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Anisotropy-based bounded real lemma for linear discrete-time descriptor systems

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***Abstract*—For linear discrete-time descriptor systems bounded real lemma in terms of anisotropic norm was formulated and proved. This lemma connects boundedness of anisotropic norm by a given nonnegative parameter with existence of solution of specified generalized Riccati equation. Numerical example is given.**

I. Introduction

Physical control systems are affected by random disturbances. This paper deals with input random disturbances. The problem of disturbance rejection is one of the main control problems. In Linear Quadratic Gaussian (LQG) control problem the Gaussian white noise sequence is considered as a random disturbance. However, noises, affecting in real control systems, are "colored", i.e. correlated. This means that LQG-controllers cannot reach desirable performance for closed-loop systems in presence of strongly correlated noises. On the other hand, H_∞-control theory assumes that the input disturbance is a square integrable (summable) sequence and gives the ability to solve disturbance rejection problem for strongly correlated noises. But the disadvantage of H_∞-control theory is that the H_∞-controller is too conservative if the input disturbance is a weakly correlated random signal. The concept of mean anisotropy of random sequences [1],[2] made it possible to introduce a class of random "colored" noises, limited by some numerical parameter, called the mean anisotropy level. It allows to develop methods for analysis and synthesis of stochastic control systems, that have robust performance with respect to the stochastic nature of input signals. The problems of optimal and suboptimal control for ordinary linear discrete-time systems were solved in [3],[4]. Suboptimal control allows to design closed-loop systems with less than given nonnegative noise rejection level (so called anisotropic norm). But some of physical control objects cannot be described only by differential or difference equations. Mathematical models of such systems contain algebraic equations. Algebraic equations in the model of the system appear as constraints when the system variables have the meaning of physical processes. In literature such systems are called descriptor systems (singular systems, semistate systems, degenerate systems, and others). They are a general case of ordinary systems, described by ordinary differential or difference equations. Because of the algebraic relations between state variables, the system becomes singular and has specific behavior, different from ordinary systems, for example, impulsive behavior in continuous time case or noncausal behavior in discrete time case. So it is necessary to generalize mathematical methods, developed for ordinary systems. This paper generalizes anisotropy-based bounded real lemma on descriptor systems.

II. Background

This section covers basic concepts of discrete-time descriptor systems and anisotropic theory.

A. Elements of descriptor systems theory

In this section, basics of descriptor systems theory are represented. See [5], [6] for more details.

In linear case discrete-time descriptor systems can be written as

$$\begin{aligned} Ex(k+1) &= Ax(k)+Bu(k), \\ y(k) &= Cx(k)+Du(k) \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^q$ are the input and output signals, respectively. A,B,C,D are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ can be singular, such system is called singular.

Assume that rank $(E) = r \leq n$ for the system (1). Some basic properties of descriptor systems are associated with matrices E and A. In different literature [6], [7], they can be represented as a matrix pencil $(zE-A)$ or a pair (E,A). In this paper, all basic definitions are given in terms of the pair (E,A).

Definition 1: The pair (E,A) is said to be regular if there exists a scalar λ such that $\det(\lambda E - A) \neq 0$.

Regularity of the pair (E,A) is a necessary and sufficient condition of existence and uniqueness of solution of the system (1). The following lemma [6] provides necessary and sufficient conditions of regularity for the system (1):

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Lemma 1: The pair (E,A) is regular if and only if there exist invertible matrices Q and U such that

$$\bar{E} = QEU = \mathrm{diag}(I_r,N), \quad \bar{A} = QAU = \mathrm{diag}(A_1,I_{n-r}) \tag{2}$$

where $A_1 \in \mathbb{R}^{r\times r}$, N is a nilpotent matrix, that is, $N^h = 0$ for some positive integer h. The minimum h_0 such that $N_0^h = 0$ is called the index of N.

The index of the system (1) in equivalent form (2) is called the index of the nilpotent N. $(\bar{E},\bar{A})$ is called the Weierstrass canonical form of (E,A).

Definition 2: The system (1) is called causal if its solution $x(k)$ depends only on $u(k),...,u(0)$ and $x(k-1),...,x(0)$ for any consistent initial conditions. It is true if the index of the nilpotent N is equal to 1.

Definition 3: The system (1) is called stable if $\rho(E,A) < 1$ where

$$\rho(E,A) \triangleq \frac{\max|\lambda|}{\lambda\in z|\det(zE-A)=0}$$

is a generalized spectral radius for the pair (E,A).

Definition 4: The system (1) is said to be admissible if the pair (E,A) is regular, and the system (1) is stable and causal.

Definition 5: The transfer function of the system (1) is given by $P(z) = C(zE-A)^{-1}B+D$.

Definition 6: Let $\mathbb{L}_2^{p\times m}(\Gamma)$ (where Γ is a unit circle on the complex plane) be the Hilbert space of matrix-valued functions $F:\Gamma\to\mathbb{C}^{p\times m}$ that have bounded $\mathbb{L}_2^{p\times m}(\Gamma)$-norm

$$\|P\|_{\mathbb{L}_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{tr}\left(P^*(e^{i\omega})P(e^{i\omega})\right)d\omega\right)^{\frac{1}{2}}.$$

A subspace of $\mathbb{L}_2^{p\times m}(\Gamma)$ which consists of all rational transfer functions that have no poles in the exterior of the closed unit disk is denoted by H_2. The H_2-norm of a transfer function $P(z)\in H_2$ is defined by

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi}\|P(e^{i\omega})\|d\omega\right)^{\frac{1}{2}}.$$

Definition 7: Let $\mathbb{L}_\infty^{p\times m}(\Gamma)$ be the Banach space of matrix-valued functions $F:\Gamma\to\mathbb{C}^{p\times m}$ that are (essentially) bounded on Γ. The subspace of $\mathbb{L}_\infty^{p\times m}(\Gamma)$ denoted by H_∞ consists of all rational transfer functions that are analytic in the exterior of the closed unit disk. The H_∞-norm of the transfer function $P(z)\in H_\infty$ is defined by

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]}\sigma_{max}\left(P(e^{i\omega})\right) = \sup_{\omega\in[0,2\pi]}\|(P(e^{i\omega})\|_2.$$

B. Mean anisotropy and anisotropic norm of linear systems

Now, we provide some background material on the anisotropic analysis of linear discrete systems. The concepts of the mean anisotropy of Gaussian random sequences and of the anisotropic norm of linear systems were introduced in [1], [8].

Let $W = (w_k)_{k\geq 0}$ be a stationary sequence of square integrable vectors with values in $\mathbb{R}^m$, which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with the interval $[0,N-1]$, into a random vector $W_{0:N-1} = [\ w_0;\ \ldots;\ w_{N-1}\ ]$ we assume that $W_{0:N-1}$ is absolutely continuously distributed for every $N\geq 0$.

The anisotropy $\mathbf{A}(W)$ is defined as the minimal value of the relative entropy [3] with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrices described by:

$$\mathbf{A}(W) = \frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}(|W|^2) - h(W)\right)$$

where $h(W) = \mathbf{E}\ln f(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw$.

The mean anisotropy of the sequence W is defined by

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{3}$$

It is shown in [2] that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}(w_0;(w_k)_{k<0}) \tag{4}$$

where $\mathbf{I}(w_0;(w_k)_{k<0}) = \lim_{s\to-\infty}\mathbf{I}(w_0;W_{s:-1})$ is the Shannon mutual information [9] between w_0 and the past history $(w_k)_{k<0}$ of the sequence W.

Now, suppose that the stationary random sequence W is Gaussian. Then,

$$\mathbf{I}(w_0;(w_k)_{k<0}) = \frac{1}{2}\ln\det(\mathbf{cov}(w_0)\mathbf{cov}(\tilde{w}_0)^{-1}) \tag{5}$$

where

$$\tilde{w}_0 = w_0 - \mathbf{E}(w_0|(w_k)_{k<0}) \tag{6}$$

is the error of mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$, provided by the conditional expectation.

Suppose W is generated from V by a shaping filter G as

$$w(j) = \sum_{k=0}^{+\infty} g(k)v(j-k), \qquad j = \ldots,-1,0,1,\ldots$$

The impulse response of the filter $g(k)\in\mathbb{R}^{m\times m}$ is assumed to be square summable over $k\geq 0$, ensuring the mean square convergence of the series.

The transfer function of the filter $G(z) = \sum_{k=0}^{+\infty} g(k)z^k$ is supposed to belong to the Hardy space $H_2^{m\times m}$ – matrix-valued functions, analytic in the disc $|z|<1$ on the complex plane. The space is equipped with the H_2-norm, defined by

$$\|G\|_2 = \left(\sum_{k=0}^{+\infty}\mathrm{Trace}\left(g(k)g(k)^{\mathrm{T}}\right)\right)^{1/2} = \\ = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Trace}\,S(\omega)d\omega\right)^{1/2} \tag{7}$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}(\omega)^*$ $(-\pi \leq \omega \leq \pi)$ is a spectral density of W, $\widehat{G}(\omega) = G(\mathrm{e}^{i\omega})$ is the boundary value of the transfer function G.

The covariance matrix of the prediction error (5) and the spectral density $S(\omega)$ are related by the Szegö-Kolmogorov formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S(\omega)d\omega = \ln\det \mathbf{cov}(\tilde{w}_0). \quad (8)$$

By using (4) - (6), the Szegö limit theorem [10], and (8), the mean anisotropy (3) of the stationary Gaussian random sequence $W = GV$ may be computed in terms of the spectral density $S(\omega)$ and H_2-norm of the shaping filter G as

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det \frac{mS(\omega)}{\|G\|_2^2}d\omega = = -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\tilde{w}_0)}{\|G\|_2^2}. \quad (9)$$

It characterizes the divergence between the signal and the Gaussian white noise sequence. For more information see [1], [2].

Let $Y = PW$ be an output of the linear system $P \in H_\infty^{p\times m}$, its transfer function $P(z)$ is analytic in the disc $|z| < 1$. $P(z)$ has a finite H_∞-norm.

Definition 8: The a-anisotropic norm of the system P for a given parameter $a \geq 0$ is defined by

$$|||P|||_a = \sup\{\|PG\|_2/\|G\|_2 : \ G \in \mathbf{G}_a\}. \quad (10)$$

The fraction on the right-hand side of (10) can also be interpreted as the ratio of the power norms of the output Y and the input W against the class of shaping filters

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \ \overline{\mathbf{A}}(G) \leq a\right\}.$$

So the a-anisotropic norm $|||P|||_a$ describes a "stochastic gain" of the system P with respect to W.

III. Problem statement. Anisotropy-based bounded real lemma

Let's state the anisotropic analysis problem for descriptor systems. The system is given by the following equations:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k) \end{aligned} \quad (11)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the input signal, $y(k) \in \mathbb{R}^q$ is the output signal.

The system P is admissible. Suppose w is a stationary Gaussian random sequence whose mean anisotropy does not exceed $a \geq 0$, i.e. w is generated from the m-dimensional Gaussian white noise v (with zero mean and an identity covariance matrix) by an unknown shaping filter G which belongs to the family:

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \ \overline{\mathbf{A}}(G) \leq a\right\}.$$

The problem is to check the condition $|||P|||_a < \gamma$ for the given system P, parameters $a \geq 0$ and $\gamma > 0$.

To proof the main result it's necessary to define all-pass systems.

Definition 9: A system with a transfer function P such that $P^*P = I_m$ is called all-pass system.

Lemma 2: [12] The system (11) is all-pass system if there exists $\widehat{R} = \widehat{R}^{\mathrm{T}}$ which satisfies the conditions $E^{\mathrm{T}}\widehat{R}E \geq 0$ and

$$\begin{aligned} B^{\mathrm{T}}\widehat{R}B + D^{\mathrm{T}}D &= I, \\ B^{\mathrm{T}}\widehat{R}A + D^{\mathrm{T}}C &= 0, \\ A^{\mathrm{T}}\widehat{R}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\widehat{R}E &= 0. \end{aligned}$$

For the system (11) the following assumption holds true:

Assumption 1: We suppose that for the system (11) the condition

$$\operatorname{rank}\begin{pmatrix} A & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (12)$$

is satisfied.

Theorem 1: Let $P \in H_\infty^{p\times m}$ be an admissible system with the state-space realization (1), where $\rho(E,A) < 1$. For the given scalar quantities $a \geq 0$ and $\gamma > 0$ the a-anisotropic norm is bounded by γ, that is $|||P|||_a \leq \gamma$ if there exists $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ such that the inequality

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) \geq a \quad (13)$$

is satisfied for the matrix Σ associated with the stabilizing solution $\widehat{R} = \widehat{R}^{\mathrm{T}}$ of the algebraic Riccati equation

$$\begin{aligned} E^{\mathrm{T}}\widehat{R}E &= A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, && (14) \\ L &= \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C), && (15) \\ \Sigma &= (I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, && (16) \end{aligned}$$

besides $E^{\mathrm{T}}\widehat{R}E \geq 0$.

Proof: The power norm ratio $\|PG\|_2/\|G\|_2$ on the right-hand side of (10) and the mean anisotropy $\overline{\mathbf{A}}(G)$ in (9) are both invariant under the scaling of the shaping filter G. Assuming the system P to be fixed, they are completely specified by the normalized spectral density [11]:

$$\Pi(\omega) = \frac{mS(\omega)}{\|G\|_2^2} = \frac{2\pi mS(\omega)}{\int_{-\pi}^{\pi}\operatorname{tr}S(v)dv}, \quad (17)$$

then

$$\overline{\mathbf{A}}(G) = \alpha(\Pi) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\Pi(\omega)d\omega, \quad (18)$$

$$\frac{\|PG\|_2}{\|G\|_2} = \nu(\Pi) = \\ = \left(\frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}(\Lambda(\omega)\Pi(\omega))d\omega\right)^{1/2} \quad (19)$$

where the function Π, defined on the interval $[-\pi, \pi)$ by (17), takes values in the set of positive definite Hermitian matrices of order m and satisfies the condition $\int_{-\pi}^{\pi} \mathrm{Trace}\,\Pi(\omega)d\omega = 2\pi m$. Let $\hat{\Pi}$ denote the set of normalized spectral densities Π, and the function $\Lambda(\omega)$ is given by

$$\Lambda(\omega) = \hat{P}(\omega)^* \hat{P}(\omega). \quad (20)$$

Note that the squared functional $\nu(\Pi)^2$ is linear, and $\alpha(\Pi)$ is strictly convex with respect to Π. The strict convexity of α follows from the strict concavity of $\ln\det(\cdot)$ considered on a convex cone of positive defined matrices [13]. The strict convexity of α can also be obtained directly from the positive definiteness of its second variation

$$\delta^2\alpha(\Pi) = \\ \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{tr}(\delta\Pi(\omega)\Pi(\omega)^{-1}\delta\Pi(\omega)\Pi(\omega)^{-1})d\omega = \\ = \frac{1}{4\pi}\int_{-\pi}^{\pi} \|\Pi(\omega)^{-1/2}\delta\Pi(\omega)\Pi(\omega)^{-1/2}\|^2 d\omega \quad (21)$$

where $\delta\Pi$ is the variation of Π, and $\|M\| = (\mathrm{Trace}\,(M^*M))^{1/2}$ denotes the Frobenius norm of a matrix. In the equation (21) we have used the Frechet derivative $d\ln|\det\Sigma|/d\Sigma = \Sigma^{-1}$ and the first variation of the inverse nonsingular matrix $\delta(\Sigma^{-1}) = -\Sigma^{-1}(\delta\Sigma)\Sigma^{-1}$. Thus, the minimum value of the mean anisotropy of the disturbance W required to achieve a given level $\gamma > 0$ for the power norm ratio of the system is

$$\min_{\Pi\in\hat{\Pi}:\nu(\Pi)\geq\gamma} \alpha(\Pi) = \\ = -\frac{1}{4\pi} \max_{\Pi\in\hat{\Pi}:\nu(\Pi)^2\geq\gamma^2} \int_{-\pi}^{\pi} \ln\det\Pi(\omega)d\omega = \\ = \min_{0\leq q<\|P\|_\infty^{-2}:\mathscr{N}(q)\geq\gamma} \mathscr{A}(q). \quad (22)$$

By using the method of Lagrange multipliers, the first minimum in (22) is shown to be achieved at a spectral density which is proportional to

$$S_q(\omega) = (I_m - q\Lambda(\omega))^{-1} \quad (23)$$

where q is a subsidiary variable satisfying $0 \leq q < \|P\|_\infty^{-2}$.

Accordingly, the functions

$$\mathscr{A}(q) = \alpha(\Pi_q), \quad \mathscr{N}(q) = \nu(\Pi_q) \quad (24)$$

are defined by evaluating the functionals α and ν from (18) and (19) at the normalized spectral density

$$\Pi_q(\omega) = \frac{2\pi m S_q(\omega)}{\int_{-\pi}^{\pi} \mathrm{tr}\,S_q(\nu)d\nu}, \quad (25)$$

associated with (23) and (17). Excluding from consideration the trivial case where the function Λ in (20) is a constant matrix, $\mathscr{A}(q)$ and $\mathscr{N}(q)$ are both strictly increasing in q (see [14],[15]).This allows the minimum required mean anisotropy in (22) to be computed as $\mathscr{A}(\mathscr{N}^{-1}(\gamma))$, where $\mathscr{N}^{-1}$ denotes the functional inverse of $\mathscr{N}$. Therefore, the inequality $|||P|||_a \leq \gamma$ is equivalent to $\mathscr{A}(\mathscr{N}^{-1}(\gamma)) \geq a$. Now, (23) implies that $\Lambda(\omega) = (I_m - S_q(\omega)^{-1})/q$ and, hence

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}\,(\Lambda(\omega)S_q(\omega))\,d\omega = \\ = \frac{1}{q}\left(\frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}\,S_q(\omega)d\omega - 1\right), \quad (26)$$

which, in combination with the definition of the function $\mathscr{N}$ via (24), (25), and (19), yields

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}\,S_q(\omega)d\omega = \frac{1}{1 - q\mathscr{N}(q)^2}. \quad (27)$$

By substituting the last identity into the definition of $\mathscr{A}$ in (24), (25), and (18), it follows that the function can be represented as

$$\mathscr{A}(q) = \mathfrak{A}(q, \mathscr{N}(q)) \quad (28)$$

in terms of

$$\mathfrak{A}(q,\gamma) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det S_q(\omega)d\omega - \frac{m}{2}\ln(1 - q\gamma^2). \quad (29)$$

Since $-\ln(1 - q\gamma^2)$ is monotonically increasing in $\gamma \in [0, 1/\sqrt{q})$, then so is $\mathfrak{A}(q,\gamma)$. A remarkable property of the function $\mathfrak{A}(q,\gamma)$ is that it achieves its maximum with respect to q at the point $q = \mathscr{N}^{-1}(\gamma)$, where, in view of (28), it coincides with the function $\mathscr{A}$:

$$\max_{0\leq q<\|P\|_\infty^{-2}} \mathfrak{A}(q,\gamma) = \mathfrak{A}(\mathscr{N}^{-1}(\gamma),\gamma) = \mathscr{A}(\mathscr{N}^{-1}(\gamma)). \quad (30)$$

The significance of this property for establishing a criterion for the inequality $|||P|||_a \leq \gamma$ is explained by that (30) implies the equivalence between $\mathscr{A}(\mathscr{N}^{-1}(\gamma)) \geq a$ and the existence of $q \in [0, \|P\|_\infty^{-2})$, satisfying $\mathfrak{A}(q,\gamma) \geq a$. Therefore,

$$|||P|||_a \leq \gamma \Longleftrightarrow \mathfrak{A}(q,\gamma) \geq a \text{ for some } q. \quad (31)$$

The property (30) is verified by differentiating the function $\mathfrak{A}$ from (29) with respect to its first argument:

$$\partial\mathfrak{A}(q,\gamma)/\partial q = \\ = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det(I_m - q\Lambda(\omega))/\partial q d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} = \\ -\frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{tr}\,(\Lambda(\omega)S_q(\omega))\,d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} - \\ -\frac{m\mathscr{N}^2(q)}{2(1-q\mathscr{N}^2(q))} + \frac{m\gamma^2}{2(1-q\gamma^2)} = \\ \frac{m(\gamma^2 - \mathscr{N}^2(q))}{2(1-q\gamma^2)(1-q\mathscr{N}^2(q))}. \quad (32)$$

The function $\mathscr{N}$ is strictly monotonic, the representation (30) implies that $\partial\mathfrak{A}(q,\gamma)/\partial q$ is positive for $q < \mathscr{N}^{-1}(\gamma)$ and negative for $q > \mathscr{N}^{-1}(\gamma)$, which establishes (30). It now remains to represent the inequality $\mathfrak{A}(q,\gamma) \geq a$ for the function (29) in terms of the state-space dynamics of the system P. We denote that (23) describes the parametric family of the worst-case spectral densities of the input disturbance W for the admissible values of q. Since the subsidiary variable q will be fixed for the rest of the proof, we use the notation

$$S_\star(\omega) = (I_m - q\Lambda(\omega))^{-1}. \tag{33}$$

We will obtain a state-space representation of the worst-case disturbance $W_\star$ with the spectral density $S_\star$. In view of (20), the relation (33) is equivalent to

$$\widehat{\Theta}(\omega)^*\widehat{\Theta}(\omega) = I_m, \quad -\pi \leq \omega < \pi \tag{34}$$

where $\hat{\Theta}$ is a boundary value of the transfer function of the system

$$\Theta = \begin{bmatrix} \sqrt{q}\widehat{P}(\omega) \\ \widehat{G}_\star^{-1}(\omega) \end{bmatrix}. \tag{35}$$

Here, $\widehat{G}_\star$ is a shaping filter which, in accordance with [14], factorizes the worst-case spectral density (33) as $S_\star = \widehat{G}_\star\widehat{G}_\star^*$. The property (34) means that the system Θ is inner.

Let $L \in \mathbb{R}^{m\times n}$ be a matrix such that the pair $(E, A+BL)$ is admissible, $\Sigma \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix. We consider the worst-case input disturbance $W_\star = G_\star V$, which can be generated as

$$w_\star(k) = Lx(k) + \Sigma^{1/2}v(k). \tag{36}$$

The state-space representation of the shaping filter $G_\star$ is

$$G_\star = \left[E, \begin{array}{cc} A+BL & B\Sigma^{1/2} \\ L & \Sigma^{1/2} \end{array}\right]. \tag{37}$$

Since $G_\star$ is invertible, its inverse is described by

$$G_\star^{-1} = \left[E, \begin{array}{cc} A & B \\ -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array}\right]. \tag{38}$$

The state-space realization of the closed-loop system Θ is

$$\Theta = \left[E, \begin{array}{c|c} A & B \\ \hline q^{1/2}C & q^{1/2}D \\ -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array}\right]. \tag{39}$$

According to lemma 2, there exists a matrix $\widehat{R} = \widehat{R}^{\mathrm{T}}$, satisfying the condition $E^{\mathrm{T}}\widehat{R}E \geq 0$ such that

$$B^{\mathrm{T}}\widehat{R}B + \begin{bmatrix} q^{1/2}D^{\mathrm{T}} & (\Sigma^{-1/2})^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} q^{1/2}D \\ \Sigma^{-1/2} \end{bmatrix} = I, \tag{40}$$

$$B^{\mathrm{T}}\widehat{R}A + \begin{bmatrix} q^{1/2}D^{\mathrm{T}} & (\Sigma^{-1/2})^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} = 0, \tag{41}$$

$$A^{\mathrm{T}}\widehat{R}A + \begin{bmatrix} q^{1/2}C^{\mathrm{T}} & -L^{\mathrm{T}}(\Sigma^{-1/2})^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} - E^{\mathrm{T}}\widehat{R}E = 0. \tag{42}$$

Since Σ is a positive definite symmetric matrix, from the equations (40) and (41) we obtain

$$\Sigma = (I - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, \tag{43}$$

$$L = \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C). \tag{44}$$

These equations coincide with the equalities (15) and (16).

The expression (42) can be rewritten as

$$E^{\mathrm{T}}\widehat{R}E = A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L. \tag{45}$$

Since the worst-case input disturbance is described by (36), where $v(k)$ is a white noise sequence with the identity covariance matrix, the prediction error (6) takes the form $\tilde{w}(0) = \Sigma^{1/2}v(0)$ and, hence, $\mathbf{cov}(\tilde{w}(0)) = \Sigma$. Therefore, in combination with the Szegö-Kolmogorov formula (8), it implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_\star(\omega)d\omega = \ln\det\Sigma.$$

By substituting this equation in (29), we obtain

$$\mathfrak{A}(q,\gamma) = -\frac{1}{2}\ln\det\left((1-q\gamma^2)\Sigma\right).$$

Hence, the condition $\mathfrak{A}(q,\gamma) \geq a$ is equivalent to the inequality (13) for the matrix Σ, associated with generalized Riccati equation (14)-(16).

■

IV. Numerical example

To check whether the anisotropic norm of a given descriptor system is bounded by a set parameter γ, we apply the conditions of theorem 1.

Let the system P be described by $E = \begin{pmatrix} 0.9 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0.7 & -0.3 \\ 0.1 & 0.3 \end{pmatrix}$, $B = \begin{pmatrix} -0.02 \\ 0.07 \end{pmatrix}$,

$$C = \begin{pmatrix} 0.50 & 0.09 \end{pmatrix}, D = 0.035.$$

$$\operatorname{rank}\begin{pmatrix} A & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 2.$$

The system is admissible, because $\rho(E,A) = 0.889$.

H_∞-norm of the transfer function $\|P\|_\infty$ is equivalent to 0.249. To satisfy the conditions of theorem 1 the subsidiary parameter q should be $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ for the given mean anisotropy

TABLE I. CONDITIONS OF ANISOTROPY-BASED BOUNDED REAL LEMMA FOR $a = 0.08$ AND $|||P|||_a = 0.1197$

γ	0.130	0.120	0.110
$[0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$	$[0, 16.15)$		
q	10.756	13.767	-7605.318
$E^{\mathrm{T}}\widehat{R}E$	$\begin{pmatrix} 14.361 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 20.969 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -693.814 & 0 \\ 0 & 0 \end{pmatrix}$

TABLE II. CONDITIONS OF ANISOTROPY-BASED BOUNDED REAL LEMMA FOR $a = 1.08$ AND $|||P|||_a = 0.2358$

γ	0.250	0.240	0.236	0.234
$[0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$	$[0, 16.00)$	$[0, 16.15)$		
q	14.272	15.516	16.076	16.363
$E^{\mathrm{T}}\widehat{R}E$	$\begin{pmatrix} 22.443 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 27.348 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 32.030 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 33.740 - 4.103i & 0 \\ 0 & 0 \end{pmatrix}$

level a and parameter γ, where $\|P\|_\infty^{-2} = 16.15$, and the inequality $E^{\mathrm{T}}\widehat{R}E \geq 0$ should be true for the matrix $\widehat{R}$.

Let's consider some numerical experiments for different values of a and γ.

The results are given in the tables I and II. As we can see, the conditions of theorem 1 are satisfied for $|||P|||_a < \gamma$. For $|||P|||_a > \gamma$ the conditions get broken not only on q, but also on $\widehat{R}$.

Therefore, theorem 1 can be used for anisotropic norm calculation with any set accuracy.

V. CONCLUSION

Anisotropy-based bounded real lemma for descriptor systems defines conditions of anisotropic norm boundedness for admissible descriptor systems. This conditions consist of solvability of generalized Riccati equation under inequality constraint. Note that obtained conditions are equivalent to the conditions [11] for ordinary systems when $E = I$. They can be implemented for anisotropic norm calculation with any set accuracy. On basis of anisotropy-based bounded real lemma for descriptor systems the problem of suboptimal regulators' synthesis can be solved.

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