A class of generalized Newton maps with connected Julia set.

Khudoyor Mamayusupov

Inha University in Tashkent, Uzbekistan; Institute of Mathematics Polish Academy of Science, Poland; Jacobs University Bremen, Germany, xudoyorm@gmail.com.

Introduction: A century ago, after extensive work by P. Fatou and G. Julia, the iteration theory of rational functions on the Riemann sphere was born. The field utilizes powerful and beautiful tools from topology, geometry, complex analysis, group theory, combinatorics and many other fields of mathematics. One of the best known classes of rational functions is the class of Newton maps for polynomials. We generalize the notion of Newton map allowing the multipliers at the fixed points to be any complex numbers. We call these type of functions Formal Newton maps. We prove that the Julia set is connected for a class of formal Newton maps and associate postcritically minimal Newton maps to them.

Definitions: Set C_f ={critical points of f}={z:deg(f,z)>1} and $P_f = \bigcup_{n\geq 1} f^{on}(C_f)$. The set P_f is called the post-critical set of f. Denote by $J(f)$ the Julia set of f, the set of points z where there is no neighborhood of z on which the iterates $\{f^{on}\}$ is normal. Its complement is called the Fatou set. The function f is called *post-critically finite* (*PCF*) if P_f is finite. The function f is called *geometrically finite* if the intersection $P_f \cap J(f)$ is a finite set. Please refer to [1], [2] and [4] for other terms and tools used in the paper.

Newton map: Let f be an entire function (polynomial or transcendental). A meromorphic function given by $N_f(z) := z - \dfrac{f(z)}{f'(z)}$ is called the **Newton map** of $f(z)$.

For the entire function $p(z)e^{q(z)}$ one can easily compute that its Newton map is $z - \dfrac{p(z)}{p'(z) + p(z)q'(z)}$, which is a rational function with all finite fixed points attracting and these are the roots of a polynomial $p(z)$. The point at ∞ is parabolic with the multiplier +1.

Define the notion of "**post-critical minimality**" for Newton maps with the parabolic fixed point at ∞. The critical orbits are finite on the Julia set and on attracting Fatou domains, for critical points in the basin of ∞ there exist minimal critical orbit relations: in every immediate basin of ∞ there exists a unique (possibly with higher multiplicity) critical point and all other critical points in the basin of ∞ will land to the critical point in one of the immediate basins of ∞ in minimal iterate, so that there exist no other types of Fatou components.

Theorem 1. (Rational function with single fixed point is Newton map). If there is a single fixed point, let it be at ∞, of a rational function F of degree ≥ 2 then it has a normal form $F(z) = z - \dfrac{1}{q(z)}$ for some polynomial q. Moreover, $F = N_f$ the Newton map of an exponential function $f = e^{\int q(w)dw}$. In particular, the fixed point at ∞ is necessarily parabolic with the multiplier +1.

Proof. Assume a rational function F of degree at least 2 is given. Let its single fixed point be at ∞, otherwise we conjugate by a Möbius map and send it to ∞. Then the rational function

$\frac{1}{z-F(z)}$ has a single pole at ∞. Thus, it is a polynomial, denote it by q, then $F(z)=z-\frac{1}{q(z)}$.

Observe that F is a Newton map of an entire function $e^{\int q(w)dw}$. Then ∞ is necessarily a parabolic fixed point with $\deg(\int q(w)dw)=\deg q+1$ number of petals. □

Formal Newton Maps: One can generalize the definition of Newton map to obtain a large family of rational functions in the following way:

Let complex numbers $a_i \neq 0$ and z_i for $1 \leq i \leq d$ be given. Define formal Newton map

$$f(z)=z-\frac{1}{\sum_{i=1}^{d}\frac{a_i}{z-z_i}} \tag{1}$$

It is the "Newton map" for a formal expression $\prod_{i=1}^{d}(z-z_i)^{a_i}$. If z_i are all different from each other, then the degree of a rational function f of the form (1) is equal to d. In this way, we obtain all rational functions of degree d with $d+1$ distinct fixed points. The points z_i are fixed by f with multipliers $1-\frac{1}{a_i}$ and are attracting if $\left|1-\frac{1}{a_i}\right|<1$ for $1 \leq i \leq d$. By changing the multipliers at these fixed points through quasiconformal surgery [1], we can convert all of them to superattracting. The resulting rational function is clearly a Newton map of a polynomial, implying that in this special case the Formal Newton map has a simple fixed point, which is necessarily repelling, other than z_i for $1 \leq i \leq d$ and its Julia set is connected. If $a_i=1$ for $1 \leq i \leq d$ we obtain Newton maps of polynomials from the above formula. We have the following:

Theorem 2 (Connectivity of Julia set of formal Newton map). Let complex numbers a_i and z_i be given and assume $\left|1-\frac{1}{a_i}\right|<1$ for $1 \leq i \leq d$, and $z_i \neq z_j$ for $i \neq j$, with $d \geq 2$. Then the Julia set is connected for $f(z)=z-\frac{1}{\sum_{i=1}^{d}\frac{a_i}{z-z_i}}$ a formal Newton map. There is a post-critically finite Newton map of polynomial corresponding to f, provided the Julia critical points of f have finite orbits, i.e. when f is geometrically finite. This correspondence is quasiconformal and conjugates the dynamics on some neighborhood of the Julia set of f.

Proof. Assume that $\left|1-\frac{1}{a_i}\right|<1$ for $1 \leq i \leq d$, and $z_i \neq z_j$ for $i \neq j$, with $d \geq 2$. Then f has attracting fixed points at $z=z_i$ for $1 \leq i \leq d$ the multipliers of which can be changed to zero by quasiconformal surgery. The resulting function has the same degree as the original function and is the Newton map of polynomial since its all but one fixed points are superattracting, hence $J(f)$ is connected by Shishikura's theorem [6]. Let $z \neq z_i$, then the ratio $\frac{a_i}{z-z_i}$ is never zero or infinite, thus $\left|\frac{1}{\sum_{i=1}^{d}\frac{a_i}{z-z_i}}\right|$ is never zero, hence z is not a fixed point of f. Since every rational function has a **weakly repelling** fixed point (parabolic with the multiplier +1 or repelling fixed point), then ∞ is the one for f, which necessarily is repelling. Since the Newton map with d superattracting fixed points has degree d so does the function f.

When f is geometrically finite and satisfies conditions of the theorem then quasiconformal surgery, which makes attracting fixed points superattracting, produces a post-critically finite Newton map. In particular, the dynamics on some neighborhood of $J(f)$ is conjugate to that of the the post-critically finite Newton map. □

Formal Newton maps form a reasonably large family of rational functions. Moreover, all rational function of degree d with $d+1$ distinct fixed points (all those where all fixed points are simple) can be obtained as a formal Newton map. The value of a_i is uniquely defined by the desired multiplier at the fixed point $z = z_i$. We can further generalize the formula for the Formal Newton maps by taking the Newton map of formal expression of the form: For an integer $d \geq 3$ consider $\prod_{i=1}^{d-n}(z-z_i)^{a_i}e^{Q(z)}$ where $a_i \neq 0$ and z_i for $1 \leq i \leq d-n$ and Q a polynomial of degree $n \geq 1$. Its "Newton map" is:

$$F(z) = z - \frac{1}{\sum_{i=1}^{d-n}\frac{a_i}{z-z_i} + Q'(z)} \tag{2}$$

which is a well defined rational function of degree $d \geq 3$. In the case when $\left|1-\frac{1}{a_i}\right| < 1$ for $1 \leq i \leq d-n$ as above by changing the multipliers at the fixed points $z = z_i$ we can derive that the function (2) has a fixed point at ∞, which is necessarily parabolic with the multiplier +1, and its Julia set is connected. In this general case we summarize the result as following:

Theorem 3 (Connectivity of Julia set, general case). Let for a pair of integers $d \geq 2$ and $1 \leq n \leq d$, complex numbers $a_i \neq 0$ and z_i for $1 \leq i \leq d-n$ and a polynomial Q of degree n be given. Assume $\left|1-\frac{1}{a_i}\right| < 1$ for all $1 \leq i \leq d-n$ and $z_i \neq z_j$ for $i \neq j$. Then the Julia set of the formal Newton maps (2) is connected. There exists a post-critically minimal Newton map N_{pe^q} with p and q polynomials and a quasiconformal map φ such that $\varphi \circ F = N_{pe^q} \circ \varphi$ on $J(f)$ provided the Julia critical points of F have finite orbits, i.e. when F is geometrically finite.

Proof. A similar argument as in the proof of Theorem 2 shows that the point ∞ is a weakly repelling fixed point of F, which necessarily is parabolic with the multiplier +1. Since ∞ is the only fixed point with this property $J(f)$ is connected by Shishikura's theorem [6]. To get a Newton map we apply a surgery tool [1] for $1 \leq i \leq n$ we convert the attracting fixed points $z = z_i$ to superattracting. The resulting rational function is a Newton map with a parabolic fixed point by [5]. Both rational functions are quasiconformally conjugate to each other away from the compact sets containing attracting fixed points, in particular the conjugacy holds in the neighborhood of the Julia set. In order to obtain a post-critically minimal Newton map we can use the theorem of McMullen [3] provided all critical points on $J(f)$ have finite orbits. □

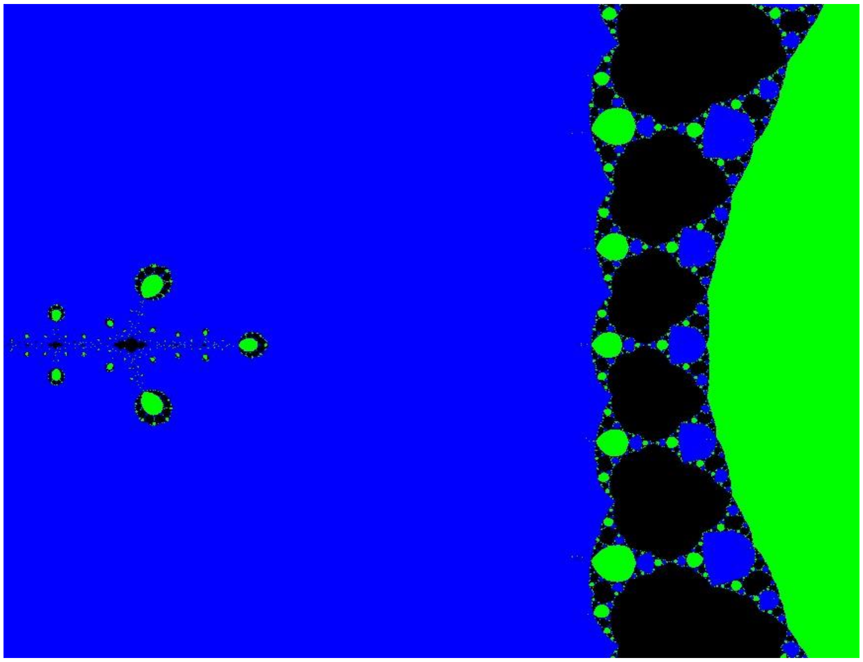

Figure. The Julia set of the degree 4 formal Newton map with superattracting fixed points at 0 and 1 and repelling fixed points with real multipliers at .5 , -1, and ∞. It has a two cycle with the basin in black. The basin of 0 is in blue and the basin of 1 is in green.

Literature

1. B. Branner, N. Fagella, Quasiconformal surgery in Holomorphic dynamics, Cambridge University Press, 2014.
2. K. Mamayusupov, On Postcritically Minimal Newton maps, PhD thesis, 2015.
3. C. McMullen, Automorphisms of rational maps, In Holomorphic Functions and Moduli. MSRI Publications, 1986.
4. J. Milnor, Dynamics in one complex variable, 3rd edition. Annals of Mathematics Studies, 160. Princeton University Press, Princeton, NJ, 2006.
5. Rückert, D. Schleicher, On Newton's method for entire functions. Journal of the London Mathematical Society, 3, 2007, pages 659—676.
6. M. Shishikura, The connectivity of the Julia set of rational maps and fixed points, D. Schleicher, editor, Complex Dynamics, Family and Friends, pages 257-276. A. K. Peters Ltd., Wellesley, MA, 2009.