2013 International Conference on Process Control (PC)
June 18–21, 2013, Štrbské Pleso, Slovakia

Computation of anisotropic norm for descriptor systems using convex optimization

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***Abstract*—For linear discrete-time descriptor systems anisotropy-based bounded real lemma in terms LMI was formulated and proved. The algorithm of anisotropic norm computation using convex optimization is given. A numerical example is considered.**

I. Introduction

Descriptor systems, which have both dynamical and algebraic constraints, often appear in various engineering systems and control applications, including aircraft stabilization, chemical engineering systems, lossless transition lines, etc. The increased practical interest for a more general description reversed necessity of developing generalized methods of analysis and synthesis control for descriptor systems. The development of such algorithms has been an area of active research in the last years.

Besides, many standard state-space systems problems lead naturally to descriptor systems formulations. They can be solved reliably only by using descriptor systems computational techniques. Notable examples are certain model reduction problems, the solution of algebraic Riccati equations, the spectral and J-spectral factorization problems. For an overview of computational methods and software for descriptor systems see [1].

Generalization and development of analysis and synthesis algorithms for discrete-time descriptor systems is rather difficult because many algorithms are based on the solution of the discrete generalized Riccati equation.

The aim of this paper is to determine the LMI-conditions for boundedness of anisotropic norm of descriptor systems and to provide software for computation of anisotropic norm. Analysis of anisotropy-based performance for descriptor system (as for normal system) is a great interest, because of anisotropic norm lies between H_2 and H_∞-norm.

II. Background

This section covers basic concepts of discrete-time descriptor systems and anisotropic theory.

A. Elements of descriptor systems theory

Discrete-time descriptor systems are described by the following equations:

$$\begin{aligned} Ex(k+1) &= Ax(k)+Bw(k), \\ y(k) &= Cx(k)+Dw(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^q$ are the input and output signals, respectively. A,B,C,D are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ can be singular, such system is called singular. Assume that rank $(E) = nf \leq n$. Some basic properties of descriptor systems are associated with matrices E and A. In different literature [2], [3] it can be matrix pencil $(zE-A)$ or pair (E,A).

Definition 1: The pair (E,A) is said to be regular if there exists a scalar λ such that $\det(\lambda E - A) \neq 0$.

The regularity of the pair is (E,A) a necessary and sufficient condition of existence and uniqueness of the solution of the system (1) . The following lemma [2] provides necessary and sufficient conditions of regularity for the system (1):

Lemma 1: If the pair (E,A) is regular, than there exist invertible matrices Q and V such that

$$QEZ = \operatorname{diag}(I,N), \quad QAZ = \operatorname{diag}(A_1,I) \tag{2}$$

where $A_1 \in \mathbb{R}^{nf\times nf}$, N is a nilpotent matrix.

The matrix N is called a nilpotent matrix of the index h, if $N^h = 0$ and $N^{h-i} \neq 0$, $i = 1,2,\ldots,h$. The index of the system (1) in an equivalent form (2) is called the index of the nilpotent N.

According to lemma 1 the equations (1) can be written in the following form:

$$\begin{aligned} x_1(k+1) &= A_1x_1(k)+B_1w(k), && (3)\\ Nx_2(k+1) &= x_2(k)+B_2w(k), \\ y(k) &= C_1x_1(k)+C_2x_2(k)+Dw(k). && (4) \end{aligned}$$

Definition 2: Generalized spectral radius for the system (1) or for the pair (E,A) is:

$$\rho(E,A) \triangleq \frac{\max|\lambda|}{\lambda\in z|\det(zE-A)=0}.$$

- The system (1) is called causal if its solution $x(k)$ depends only on $u(k-1),...,u(0)$ $x(k-1),...,x(0)$ for any consistent initial conditions. It is true if the index of the nilpotent N is equal to 1.
- The system (1) is called stable if $\rho(E,A)<1$.
- The system (1) is said to be admissible if the pair (E,A) is regular and the system (1) is stable and causal.

Definition 3: The transfer function of the system (1) is given by

$$P(z)=C(zE-A)^{-1}B+D. \tag{5}$$

Definition 4: The H_2-norm of the transfer function $P(z)$ is defined by

$$\|P\|_2=\left(\frac{1}{2\pi}\int_0^{2\pi}\operatorname{tr}\left(P^*(e^{i\omega})P(e^{i\omega})\right)d\omega\right)^{\frac{1}{2}}= =\left(\frac{1}{2\pi}\int_0^{2\pi}\|P(e^{i\omega})\|d\omega\right)^{\frac{1}{2}}. \tag{6}$$

Definition 5: The H_∞-norm of the transfer function $P(z)\in H_\infty$ is defined by

$$\|P\|_\infty=\sup_{\omega\in[0,2\pi]}\sigma_{max}\left(P(e^{i\omega})\right)=\sup_{\omega\in[0,2\pi]}\|(P(e^{i\omega})\|_2.$$

Clearly, the H_∞-norm of $P(z)$ is finite only if $P(z)$ is proper.

B. Mean anisotropy and anisotropic norm

Now we provide a background material on the anisotropic analysis of linear discrete systems.The concepts of mean anisotropy of gaussian random sequences and of the anisotropic norm of linear systems were introduced in [4], [5]. An extended exposition of the mean anisotropy and the anisotropic norm can be found in [6].

Let $W=(w_k)_{-\infty<k<\infty}$ be a stationary sequence of square integrable vectors with values in $\mathbb{R}^m$ which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with a time interval $[s,t]$, into a random vector

$$W_{s:t}=\begin{bmatrix} w_s\\ \vdots\\ w_t\end{bmatrix}, \tag{7}$$

we assume that $W_{0:N}$ is absolutely continuously distributed for any $N\geq 0$. The anisotropy A(W) is defined as the minimal value of the relative entropy [3] with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrices described by:

$$A(W)=\frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}(|W|^2)-h(W)\right),$$

where

$$h(W)=\mathbf{E}\ln f(W)=-\int_{\mathbb{R}^m}f(w)\ln f(w)dw.$$

The mean anisotropy of the sequence W is defined by

$$\overline{\mathbf{A}}(W)=\lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N})}{N}. \tag{8}$$

It is shown in [6] that

$$\overline{\mathbf{A}}(W)=\mathbf{A}(w_0)+\mathbf{I}(w_0;(w_k)_{k<0}) \tag{9}$$

where $\mathbf{I}(w_0;(w_k)_{k<0})=\lim_{s\to-\infty}\mathbf{I}(w_0;W_{s:-1})$ is the Shannon mutual information [7] between w_0 and the past history $(w_k)_{k<0}$ of the sequence W.

Now, suppose that the stationary random sequence W is Gaussian. Then

$$\mathbf{I}(w_0;(w_k)_{k<0})=\frac{1}{2}\ln\det(\mathbf{cov}(w_0)\mathbf{cov}(\tilde{w}_0)^{-1}), \tag{10}$$

where

$$\tilde{w}_0=w_0-\mathbf{E}(w_0|(w_k)_{k<0}) \tag{11}$$

is the error of the mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$, provided by the conditional expectation.

Suppose W is generated from V by a shaping filter G as

$$w(j)=\sum_{k=0}^{+\infty}g(k)v(j-k),\qquad j=\ldots,-1,0,1,\ldots,$$

The impulse response of the filter $g(k)\in\mathbb{R}^{m\times m}$ is assumed to be square summable over $k\geq 0$, ensuring the mean square convergence of the series.

The transfer function of the filter $G(z)=\sum_{k=0}^{+\infty}g(k)z^k$ is supposed to belong to the Hardy space $H_2^{m\times m}$ – matrix-valued functions, analytic in the disc $|z|<1$ of the complex plane. The space is equipped with the H_2-norm, defined by

$$\|G\|_2=\left(\sum_{k=0}^{+\infty}\operatorname{tr}(g(k)g(k)^{\mathrm{T}})\right)^{1/2}= =\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{tr}(\widehat{G}(\omega)\widehat{G}(\omega)^*)d\omega\right)^{1/2}= =\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{tr}S(\omega)d\omega\right)^{1/2} \tag{12}$$

where $\widehat{G}(\omega)=G(e^{i\omega})$ is the boundary value of the transfer function G,
$S(\omega)=\widehat{G}(\omega)\widehat{G}(\omega)^*$, $-\pi\leq\omega\leq\pi$ is the spectral density of W.

The covariance matrix of the prediction error (11) and the spectral density $S(\omega)$ are related by the Szegö-Kolmogorov formula (13):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\det S(\omega)d\omega=\ln\det\mathbf{cov}(\tilde{w}_0). \tag{13}$$

By using (9) - (11), the Szegö limit theorem [8] and (13), the mean anisotropy (8) of the stationary Gaussian random sequence $W = GV$ may be computed in terms of the spectral density $S(\omega)$ and H_2-norm of the shaping filter G as

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2} d\omega = = -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\tilde{w}_0)}{\|G\|_2^2}. \quad (14)$$

It characterizes the divergence between the signal and the Gaussian white noise sequence. For more information see [5], [6].

Let $Y = FW$ be an output of the linear system $F \in H_\infty^{p\times m}$, its transfer function $F(z)$ is analytic in the disc $|z| < 1$. $F(z)$ has a finite H_∞-norm.

Definition 6: The a-anisotropic norm of the system F for a given parameter $a \geq 0$ is defined by

$$|||F|||_a = \sup\{\|FG\|_2/\|G\|_2 : \ G \in \mathbf{G}_a\}. \quad (15)$$

The fraction on the right-hand side of (15) can also be interpreted as the rario of the power norms of the output Y and the input W against the class of shaping filters

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \ \overline{\mathbf{A}}(G) \leq a\right\}.$$

So the a-anisotropic norm $|||F|||_a$ describes a "stochastic gain" of the system F with respect to W.

Lemma 2: [9] The pair (E,A) is regular, causal, and stable if and only if there exists an invertible symmetric matrix $R \in \mathbb{R}^{n\times n}$ such that $E^{\mathrm{T}}RE \geq 0$, and

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE < 0.$$

Theorem 1: [3] The pair (E,A) is admissible if and only if there exist matrices $R > 0$ and Q such that

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + QS^{\mathrm{T}}A + A^{\mathrm{T}}SQ^{\mathrm{T}} < 0, \quad (16)$$

where $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^{\mathrm{T}}S = 0$.

Lemma 3: (Bounded real lemma for descriptor systems [9]) The following two statements are equivalent:

1) A is a stable matrix (i.e., all eigenvalues of A lie inside a unit disk) and $\|C(zE-A)^{-1}B + D\|_\infty < \gamma$.
2) There exists a symmetric matrix R with $E^{\mathrm{T}}RE > 0$ satisfying

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + C^{\mathrm{T}}C + (A^{\mathrm{T}}RB + C^{\mathrm{T}}D) \times(\gamma^2 I - B^{\mathrm{T}}RB - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}RA + D^{\mathrm{T}}C) < 0 \quad (17)$$

with $\gamma^2 I - B^{\mathrm{T}}RB - D^{\mathrm{T}}D > 0$.

Theorem 2: Let $P \in H_\infty^{p\times m}$ be an admissible system with the state-space realization (1) where $\rho(E,A) < 1$. For the given scalar quantities $a \geq 0$ and $\gamma > 0$ the a-anisotropic norm is bonded by γ, that is

$$|||P|||_a \leq \gamma$$

if there exists the stabilizing solution $\widehat{R} = \widehat{R}^{\mathrm{T}}$ of the algebraic Riccati equation

$$E^{\mathrm{T}}\widehat{R}E = A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, \quad (18)$$

$$L = \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C), \quad (19)$$

$$\Sigma = (I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, \quad (20)$$

besides

$$E^{\mathrm{T}}\widehat{R}E \geq 0,$$

and $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ satisfies the equation

$$-\frac{1}{2}\ln\det((1-q\gamma^2)\Sigma) \geq a. \quad (21)$$

III. Main Result

The main result of this paper is given by the theorem. Consider a descriptor system with a state-space realization:

$$Ex(k+1) = Ax(k) + Bw(k), \quad (22)$$
$$y(k) = Cx(k) + Dw(k).$$

Assumption 1: We suppose that the relation

$$\mathrm{rank}\,(E) = \mathrm{rank}\,([\ E \quad B\ ]) \quad (23)$$

holds true for the system (22).

Lemma 4: Let assumption 1 be true for the stable system (22). Let T be a solution of the following generalized Lyapunov equation

$$A^{\mathrm{T}}TA - E^{\mathrm{T}}TE + C^{\mathrm{T}}C = 0, \quad (24)$$

then the H_2-norm of the system (22) can be computed as

$$\|P\|_2 = \sqrt{\mathrm{tr}(B^{\mathrm{T}}TB + D^{\mathrm{T}}D)}. \quad (25)$$

Proof: The assumption 1 means that there exists a matrix Q such that

$$QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

The H_2-norm of the descriptor system can be computed using the formula

$$\sqrt{\mathrm{tr}(B^{\mathrm{T}}G_oB + D^{\mathrm{T}}D)}$$

where G_o is observability Gramian, which can be found from the solution of generalized projected Lyapunov equation

$$A^{\mathrm{T}}G_oA - E^{\mathrm{T}}G_oE = -P_r^{\mathrm{T}}C^{\mathrm{T}}CP_r + (I-P_r)^{\mathrm{T}}C^{\mathrm{T}}C(I-P_r),$$
$$G_o = (I-P_l)^{\mathrm{T}}G_o(I-P_l) \quad (26)$$

with $P_r = Z^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Z$ and $P_l = Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$. By some algebraic manipulations it is easy to show that the equation (26) is equivalent to the equations [10]

$$\begin{aligned} A_1^{\mathrm{T}} G_{co} A_1 - G_{co} &= -C_1^{\mathrm{T}} C_1, & (27)\\ G_{no} - N^{\mathrm{T}} G_{no} N &= C_2^{\mathrm{T}} C_2. & (28) \end{aligned}$$

Here $G_{co} \in \mathbb{R}^{nf\times nf}$ and $G_{no} \in \mathbb{R}^{(n-nf)\times(n-nf)}$ are causal and noncausal observability Gramians respectively with $G_o = \mathrm{diag}(G_{co}, G_{no})$. The H_2-norm of the system can be computed as $\|p\|_2 = \sqrt{\mathrm{tr}\left(B_1^{\mathrm{T}} G_{co} B_1 + B_2^{\mathrm{T}} G_{no} B_2 + D^{\mathrm{T}} D\right)}$.

Choosing $T = Q^{\mathrm{T}} \tilde{T} Q$ from (25) we get

$$\begin{aligned} &\sqrt{\mathrm{tr}(B^{\mathrm{T}} T B + D^{\mathrm{T}} D)} = \\ &\qquad = \sqrt{\mathrm{tr}(B^{\mathrm{T}} Q^{\mathrm{T}} \tilde{T} Q B + D^{\mathrm{T}} D)} = \\ &= \sqrt{\mathrm{tr}(\begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \\ \tilde{T}_3 & \tilde{T}_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D^{\mathrm{T}} D)} = \\ &\qquad = \sqrt{\mathrm{tr}(B_1^{\mathrm{T}} \tilde{T}_1 B_1 + D^{\mathrm{T}} D)}. \quad (29) \end{aligned}$$

∎

Theorem 3: Let $w(k)$ be a stationary Gaussian random sequence whose mean anisotropy does not exceed known $a \geq 0$. Consider that assumption 1 holds. For the admissible system $P \in H_\infty^{p\times m}$ with a state space realization (22) a-anisotropic norm is bounded by a positive scalar $\gamma > 0$, i.e.

$$|||P|||_a \leq \gamma \quad (30)$$

if there exists a scalar $q \in (0, \min(\gamma^{-2}, \|P\|_\infty))$ and symmetric matrix R satisfying

$$ERE^{\mathrm{T}} \geq 0,$$
$$-(\det(I_m - B^T RB - qD^T D))^{1/m} < -(1-q\gamma^2)e^{2a/m}, \quad (31)$$

$$\begin{bmatrix} A^{\mathrm{T}} RA - E^{\mathrm{T}} RE & A^{\mathrm{T}} RB \\ B^{\mathrm{T}} RA & B^{\mathrm{T}} RB - I_m \end{bmatrix} + q\begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0 \quad (32)$$

Proof: The proof of the theorem consist of the following:

1) We shall show that the inequality (21) can be rewritten in the form (31). Then we shall show that (31) can be written as a convex set.
2) We shall prove that (32) holds true for an admissible system P.
3) On the last step we shall show that the pair $(E, A+BL) = (E, A + B(I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C))$ is admissible. Note that the pair $(E, A+BL)$ is connected to the worst case shaping filter [19] and has to be admissible too.

Using logarithm properties, we transform the inequality (21) and get

$$-\ln\det((1-q\gamma^2)\Sigma) \geq 2a,$$

$$-\ln\det\Sigma - \ln\det(I_m(1-q\gamma^2)) \geq 2a,$$

then

$$-\ln\det\Sigma \geq m\ln(1-q\gamma^2) + 2a,$$

$$-1/m\ln\det\Sigma \geq \ln\left(e^{2a/m}(1-q\gamma^2)\right),$$

$$\det\Sigma^{-1/m} \geq (1-q\gamma^2)e^{2a/m}.$$

By multiplication on (-1) we get

$$-\det\Sigma^{-1/m} \leq -(1-q\gamma^2)e^{2a/m}.$$

Note that

1) The function $(\det\Psi)^p$ of the $(m\times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geq 0$ is concave upward on its argument for any $0 \leq p \leq \frac{1}{m}$ [11].
2) The function $(\det\Psi)^{\frac{1}{m}}$ of the $(m\times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geq 0$ is the geometric mean of its eigenvalues $\sqrt[m]{\lambda_1(\Psi)\ldots\lambda_m(\Psi)}$.
3) It is shown in [11] that the set

$$\{(\lambda_1, \lambda_2, t) \in \mathbb{R}^3 | \lambda_1, \lambda_2 \geq 0, t \leq \sqrt{\lambda_1\lambda_2}\}$$

can be represented as a second-order cone

$$\left\{(\lambda_1, \lambda_2, t) | \exists\tau : t \leq \tau; \tau \geq 0, \left\| \begin{bmatrix} \tau \\ \frac{\lambda_1-\lambda_2}{2} \end{bmatrix} \right\|_2 \leq \frac{\lambda_1-\lambda_2}{2} \right\}, \quad (33)$$

and the set $\{(\lambda_1, \ldots, \lambda_{2^l}, t) \in \mathbb{R}^{2^l+1} | \lambda_i \geq 0, i = 1, \ldots, 2^l, t \leq (\lambda_1\lambda_2\ldots\lambda_{2^l})^{1/2^l}\}$ is an intersection of finite numbers of second-order cones.

Consider a linear matrix inequality. Using Schur complement we have

$$\begin{aligned} &A^{\mathrm{T}} RA - E^{\mathrm{T}} RE + qC^{\mathrm{T}} C + \\ &\quad + (ARB^{\mathrm{T}} + qC^{\mathrm{T}} D)(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D)^{-1} \times \\ &\qquad \times (BRA^{\mathrm{T}} + qD^{\mathrm{T}} C) \leq 0. \quad (34) \end{aligned}$$

Taking into account that $P \in H_\infty^{p\times m}$ we have

$$(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D) > 0.$$

The system P is admissible then from lemma 2 there exists such R that (34) (or equivalently (32)) holds true. Now, if (34) is satisfied then there exists a nonnegative matrix Ξ such that

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + qC^{\mathrm{T}}C + \\ + (ARB^{\mathrm{T}} + qC^{\mathrm{T}}D)(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1} \times \\ \times (BRA^{\mathrm{T}} + qD^{\mathrm{T}}C) + \Xi = 0. \quad (35)$$

The anisotropic norm of the linear system is connected with a generating filter G with a state space realization $G = \left[E, \begin{array}{cc} A+BL & B\Sigma^{1/2} \\ L & \Sigma^{1/2} \end{array}\right]$, where matrices L and Σ are defined in (19) and (20), respectively. Prove that the pair $(E, A+BL)$ is admissible.

$$(A+BL)^{\mathrm{T}}\tilde{R}(A+BL) - E^{\mathrm{T}}\tilde{R}E = \\ = A^{\mathrm{T}}\tilde{R}A - E^{\mathrm{T}}\tilde{R}E + \\ + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}A + A^{\mathrm{T}}\tilde{R}BL + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}BL. \quad (36)$$

The system (22) is admissible. Using theorem 1 there exists a matrix $\hat{R}$ such that

$$A^{\mathrm{T}}\hat{R}A - E^{\mathrm{T}}\hat{R}E + L^{\mathrm{T}}B^{\mathrm{T}}\hat{R}A + A^{\mathrm{T}}\hat{R}BL < 0.$$

It means that there exists a matrix $\tilde{R}$ such that

$$A^{\mathrm{T}}\tilde{R}A - E^{\mathrm{T}}\tilde{R}E + \\ + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}A + A^{\mathrm{T}}\tilde{R}BL + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}BL < 0 \quad (37)$$

holds true. This completes the proof. ■

Because of multiplication on q and γ^2, direct computation of the a-anisotropic norm for the given system is not possible, additional algorithms are required. Define $\eta = q^{-1}$ and $\xi = \gamma^2$. Multiplying both inequalities on η and taking into account ξ the conditions of theorem 3 for the admissible system (1) can be rewritten as

$$\eta - (e^{-2a}\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} \leq \xi, \quad (38)$$

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0, \quad (39)$$

$$E^{\mathrm{T}}\Phi E \geq 0 \quad (40)$$

where $\Phi = \eta R$. Conditions (38) and (39) are linear on ξ. They allow us to calculate the minimum value of γ by solving the following convex optimization problem: to find $\xi_* = \inf \xi$ on the set $\{\Phi, \eta, \xi\}$ that satisfies (38), (39). If the minimum value ξ_* is found, then the a-anisotropic norm of the system P can be approximately calculated as

$$|||P|||_a \approx \sqrt{\xi_*}. \quad (41)$$

Limiting cases. Consider two cases when $a = 0$ and $a \to \infty$.

If $a = 0$ then inequality (38) takes the form

$$\eta - (\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} \leq \gamma^2. \quad (42)$$

Taking into account the relation between geometric mean and arithmetic mean [14] we get

$$(\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} \leq \\ \leq \frac{1}{m}\operatorname{tr}(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D)).$$

The inequality

$$\operatorname{tr}(B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D) < m\gamma^2 \quad (43)$$

follows from (42). It is easy to check that the inequality (39) holds true if

$$A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C < 0. \quad (44)$$

Conditions (42) and (44) are equivalent to

$$\frac{1}{\sqrt{m}}\|P\|_2 < \gamma.$$

In the case $a \to \infty$ we have $\eta \to \gamma^2$. This makes condition (38) inactive. Defining $\bar{\Phi} = \gamma\Phi$ it is easy to see that inequality (39) is equivalent to (17). It means that for $a \to \infty$ the condition $|||P|||_a < \gamma$ is equivalent to $\|P\|_\infty < \gamma$.

IV. Numerical example

The system is described by equations

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.7 & -3.25 & -0.7 & 0 \\ 1.8 & 0.4 & -6.4 & 2.6 \\ 1 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix}, B = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.4 & 0.45 & 0.6 \end{bmatrix}, D = \begin{bmatrix} 0.2 & 1 \end{bmatrix}.$$

The rank condition (23) holds true. The results of anisotropic norm computation using LMI-based algorithm and Riccati-based algorithm are presented on fig. 1. Fig. 2 demonstrates an absolute error between values. Simulation results show that suggested algorithm allows to compute anisotropic norm of descriptor systems with high accuracy and without using algebraic transformation for the given system instead of the algorithm in [15].

V. Conclusion

In this paper, anisotropy-based bounded real lemma for descriptor systems in terms of LMI is given and proved. The obtained conditions make it possible to check whether the anisotropic norm of the system (1) is bounded by a given numerical value. Note that these conditions are sufficient and more conservative than the conditions of anisotropic-norm boundedness in theorem 2. The lifting of restrictions is the topic for further work.

The criterion is written as a system of inequalities which consists of LMI and the inequality concerning the determinant of the positive defined matrix and

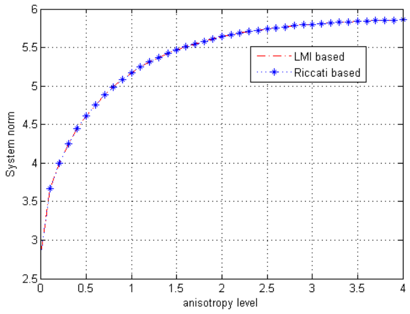

Fig. 1. Anisotropic norm of the system computed using Riccati and LMI algorithms.

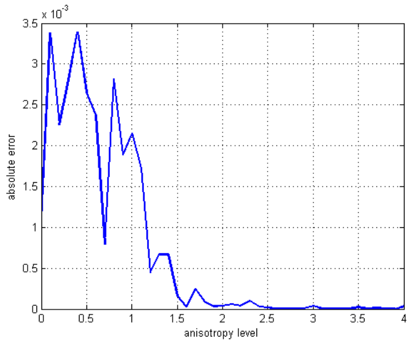

Fig. 2. Absolute error between norm values computed different algorithms.

the scalar parameter. Anisotropy-based bounded real lemma for descriptor systems in terms of LMI is an important result for anisotropic suboptimal controllers synthesis using the methods of convex optimization and semidefinite programming. Such controllers guarantee that the anisotropic norm of the closed-loop system is bounded by a given value.

ACKNOWLEDGMENT

A. Belov is supported by the Russian Foundation for Basic Research (grant 11-08-00714-a) and Program for Fundamental Research No. 14.

O. Andrianova is supported by the Russian Foundation for Basic Research (grant 12-07-00329).

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