldots, t_T\}, \ T \text{ is admitted to be } \infty
\]
Corollary

The main Markov property for this particular case looks as follows:

- **in continuous time:** for any \( i, j \in \mathcal{X} \) and any \( s_1 < \cdots < s_m < s \leq t \in \mathcal{T} \)

\[
P \{ x(t, \omega) = j \mid x(s_1, \omega) = i_i, \ldots, x(s_m, \omega) = i_m, x(s, \omega) = i \} = \text{a.s.} \ P \{ x(t, \omega) = j \mid x(s, \omega) = i \}
\]

- **in discrete time:** for any \( i, j \in \mathcal{X} \) and any \( n = 0, 1, 2, \ldots \)

\[
P \{ x(t_{n+1}, \omega) = j \mid x(t_0, \omega) = i_0, \ldots, x(t_m, \omega) = i_m, x(t_n, \omega) = i \} = \text{a.s.} \ P \{ x(t_{n+1}, \omega) = j \mid x(t_n, \omega) = i \} := \pi_{j|i}(n)
\]
Homogeneous

**Definition**

A Markov Chain is said to be Homogeneous (Stationary) if the transition probabilities are constant, that is,

$$
\pi_{j|i}(n) = \pi_{j|i} = \text{const for all } n = 0, 1, 2, \ldots
$$
Finite Markov Chains

Transition Matrix

- Transition matrix $\Pi$:

$$
\Pi = \begin{bmatrix}
\pi_{1|1} & \pi_{2|1} & \cdots & \pi_{N|1} \\
\pi_{1|2} & \pi_{2|2} & \cdots & \pi_{N|2} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{1|N} & \pi_{2|N} & \cdots & \pi_{N|N}
\end{bmatrix} = [\pi_{j|i}]_{i,j=1,...,N}
$$

- Stochastic property

$$
\sum_{j=1}^{N} \pi_{j|i} = 1 \text{ for all } i = 1, \ldots, N
$$
By the Bayes formula

\[ P \{ A \} = \sum_i P \{ A \mid B_i \} P \{ B_i \} \]

it follows

\[ P \{ x(t_{n+1}, \omega) = j \} = \sum_{i=1}^{N} P \{ x(t_{n+1}, \omega) = j \mid x(t_n, \omega) = i \} \pi_{j|i} P \{ x(t_n, \omega) = i \} \]

Defining \( p_i(n) := P \{ x(t_n, \omega) = i \} \), we can write the Dynamic Model of Finite Markov Chain as

\[ p_j(n+1) = \sum_{i=1}^{N} \pi_{j|i} p_i(n) \]
In the vector format, Dynamic Model of Finite Markov Chain can be represented as follows

\[
p(n + 1) = \Pi^T p(n), \quad p(n) := (p_1(n), \ldots, p_N(n))^T
\]

Iteration back implies

\[
p(n + 1) = \Pi^T p(n) = (\Pi^T)^{n+1} p(0)
\]
Definition
A Markov Chain is called **ergodic** if all its states are returnable.

The result below shows that homogeneous ergodic Markov chains possess some additional property:

*after a long time such chains "forget" the initial states from which they have started.*
Theorem (the ergodic theorem)

Let for some state \( j_0 \in X \) of a homogeneous Markov chain and some \( n_0 > 0, \delta \in (0, 1) \) for all \( i \in (1, \ldots, N) \)

\[
(\Pi^{n_0})_{j_0|i} \geq \delta > 0
\]

i.e., after \( n_0 \)-times multiplications \( \Pi \) by itself at least one column of the matrix \( \Pi^{n_0} \) has all nonzero elements. Then for any initial state distribution \( P \{ x(t_0, \omega) = i \} \) and for any \( i, j \in (1, \ldots, N) \) there exists the limit

\[
p_j^* := \lim_{n \to \infty} (\Pi^n)_{j|i} > 0
\]

such that for any \( t \geq 0 \) this limit is reachable with an exponential rate, namely,

\[
| (\Pi^n)_{j|i} - p_j^* | \leq (1 - \delta)^{t_n/n_0} = e^{-\alpha[t_n/n_0]}, \alpha := |\ln (1 - \delta)|
\]
Show that the Finite Markov Chain with the transition matrix

\[
\Pi := \begin{bmatrix}
0 & 0.3 & 0 & 0.7 \\
1 & 0 & 0 & 0 \\
0.1 & 0 & 0.9 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

is ergodic. Indeed, after 2 steps \((n_0 = 2)\)

\[
\Pi^2 = \begin{bmatrix}
0.3 & 0.7 & 0 & 0 \\
0 & 0.3 & 0 & 0.7 \\
0.09 & 0.03 & 0.81 & 0.07 \\
1.0 & 0 & 0 & 0
\end{bmatrix}, \quad \Pi^3 = \begin{bmatrix}
0.7 & 0.09 & 0 & 0.21 \\
0.3 & 0.7 & 0 & 0 \\
0.111 & 0.097 & 0.729 & 0.063 \\
0 & 0.3 & 0 & 0.7
\end{bmatrix}
\]

\(\Pi^3 = \Pi^{1+n_0}\) contains the column \(j = 2\) with strictly positive elements.
Corollary

If for a homogeneous finite Markov chain with transition matrix $\Pi$ the ergodicity coefficient $k_{\text{erg}}(n_0)$ is strictly positive, that is,

$$k_{\text{erg}}(n_0) := 1 - \frac{1}{2} \max_{i,j=1,...,N} \sum_{m=1}^{N} \left| (\Pi^{n_0})_{m|i} - (\Pi^{n_0})_{m|j} \right| > 0$$

then this chain is ergodic.

Th following simple estimate holds

$$k_{\text{erg}}(n_0) \geq \min_{i=1,...,N} \max_{j=1,...,N} (\Pi^{n_0})_{j|i} := k_{\text{erg}}(n_0)^{-}$$

Corollary

Si, if $k_{\text{erg}}^{-}(n_0) > 0$, then the chain is ergodic.
Corollary

For any $j \in (1, 2, \ldots, N)$ of an ergodic homogeneous finite Markov chain the components $p_j^*$ of the stationary distribution, satisfy the following ergodicity relations

$$p_j^* = \sum_{i \in \mathcal{X}} \pi_{j|i} p_i^*$$

$$\sum_{i \in \mathcal{X}} p_i^* = 1, \quad p_i^* > 0 \quad (i = 1, 2, \ldots, N)$$

or equivalently, in the vector format

$$p^* = \Pi^t p^*, \quad p^* := (p_1^*, \ldots, p_N^*), \quad \Pi := \|\pi_{j|i}\|_{i,j=1,...,N}$$

that is, the positive vector $p^*$ is the eigenvector of the matrix $\Pi^t (t)$ corresponding to its eigenvalue equal to 1.
Let $\Pi_k(n) := \|\pi_{j|i,k}(n)\|_{i,j=1,...,N}$ be the transition matrix with the elements

$$\pi_{j|i,k}(n) := P \{ x(t_{n+1}, \omega) = j \mid x(t_n, \omega) = i, a(t_n, \omega) = k \}, \ k = 1, ..., K$$

where the variable $a(t_n, \omega)$ is associated with a control action (decision making) from the given set of possible controls $(1, ..., K)$. Each control action $a(t_n, \omega) = k$ may be selected (realized) in state $x(t_n, \omega) = i$ with the probability

$$d_{ki}(n) := P \{ a(t_n, \omega) = k \mid x(t_n, \omega) = i \}$$

fulfilling the stochastic constraints

$$d_{ki}(n) \geq 0, \sum_{k=1}^{N} d_{ki}(n) = 1 \text{ for all } i = 1, ..., N$$
Controllable Markov Chains

What are control strategies (decision making) for Finite Markov Chain processes?

**Definitions**

A sequence \( \{d(0), d(1), \ldots \} \) of a stochastic matrices

\[
d(n) := \|d_{ki}(n)\|_{i=1,\ldots,N; k=1,\ldots,K}
\]

with the elements satisfying the stochastic constrains is called a **control strategy** or **decision making process**.

If \( d(n) = d \) is a constant stochastic matrix such startegy is named **stationary** one.
Controllable Markov Chains
Pure and mixed strategies

Definition
If each row of the matrix $d$ contains one element equal to 1 and others equal to zero, i.e., $d_{ki} = d_{k_0i} \delta_{k,k_0}$ where $\delta_{k,k_0}$ is the Kronecker symbol $\delta_{k,k_0} := \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq k_0 \end{cases}$, then the strategy is referred to as pure, if at least in one row this is not true, then strategy is called mixed.

Example

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ - a pure strategy; $\begin{bmatrix} 0 & 0.2 & 0 & 0.8 \\ 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ - a mixed strategy
Controllable Markov Chains

Structure of a controllable Markov Chain

Figure: Structure of a controllable Markov Chain.
Controllable Markov Chains

Again by the Bayes formula \( P \{ A \} = \sum_i P \{ A \mid B_i \} P \{ B_i \} \) we have

\[
\pi_{j|i}(n) := P \{ x(t_{n+1}, \omega) = j \mid x(t_n, \omega) = i \} = \sum_{k=1}^{N} P \{ x(t_{n+1}) = j \mid x(t_n) = i, a(t_n) = k \} P \{ a(t_n) = k \mid x(t_n) = i \} d_{k|i}(n)
\]

so,

\[
\pi_{j|i}(n) = \sum_{k=1}^{N} \pi_{j|i,k}(n) d_{k|i}(n)
\]

For homogenous Finite Markov models and stationary under stationary strategies \( d_{k|i}(n) = d_{k|i} \) one has

\[
\pi_{j|i}(d) = \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i}
\]
Controllable Markov Chains

Dynamics of state probabilities

For stationary strategy \( d = \|d_{ki}\|_{i=1,\ldots,N; k=1,\ldots,K} \) we have

\[
p_j(n+1) := P \{ x(t_n, \omega) = i \} = \sum_{i=1}^{N} \pi_{j|i}(d) p_i(n)
\]

\[
= \sum_{i=1}^{N} \left( \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} \right) p_i(n)
\]

which represents the **Dynamic Model of Controllable Finite Markov Chain** under a stationary strategy \( d \). If for each \( d \) the chain is ergodic, then \( p_j(n) \xrightarrow[n \to \infty]{} p_j \) satisfying

\[
p_j = \sum_{i=1}^{N} \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} p_i
\]

or
Figure: Convergence to stationary distribution.
Controllable Markov Chains

Dynamics of state probabilities: the vector form

In the vector form the *Dynamic Model* of Controllable Finite Markov Chain (or Decision Making process) under a stationary strategy $d$ looks as

$$p = \Pi^T (d) \ p$$

$$\Pi (d) = \left\{ \sum_{k=1}^{N} \pi_{j|i, k} d_{ki} \right\}_{i=1, \ldots, N; j=1, \ldots, N}$$

**Fact**

*So, the final distribution $p$ depends also on the strategy $d$, that is, $p = p(d)$, so that*

$$p(d) = \Pi^T (d) \ p(d)$$
PART 2: Simplest Production Optimization Problem

Suppose that some company obtains for the transition

$$x(t_n, \omega) = i, \quad a(t_n, \omega) = k \rightarrow x(t_{n+1}, \omega) = j$$

from state $i$ to the state $j$, applying the control $k$, the following income

$$W_{j|i,k}, i, j = 1, ..., n, k = 1, ..., K$$

Then the average income of this company in stationary state is

$$J(d) := \sum_{i=1}^{N} \sum_{j=1}^{N} W_{j|i,k} \left( \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} \right) p_i = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i} p_i$$

where the components $p_i$ satisfies the ergodicity condition

$$p_j(d) = \sum_{i=1}^{N} \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} p_i(d)$$
The rigorous mathematical problem formulation is as follows:

\[
J(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i} p_i(d) \rightarrow \max_{d \in \mathcal{D}_{adm}}
\]

under the constrains

\[
\mathcal{D}_{adm} := \left\{ d_{k|i} : p_j(d) = \sum_{i=1}^{N} \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} p_i(d), j = 1, \ldots, N \right\}
\]

\[
d_{k|i} \geq 0, \quad \sum_{k=1}^{N} d_{k|i} = 1, \quad i = 1, \ldots, N
\]
Simplest Production Optimization Problem

Best-reply strategy

Definition

The matrix $d^{br}$ is called the **best-reply strategy** if

$$d^{br}_{\beta|\alpha} = \begin{cases} 1 & \text{if } \sum_{j=1}^{N} W_{j|\alpha,\beta} \pi_{j|\alpha,\beta} \geq \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} \\ 0 & \text{if not} \end{cases}$$

Indeed, the upper bound for $J(d)$ can be estimated as

$$J(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i} p_{i}(d) \leq \sum_{i=1}^{N} \max_{k} \left( \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} \right) p_{i}(d)$$

which is reachable for $d^{br}_{k|i} = d^{br}_{k|i}$. It is **optimal** if and only if

$$\max_{k} \left( \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} \right) = \max_{k} \left( \sum_{j=1}^{N} W_{j|s,k} \pi_{j|s,k} \right) \quad \forall i, s$$

(1)
Example (State and action spaces interpretation)

Let

- the state \( x(t_n, \omega) = i \) be associated with a number of working units (staff places);
- the action \( a(t_n, \omega) = k \) is related with the financial schedule (possible wage increase, decreasing or no changes): \( k = (-1, 0, 1) \);
- the incomes for these actions may be calculated as

\[
W_{j|i,k} = [v_0 - (v + \Delta v k) - v_1](j - i)
\]

where \( v_0 \) the price of the product, produced by a single working unit with the salary \( v \), its \( \Delta v \) adjustment and the production costs \( v_1 \) supporting this process.
Example (State and action spaces interpretation (continuation-1))

For example, for $N = 3$, $i = (10, 20, 30)$ and $v_0 = 400,000.00$, $v_1 = 20,000.00$, $r = 80,000.00$, $\Delta v = 5,000.00$ we have

\[
\begin{bmatrix}
0 & 3050,000.00 & 6100,000.00 \\
-3050,000.00 & 0 & 3050,000.00 \\
-6100,000.00 & -3050,000.00 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 3000,000.00 & 6000,000.00 \\
-3000,000.00 & 0 & 3000,000.00 \\
-6000,000.00 & -3000,000.00 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 2950,000.00 & 5900,000.00 \\
-2950,000.00 & 0 & 2950,000.00 \\
-5900,000.00 & -2950,000.00 & 0
\end{bmatrix}
\]
Example (State and action spaces interpretation (continuation-2))

Let the transition matrices $\pi_{j|i,k}$ be as follows:

\[
\begin{bmatrix}
0.5 & 0.3 & 0.2 \\
0 & 0.5 & 0.5 \\
0 & 0.5 & 0.5 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0.1 & 0.9 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
0.5 & 0.2 & 0.3 \\
0 & 0.25 & 0.75 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

$k = -1$, $k = 0$, $k = 1$
Then the matrix \[ \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} \], participating in the average income, is

\[
\begin{pmatrix}
2745000 & 5700000 & 2360000 \\
1525000 & 3000000 & 2212500 \\
-1525000 & 0 & -5900000
\end{pmatrix}
\]

and the best reply strategy is (it is non-optimal)

\[ d_{br} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix} \]

So, no changes are recommended since \( k^* = 0 \) for all states \( i \).
Optimization Problem with Additional Constrains

Problem formulation with additional constrains

**Problem**

\[
J(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i} p_i(d) \rightarrow \max_{d \in D_{adm}}
\]

under the constrains

\[
D_{adm} := \left\{ d_{k|i} : p_j(d) = \sum_{i=1}^{N} \sum_{k=1}^{N} \pi_{j|i,k} d_{k|i} p_i(d), j = 1, \ldots, N \right\}
\]

\[
d_{k|i} \geq 0, \quad \sum_{k=1}^{N} d_{k|i} = 1, \quad i = 1, \ldots, N
\]

\[
\sum_{i=1}^{N} \sum_{k=1}^{K} \left( \sum_{j=1}^{N} A^{(l)}_{j|i,k} \pi_{j|i,k} \right) d_{k|i} p_i(d) \leq b_l, \quad l = 1, \ldots, L
\]

The additional constrains may be interpreted as some financial limitations.
Optimization Problem with Additional Constrains

What can we do in this complex situation?

**Fact**

- **The best reply strategy in general is non optimal:** it may not satisfy condition (1) and the additional constrains.

- **The functional**

  \[
  J(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i} p_{i}(d)
  \]

  as well as the constrains

  \[
  p_{j}(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \pi_{j|i,k} d_{k|i} p_{i}(d), \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} A_{j|i,k}^{(l)} \pi_{j|i,k} d_{k|i} p_{i}(d) \leq b_{l}
  \]

  are extremely nonlinear functions of \( d \).

The question is: what can we do in this complex situation?
Optimization Problem with Additional Constrains

What can we do in this complex situation? Answer: c-variables!

Definition

Define new variables

\[ c_{ik} := d_{k|i}p_i(d) \]

Then the Production Optimization Problem can be express as a **Linear Programming Problem** solved by standard Matlab Toolbox:

\[
J(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{j=1}^{N} W_{j|i,k} \pi_{j|i,k} d_{k|i}p_i(d) = \sum_{i=1}^{N} \sum_{k=1}^{K} W_{ik}^\pi c_{ik} := J(c) \rightarrow \min_{c \in C_{adm}}
\]

\[
C_{adm} := \left\{ c_{ik} : \sum_{k=1}^{K} c_{jk} = \sum_{i=1}^{N} \sum_{k=1}^{K} \pi_{j|i,k} c_{ik}, j = 1, N, c_{ik} \geq 0, \sum_{i=1}^{N} \sum_{k=1}^{K} c_{ik} = 1, \sum_{i=1}^{N} \sum_{k=1}^{K} \left( \sum_{j=1}^{N} A_{j|i,k}^{(l)} \pi_{j|i,k} \right) c_{ik} \leq b_l, l = 1, L \right\}
\]
Corollary

For \textbf{c-variables} defined as \( c_{ik} := d_{k|i} p_i (d) \) and found as the solution \( c^* \) of the LPP above

1) \textit{we can recuperate the state distribution} \( p_i (d^*) \) as

\[
p_i (d^*) = \sum_{k=1}^{K} c_{ik}^* > 0 \text{ (by the ergodicity property)}
\]

2) \textit{and the optimal control strategy (or decision making) \( d_{k|i}^* \) can be recuperated as}

\[
d_{k|i}^* = \frac{c_{ik}^*}{p_i^* (d)} = \frac{c_{ik}^*}{K \sum_{k=1}^{K} c_{ik}^*}
\]
Solution of LPP

Numerical solution of LPP with the data of the previous example

\[ c_{ik}^* = \begin{bmatrix} 0.0001 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.3996 \\ 0.5996 & 0.0001 & 0.0002 \end{bmatrix} \]

\[ p_i^* (d) = (0.0003, 0.3998, 0.5999) \]

\[ d_{ki}^* = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.0002 & 0.0003 & 0.9995 \\ 0.9995 & 0.0002 & 0.0003 \end{bmatrix} \]
A controllable continuous-time Markov chain is a 4-tuple

\[ \text{CTMC} = (S, A, \mathcal{K}, Q) \]

where \( S \) is the finite state space \( \{s_1, \ldots, s_N\} \), \( A \) is the set of actions: for each \( s \in S \), \( A(s) \subset A \) is the non-empty set of admissible actions at state \( s \in S \), \( \mathcal{K} = \{(s, a) \mid s \in S, a \in A(s)\} \) is the class of admissible state-action pairs and \( Q = \) is the transition rates \( \begin{bmatrix} q(j|i,k) \end{bmatrix} \) with the elements defined as

\[
q(j|i,k) = \begin{cases} 
q(i|i,k) = - \sum_{i \neq j}^N q(j|i,k) \leq 0 & \text{if} \quad i = j \\
\geq 0 & \text{if} \quad i \neq j
\end{cases}
\]
Continuous-time controllable Markov chains

Fact

For each fixed \( k \) the matrix of the transition rates is assumed to be conservative, i.e., from the definition above it follows that \( \sum_{j=1}^{N} q(j|i,k) = 0 \) and stable, which means that \( q(i) := \max_{a(k) \in A(s(i))} q(j|i,k) < \infty \ \forall i \).

Example

\[
q(j|i,k=1) = \begin{bmatrix}
-0.5366 & 0.0888 & 0.0611 & 0.1893 & 0.1409 \\
0.0416 & -0.5689 & 0.0588 & 0.1331 & 0.0942 \\
0.2358 & 0.1929 & -0.3784 & 0.1878 & 0.2084 \\
0.0942 & 0.1929 & 0.1244 & -0.5963 & 0.0570 \\
0.1649 & 0.0942 & 0.1342 & 0.0861 & -0.5005
\end{bmatrix}
\]
Continuous-time controllable Markov chains

Transition probabilities

Definition

Let $X_s := \{ i \in \mathcal{X} : \mathbb{P} \{ x(s, \omega) = i \} \neq 0, \ s \in \mathcal{T} \}$. For $s \leq t \ (s, t \in \mathcal{T})$ and $i \in X_s$, $k \in \mathcal{A}$, $j \in \mathcal{X}$ define the conditional probabilities

\[
\pi_{j|i,k}(s,t) := \mathbb{P} \{ x(t, \omega) = j \mid x(s, \omega) = i, a(s) = k \}
\]

which we will call the transition probabilities of a given Markov chain defining the conditional probability for a process $\{ x(t, \omega) \}_{t \in \mathcal{T}}$ to be in the state $j$ at time $t$ under the condition that it was in the state $i$ at time $s < t$ and in the same tame the decision $a(s) = k$ was done. The transition probabilities to be in the state $j$ at time $t$ under the condition that it was in the state $i$ at time $s < t$

\[
\pi_{ij}(s,t \mid d) := \sum_{k=1}^{K} \pi_{j|i,k}(s,t) \mathbb{P}\{a(s) = k \mid x(s, \omega) = i\} = \sum_{k=1}^{K} \pi_{j|i,k}(s,t) d_{k|i}
\]
Continuous-time controllable Markov chains
Properties of Transition probabilities

The function \( \pi_{ij}(s, t \mid d) \) for any \( i \in X_s, j \in X \) and any \( s \leq t \ (s, t \in T) \) should satisfy the following \textit{four conditions}:

1) \( \pi_{ij}(s, t \mid d) \) is a conditional probability, and hence, is nonnegative, that is, \( \pi_{i,j}(s, t) \geq 0 \).

2) starting from any state \( i \in X_s \) the Markov chain will obligatory occur in some state \( j \in X_t \), i.e., \( \sum_{j \in X_t} \pi_{ij}(s, t \mid d) = 1 \).

3) if no transitions, the chain remains to in its starting state with probability one, that is, \( \pi_{ij}(s, s \mid d) = \delta_{ij} \) for any \( i, j \in X_s, j \in X \) and any \( s \in T \);

4) the chain can occur in the state \( j \in X_t \) passing through any intermediate state \( k \in X_u \ (s \leq u \leq t) \), i.e.,

\[
\pi_{ij}(s, t \mid d) = \sum_{k \in X_u} \pi_{ik}(s, u \mid d) \pi_{kj}(u, t \mid d)
\]

This relation is known as the Markov (or Chapman-Kolmogorov) equation.
Continuous-time controllable Markov chains
Properties of Transition probabilities for homogeneous Markov chains

Corollary

Since for homogeneous Markov chains the transition probabilities \( \pi_{i,j}(s, t) \) depend only on the difference \( (t - s) \), below we will use the notation

\[
\pi_{ij}(s - t \mid d) := \pi_{ij}(s, t \mid d)
\]  

In this case the Markov equation becomes

\[
\pi_{i,j}(h_1 + h_2 \mid d) = \sum_{k \in \mathcal{X}} \pi_{i,k}(h_1 \mid d) \pi_{k,j}(h_2 \mid d)
\]

valid for any \( h_1, h_2 \geq 0 \).
Consider now the time \( \tau \) (after the time \( s \)) just before changing the current state \( i \), i.e., \( \tau > s \).

By the *homogeneity property* it follows that distribution function of the time \( \tau_1 \) (after the time \( s_1 := s + u, x(s + u, \omega) = i \)) is the same as for the \( \tau \) (after the time \( s, x(s, \omega) = i \)) that leads to the following identity

\[
P \{ \tau > v \mid x(s, \omega) = i \} = P \{ \tau_1 > v \mid x(s_1, \omega) = i \}
\]

\[
P \{ \tau > v + u \mid x(s + u, \omega) = i \} =
\]

\[
P \{ \tau > u + v \mid x(s, \omega) = i, \tau > u \geq s \}
\]

since the event \( \{ x(s, \omega) = i, \tau > u \} \) includes as a subset the event \( \{ x(s + u, \omega) = i \} \).
Lemma on the expectation time before changing a state

Lemma

The expectation time $\tau$ (of the homogenous Markov chain $\{x(t, \omega)\}_{t \in T}$ with a discrete phase space $X$) to be in the current state $x(s, \omega) = i$ before its changing has the exponential distribution

$$P \{ \tau > v \mid x(s, \omega) = i \} = e^{-\lambda_i v}$$  \hspace{1cm} (4)$$

where $\lambda_i$ is a nonnegative constant which inverse value characterizes the average expectation time before the changing the state $x(s, \omega) = i$, namely,

$$\frac{1}{\lambda_i} = E \{ \tau \mid x(s, \omega) = i \} , \lambda_i = \left| q(i|i,k) \right| = \sum_{i \neq j}^{N} q(j|i,k)$$  \hspace{1cm} (5)$$

The constant $\lambda_i$ is usually called the "exit density".
Proof.

Define the function $f_i(u)$ as $f_i(u) := P \{ \tau > u \mid x(s, \omega) = i \}$. By the Bayes formula

$$f_i(u + v) := P \{ \tau > u + v \mid x(s, \omega) = i \} = P \{ \tau > u + v \mid x(s, \omega) = i, \tau > u \} P \{ \tau > u \mid x(s, \omega) = i \}$$
$$= P \{ \tau > u + v \mid x(s, \omega) = i, \tau > u \} f_i(u)$$

By the homogeneous property one has

$$f_i(u + v) := P \{ \tau > u + v \mid x(s, \omega) = i \} = P \{ \tau > v \mid x(s, \omega) = i \} f_i(u) = f_i(v) f_i(u)$$

which means that

$$\ln f_i(u + v) = \ln f_i(u) + \ln f_i(v)$$

$$f_i(\tau = 0) = P \{ \tau > 0 \mid x(s, \omega) = i \} = 1$$
Proof.

[Continuation of the proof] Differentiation the logarithmic identity by $u$ gives

\[
\frac{f_i'(u+v)}{f_i(u+v)} = \frac{f_i'(u)}{f_i(u)}
\]

which for $u = 0$ becomes

\[
\frac{f_i'(v)}{f_i(v)} = \frac{f_i'(0)}{f_i(0)} = \frac{f_i'(0)}{f_i(0)} := -\lambda_i \rightarrow f_i(v) = e^{-\lambda_i v}
\]

To prove (5) it is sufficient to notice that

\[
E \{ \tau \mid x(s, \omega) = i \} = \int_{t=0}^{\infty} td \left[ -f_i(t) \right] =
\]

\[
\left[ -te^{-\lambda_i t} \right]_{t=0}^{\infty} - \int_{t=0}^{\infty} \left[ -e^{-\lambda_i t} \right] dt = \int_{t=0}^{\infty} e^{-\lambda_i t} dt = \lambda_i^{-1}
\]

Lemma is proven.
Continuous-time controllable Markov chains

The Kolmogorov forward equations

For homogenous Markov chains $\pi_{ij} (s, t \mid d) = \pi_{ij} (t - s \mid d)$ and stationary strategies $P \{ a(s) = k \mid x(s, \omega) = i \} = d_{k|i}$ the Markov equation becomes (taking $s = 0$)

$$\frac{d}{dt} \pi_{ij} (t \mid d) = - \left( \sum_{i}^{N} q(j \mid i, k) \right) \pi_{ij} (t \mid d) + q(j \mid i, k) \pi_{il} (t \mid d)$$

can be written as the matrix differential equation as follows:

$$\Pi'(t \mid d) = \Pi(t \mid d) Q(d); \quad \Pi(0) = I_{N \times N}$$

$$\Pi(t \mid d) = \| \pi_{i,k} (t \mid d) \| \in \mathbb{R}^{N \times N}, \quad Q(d) = \| \sum_{k=1}^{K} q(j \mid i, k) d_{k|i} \|$$

This system can be solved by

$$\Pi(t \mid d) = \Pi(0) e^{Q(d)t} = e^{Qt} := \sum_{t=0}^{\infty} \frac{t^n Q^n (d)}{n!}$$  \quad (6)
Continuous-time controllable Markov chains

Stationary distribution

At the stationary state, the probability transition matrix is

$$\Pi(d) = \lim_{t \to \infty} \Pi(t | d)$$

**Definition**

The vector $P \in \mathbb{R}^N \left( \sum_{i=1}^{N} P_i = 1 \right)$ is called the stationary distribution vector if

$$\Pi^T(d) P = P$$

**Claim**

This vector can be seen as the long run proportion of time that the process is in state $s_{(i)} \in S$. 
**Theorem (Xianping Guo, Onesimo Hernandez Lerma, 2009)**

The stationary distribution vector $P$ satisfies the linear equation

$$Q^T (d) P = 0$$

(7)

**Fact**

The Production Optimization Problem, described by a continuous-time controllable Markov chain in stationary states, is the same Linear programming problem (LPP) as for a discrete-time model but with the additional linear constraint (7).
Conclusion

Which topics we have discussed today?

1-st Lecture Day: Basic Notions on Controllable Markov Chains Models, Decision Making and Production Optimization Problem.

- Markov Processes: Classical Definition (Markov), Mathematical Definition (Kolmogorov), Markov property in a general format
- Simplest Production Optimization Problem: c-variables, Linear Programming Problem.
- Continuous-time controllable Markov chains: Distribution function of the time just before changing the current state, the transition rates, expectation time, Additional linear constraint and LPP problem.

Thank you for your attention! See you soon!