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Robust anisotropy-based control of linear discrete-time descriptor systems with norm-bounded uncertainties ⋆

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Abstract: The paper deals with linear discrete-time time-invariant (LDTI) descriptor systems with norm-bounded parametric uncertainties. The input signal is supposed to be a "colored" noise with bounded mean anisotropy. Sufficient conditions of anisotropic norm boundedness for such class of systems are given. The control design procedure, based on these conditions, is proposed. Numerical example is given.

Keywords: Descriptor systems, suboptimal control, coloured noise, stochastic control, uncertain linear systems, robust performance, robust control.

1. INTRODUCTION

Descriptor systems describe a natural representation of physical systems and contain not only dynamical modes but also algebraic constraints on their variables. Being a general case of state-space systems (also referred to as normal systems), descriptor systems arise in control problems of constrained mechanical systems, in electrical circuit simulation and others (see (Dai, 1989; Duan, 2010) and references therein).

In recent years, robust control has become one of the most popular research areas in control theory. Considerable attention is paid to problems of robust stabilization and robust performance analysis of uncertain systems in both continuous and discrete-time cases. $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms are the most popular criteria in robust performance analysis and control of linear systems. In discrete-time $\mathcal{H}_\infty$-approach, the input signal is assumed to be square summable, i.e. it has to be with limited power. The squared $\mathcal{H}_2$-norm of a linear time-invariant system can be interpreted as the trace of its steady-state output covariance matrix under the assumption that the system is driven by the Gaussian white noise with an identity covariance matrix. So, $\mathcal{H}_2$-norm is a useful measure of performance when the system is affected by the Gaussian white noise.

Interest in stability analysis and control of descriptor systems with parametric uncertainties has grown recently due to their frequent presence in dynamical systems. Uncertainties in such systems are often causes of instability and bad performance. It is known that control of uncertain descriptor systems is much more complicated than that of the normal ones. Robust $\mathcal{H}_\infty$ control problem for discrete-time descriptor systems with parametric uncertainties is investigated in (Ji *et al.*, 2007; Xu & Lam, 2006; Chadli & Darouach, 2014; Coutinho *et al.*, 2014).

In anisotropy-based control theory, the input disturbance is assumed to be a stationary random sequence with a known mean-anisotropy level (Vladimirov *et al.*, 1995, 2006). In this case, anisotropic norm of the system defines its performance in presence of the input signal. The feature of anisotropy-based approach is that anisotropic norm of the system lies between the scaled $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm. Hence, the solution of robust control problem for anisotropy-based case automatically solves $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control problems as special limiting cases.

Problems of anisotropy-based performance analysis and suboptimal control design for descriptor systems with certain parameters are studied in (Belov & Andrianova, 2016). The obtained result provides numerically effective method of control design for descriptor systems.

The aim of this paper is to provide conditions of anisotropic norm boundedness for descriptor systems with norm-bounded parametric uncertainties and to develop methods of control design which make the closed-loop uncertain system admissible with prescribed anisotropy-based performance. The paper is organized as follows. In Section 2, basics of linear discrete-time descriptor systems and anisotropy-based control theory are given. In Section 3, problem statements and main results are discussed. A numerical example is introduced in Section 4.

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In this paper, the following denotations are used: Z^* is the Hermitian conjugate of the matrix $Z = [z_{ij}] \in \mathbb{C}^{m\times n}$: $Z^* = [z^*_{ji}] \in \mathbb{C}^{n\times m}$; $\overline{\sigma}(A)$ stands for the maximal singular value of the matrix A: $\overline{\sigma}(A) = \sqrt{\rho(A^*A)}$; sym (A) stands for symmetrization of the matrix A: $\operatorname{sym}(A) = A + A^{\mathrm{T}}$.

2. PRELIMINARIES

In this section, main definitions, concepts, and theorems from the theories of descriptor systems (Dai, 1989; Xu & Lam, 2006) and anisotropy-based control (Diamond *et al.*, 2001; Vladimirov *et al.*, 1996) are given.

2.1 LDTI descriptor systems

A state-space representation of discrete-time descriptor systems is

$$Ex(k+1) = Ax(k) + Bw(k), \tag{1}$$

$$y(k) = Cx(k) + Dw(k) \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^q$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively, E, A, B, C and D are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, i.e. rank $(E) = r < n$.

Definition 1. System (1) is called regular if $\exists\lambda \neq 0$: $\det(\lambda E - A) \neq 0$.

Regularity stands for existence and uniqueness of the solution for consistent initial conditions (Dai, 1989).

Hereinafter, we suppose that the considered systems are regular. Now we give some definitions, necessary for further understanding.

Definition 2. System (1) is called admissible if it is regular, causal ($\deg\det(zE - A) = \operatorname{rank} E$), and stable ($\rho(E, A) = \max|\lambda|_{\lambda\in\{z|\det(zE-A)=0\}} < 1$). For more information, see (Dai, 1989; Xu & Lam, 2006).

Definition 3. The transfer function of system (1)–(2) is defined by the expression

$$P(z) = C(zE - A)^{-1}B + D,\ z \in \mathbb{C}. \tag{3}$$

$\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the transfer function $P(z)$ are defined as follows

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left(P^*(e^{i\omega})P(e^{i\omega})\right)d\omega\right)^{\frac{1}{2}},$$

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \overline{\sigma}\left(P(e^{i\omega})\right).$$

For regular system (1)–(2) there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that $\overline{W}E\overline{V} = \operatorname{diag}(I_r, 0)$ (see (Dai, 1989)).

Consider the following change of variables

$$\overline{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{4}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

By left multiplying system (1)–(2) on the matrix $\overline{W}$ and using change of variables (4), one can rewrite system (1)–(2) in the form

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1w(k), \tag{5}$$

$$0 = A_{21}x_1(k) + A_{22}x_2(k) + B_2w(k), \tag{6}$$

$$y(k) = C_1x_1(k) + C_2x_2(k) + D_dw(k). \tag{7}$$

The following denotation will be used below

$$A_d = \overline{W}A\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad B_d = \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$C_d = C\overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\quad D_d = D. \tag{8}$$

Matrices $\overline{W}$ and $\overline{V}$ are found from the singular value decomposition (SVD) of the matrix E (see (Belov & Andrianova, 2016)).

2.2 Mean anisotropy and a-anisotropic norm

Anisotropy of the random vector is Kullback-Leibler information divergence from the probability density function (p.d.f.) of the vector to p.d.f. of the Gaussian white noise sequence (Vladimirov *et al.*, 1995). Mean anisotropy of the sequence stands for anisotropy averaged on discrete time.

Introduce denotations, necessary for mean anisotropy computation. The input signal $w(k)$ is assumed to be a random colored noise. Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary sequence of square-summable random m-dimensional vectors. The sequence W can be generated from the Gaussian white noise sequence $V = \{v(k)\}_{k\in\mathbb{Z}}$ with zero mean and identity covariance matrix by an admissible shaping filter with a transfer function $G(z)$.

Mean anisotropy of the sequence can be computed by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(le^{i\omega})$ is a boundary value of the transfer function $G(z)$.

Remark 1. Mean anisotropy of the random sequence W, generated by shaping filter $G(z)$, is fully defined by its parameters, so the notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(W)$ are equivalent.

Mean anisotropy of the signal characterizes its "spectral color", i.e. the difference between the signal and the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) = 0$, then the signal is the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) \to \infty$, the signal is a determinate sequence. For more information, see (Vladimirov *et al.*, 1995, 2006).

Let $Y = PW$ be an output of the linear discrete-time (normal or descriptor) system $P \in \mathcal{H}_\infty{}^{p\times q}$ with a transfer function $P(z)$, which is analytic in the identity circle $|z| < 1$, $P(z)$ has a finite $\mathcal{H}_\infty$-norm.

Definition 4. For a given constant value $a \geqslant 0$ a-anisotropic norm of the system P is defined as

$$|\!|\!| P |\!|\!|_a = \sup\left\{\|PG\|_2/\|G\|_2 :\ G \in \mathbf{G}_a\right\}, \tag{9}$$

i.e. the maximum value of the system's gain with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G \in \mathcal{H}_2{}^{q\times q} :\ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

So, a-anisotropic norm $|\!|\!| P(z) |\!|\!|_a$ describes the stochastic gain of the system $P(z)$ with respect to the input sequence W. The problem of anisotropy-based performance analysis of the admissible system (1)–(2) is to check the condition $\|P(z)\|_a < \gamma$ for a given scalar $\gamma > 0$ and a known mean anisotropy level of the input disturbance $a \geqslant 0$.

Theorem 1. (Belov & Andrianova, 2016) Let the system (1)–(2) with a transfer function $P(z) \in \mathcal{H}_\infty{}^{p\times q}$ be admissible. Suppose that

$$\operatorname{rank}\,[\,E\ \ B\,] = \operatorname{rank} E. \tag{10}$$

For given scalar values $a \geqslant 0$ and $\gamma > 0$ a-anisotropic norm of the system is bounded by γ, i.e.

$$\|P(z)\|_a < \gamma$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$,$\Psi \in \mathbb{R}^{q\times q}$, scalar values $\eta > \gamma^2$ and $\alpha > 0$, for which the following inequalities hold true

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/q} < \gamma^2, \tag{11}$$

$$\begin{bmatrix} \Psi - \eta I_q + B_d^{\mathrm{T}}\Theta B_d & D_d^{\mathrm{T}} \\ D_d & -I_p \end{bmatrix} < 0, \tag{12}$$

and (13),

where

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},\ \Gamma = [\,Q\ \ R\,].$$

The following results are used for matrix inequalities transformations below.

Lemma 2. (Petersen, 1987)
Let matrices $M \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{q\times n}$ be nonzero, and $G = G^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. The inequality

$$G + M\Delta N + N^{\mathrm{T}}\Delta^{\mathrm{T}}M^{\mathrm{T}} \leqslant 0 \tag{14}$$

is true for all $\Delta \in \mathbb{R}^{p\times q}$: $\|\Delta\|_2 \leqslant 1$ if and only if there exists a scalar value $\varepsilon > 0$ such that

$$G + \varepsilon MM^{\mathrm{T}} + \frac{1}{\varepsilon}N^{\mathrm{T}}N \leqslant 0. \tag{15}$$

Lemma 3. (Schur lemma introduced in (Boyd et al., 1994))
Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}$$

where X_{11} and X_{22} are square matrices.

If $X_{11} > 0$, then $X > 0$ if and only if

$$X_{22} - X_{12}^{\mathrm{T}}X_{11}^{-1}X_{12} > 0. \tag{16}$$

If $X_{22} > 0$, then $X > 0$ if and only if

$$X_{11} - X_{12}X_{22}^{-1}X_{12}^{\mathrm{T}} > 0. \tag{17}$$

3. PROBLEM STATEMENTS AND MAIN RESULTS

Consider the following discrete-time descriptor system:

$$Ex(k+1) = A_\Delta x(k) + B_{\Delta w}w(k) + B_u u(k), \tag{18}$$

$$y(k) = C_\Delta x(k) + D_{\Delta w}w(k) \tag{19}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^q$ is a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^p$ is the output, $u(k) \in \mathbb{R}^m$ is the control input. The matrix E is singular, i.e. $\operatorname{rank} E = r < n$. $A_\Delta = A + M_A\Delta N_A$, $B_{\Delta w} = B_w + M_B\Delta N_B$, $C_\Delta = C + M_C\Delta N_C$, $D_{\Delta w} = D_w + M_D\Delta N_D$.

The matrix $\Delta \in \mathbb{R}^{s\times s}$ is unknown norm-bounded, i.e. $\|\Delta\|_2 \leqslant 1$ (or Frobenius norm-bounded matrix as $\|\Delta\|_2 \leqslant \|\Delta\|_F$). Note that $\|\Delta\|_2 := \overline{\sigma}(\Delta) \leqslant 1$ iff $\Delta^{\mathrm{T}}\Delta \leqslant I_s$.

Introduce denotations

$$M_B^d = \overline{W}M_B = \begin{bmatrix} M_{B1}^d \\ M_{B2}^d \end{bmatrix},\ N_B^d = N_B,\ M_A^d = \overline{W}M_A,$$
$$N_A^d = N_A\overline{V},\ M_C^d = M_C,\ N_C^d = N_C\overline{V} = \left[\,N_{C1}^d\ \ N_{C2}^d\,\right],$$
$$B_{wd} = \overline{W}B_w = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_{ud} = \overline{W}B_u,\ D_{wd} = D_w.$$

Other denotations are taken from Section 2.

Suppose that

$$\operatorname{rank} E^{\mathrm{T}} = \operatorname{rank}\left[E^{\mathrm{T}}, C^{\mathrm{T}}, N_C^{\mathrm{T}}\right], \tag{20}$$

$$\operatorname{rank} E = \operatorname{rank}\,[E, B_w, M_B]. \tag{21}$$

In this paper we consider two problems:

- anisotropy-based analysis of system (18)–(19). This problem is investigated in subsection 3.1;
- state-feedback anisotropy-based control design for (18)–(19). The solution of this problem is given in subsection 3.2.

3.1 Anisotropy-based analysis for uncertain descriptor systems

In anisotropy-based analysis problem the control input is assumed to be zero, i.e. $B_u = 0$. The output $y(k)$ is considered as a measurable output. System (18)–(19) is supposed to be admissible for all Δ from the given set. Its transfer function is given by $P_\Delta(z) = C_\Delta(zE - A_\Delta)^{-1}B_{\Delta w} + D_{\Delta w}$.

For known values $a \geqslant 0$ and $\gamma > 0$ the problem is to find the conditions, which allow to check that the inequality

$$\|P_\Delta(z)\|_a < \gamma$$

holds true.

Theorem 4. (Andrianova & Belov, 2016) For given scalars $a \geqslant 0$ and $\gamma > 0$ system (18)–(19) is admissible and its a-anisotropic norm $\|P_\Delta(z)\|_a < \gamma$, if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and matrices $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$,$\Psi \in \mathbb{R}^{q\times q}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$, $\Upsilon \in \mathbb{R}^{r\times r}$, $\Upsilon > 0 : \Upsilon L = I_r$, such that

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/q} < \gamma^2, \tag{22}$$

$$\begin{bmatrix} \mho + \varepsilon_1 N_1^{\mathrm{T}}N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon_1 I \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}}N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I \end{bmatrix} < 0. \tag{24}$$

Here

$$\mho = \begin{bmatrix} \Psi - \eta I_m & D_{wd}^{\mathrm{T}} & B_1^{\mathrm{T}} \\ D_{wd} & -I_p & 0 \\ B_1 & 0 & -\Upsilon \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 \\ M_D & 0 \\ 0 & M_{B1}^d \end{bmatrix},\ N_1 = \begin{bmatrix} N_D & 0 & 0 \\ N_B^d & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2}Q-\frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & L^{\mathrm{T}}-Q^{\mathrm{T}}-\frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} + \alpha A_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_q & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_d^{\mathrm{T}} + \alpha B_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ L-Q-\frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & -Q-Q^{\mathrm{T}} & 0 \\ 0 & C_d + \alpha C_d \Pi A_d & D_d + \alpha C_d \Pi B_d & 0 & -I_p \end{bmatrix} < 0 \tag{13}$$

$$\Sigma = \begin{bmatrix} -\frac{1}{2}Q-\frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_{wd} & L^{\mathrm{T}}-Q^{\mathrm{T}}-\frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_{wd} & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} \\ B_{wd}^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_{wd}^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_q & B_{wd}^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_{wd}^{\mathrm{T}} \\ L-Q-\frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_{wd} & -Q-Q^{\mathrm{T}} & 0 \\ 0 & C_d & D_{wd} & 0 & -I_p \end{bmatrix} \tag{25}$$

$$M_2 = \begin{bmatrix} \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ \Pi M_A^d & \Pi M_B^d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ 0 & 0 & M_C^d & M_D \end{bmatrix},\ N_2 = \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & 0 & N_B^d & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},\ \Gamma = [\, Q \ \ R \,].$$

The matrix Σ is defined by (25).

Proof. Under the assumptions (20) and (21) $B_2 = 0$ and $C_2 = 0$. It's easy to check that in (13) $\alpha C_d \Pi A_d = 0$ and $\alpha C_d \Pi B_{wd} = 0$. Consider the inequality (12) from Theorem 1. Taking into account $B_2 = 0$, transform the expression $B_{wd}^{\mathrm{T}}\Theta B_{wd} = [\, B_1^{\mathrm{T}} \ 0 \,] \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = B_1^{\mathrm{T}} L B_1 > 0$. So the inequality (12) is equal to

$$\begin{bmatrix} \Psi - \eta I_q + B_1^{\mathrm{T}} L B_1 & D_{wd}^{\mathrm{T}} \\ D_{wd} & -I_p \end{bmatrix} < 0,$$

using Schur complement and denoting $\Upsilon = L^{-1}$, we have

$$\begin{bmatrix} \Psi - \eta I_q & D_{wd}^{\mathrm{T}} & B_1^{\mathrm{T}} \\ D_{wd} & -I_p & 0 \\ B_1 & 0 & -\Upsilon \end{bmatrix} < 0. \tag{26}$$

Now we write the inequality of the form (26) for the system (18)–(19) with norm-bounded uncertainties:

$$\begin{bmatrix} \Psi - \eta I_q & (D_{wd} + M_D \Delta N_D)^{\mathrm{T}} & (B_1 + M_{B1}^d \Delta N_B^d)^{\mathrm{T}} \\ D_{wd} + M_D \Delta N_D & -I_p & 0 \\ B_1 + M_{B1}^d \Delta N_B^d & 0 & -\Upsilon \end{bmatrix} < 0 \tag{27}$$

or

$$\mho + \mathrm{sym}\,(M_1 \Delta N_1) < 0. \tag{28}$$

Using the conditions of Lemmas 2 and 3, we can rewrite inequality (28) as (23). Now we transform expression (13) for system (18)–(19)

$$\Sigma + \mathrm{sym}\,(M_2 \Delta N_2) < 0. \tag{29}$$

Applying Lemmas 2 and 3 to inequality (29), we get

$$\Sigma + \frac{1}{\varepsilon_2} M_2 M_2^{\mathrm{T}} + \varepsilon_2 N_2^{\mathrm{T}} N_2 < 0,$$

$$\Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 - M_2 (-\varepsilon_2 I)^{-1} M_2^{\mathrm{T}} < 0,$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I \end{bmatrix} < 0.$$

The last inequality coincides with (24). Expression (22) is equal to (11). Consequently, the conditions of Theorem 1 hold true for system (18)–(19), it means that its anisotropic norm is bounded by a positive scalar value, i.e. $|\!|\!| P_\Delta(z) |\!|\!|_a < \gamma$.

The mutual inverse matrices search procedure can be found in (Balandin & Kogan, 2005).

Remark 2. If $M_B = 0$ and $N_B = 0$, then the conditions of Theorem 4 become simpler:

$$\eta - (\mathrm{e}^{-2a} \det(\Psi))^{1/q} < \gamma^2,$$

$$\begin{bmatrix} \mho + \varepsilon_1 N_1^{\mathrm{T}} N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon_1 I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I \end{bmatrix} < 0.$$

Here

$$\mho = \begin{bmatrix} \Psi - \eta I_q + B_1^{\mathrm{T}} L B_1 & D_{wd}^{\mathrm{T}} \\ D_{wd} & -I_p \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 \\ M_D \end{bmatrix},\ N_1 = [\, N_D \ \ 0 \,].$$

In this case, the algorithm of mutually inverse matrices computation in order to find Υ is no longer required.

3.2 State-space anisotropy-based robust control design for uncertain descriptor systems

In state-space anisotropy-based robust control design problem the system is supposed to be noncausal and unstable, $y(k)$ stands for controllable output. The problem is to find a feedback gain $u(k) = Fx(k)$ such that the closed-loop system with a transfer function

$$P_\Delta^{cl}(z) = C_\Delta (zE - (A_\Delta + B_u F))^{-1} B_{\Delta w} + D_{\Delta w}$$

is admissible and

$$|\!|\!| P_\Delta^{cl}(z) |\!|\!|_a < \gamma$$

for all Δ from the given set.

Assume that

(1) system (18) is causally controllable;
(2) system (18) is stabilizable;
(3) a mean anisotropy of the input disturbance is bounded: $\overline{\mathbf{A}}(W) \leqslant a$ (a is a known value);
(4) a scalar value $\gamma > 0$ is given;
(5) $p \leqslant q$.

The definitions of causal controllability and stabilizability can be found in (Dai, 1989) and (Xu & Lam, 2006).

The following theorem defines the control design procedure.

Theorem 5. For a given scalar $\gamma > 0$ and a known mean anisotropy level a ($\overline{\mathbf{A}}(W) \leqslant a$) the control design problem is solvable if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and matrices $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Psi \in \mathbb{R}^{p\times p}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$, $\Upsilon \in \mathbb{R}^{r\times r}$, $\Upsilon > 0$, and $Z \in \mathbb{R}^{n\times m}$ such that

$$\Upsilon L = I_r \tag{30}$$

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/p} < \gamma^2, \tag{31}$$

$$\begin{bmatrix} \mho + \varepsilon_1 M_1^{\mathrm{T}} M_1 & N_1 \\ N_1^{\mathrm{T}} & -\varepsilon_1 I_{2s} \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \Lambda + \varepsilon_2 M_2^{\mathrm{T}} M_2 & N_2 \\ N_2^{\mathrm{T}} & -\varepsilon_2 I_{4s} \end{bmatrix} < 0 \tag{33}$$

where

$$\mho = \begin{bmatrix} \Psi - \eta I_p & D_{wd} & C_1 \\ D_{wd}^{\mathrm{T}} & -I_q & 0 \\ C_1^{\mathrm{T}} & 0 & -\Upsilon \end{bmatrix},$$

$$M_1 = \begin{bmatrix} (M_D)^{\mathrm{T}} & 0 & 0 \\ (M_C^d)^{\mathrm{T}} & 0 & 0 \end{bmatrix},\ N_1 = \begin{bmatrix} 0 & 0 \\ (N_D)^{\mathrm{T}} & 0 \\ 0 & (N_{C1}^d)^{\mathrm{T}} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & (M_A^d)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & (M_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & (M_B^d)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & (M_D)^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \tag{34}$$

$$N_2 = \begin{bmatrix} \Gamma(N_A^d)^{\mathrm{T}} & \Gamma(N_C^d)^{\mathrm{T}} & 0 & 0 \\ \Pi(N_A^d)^{\mathrm{T}} & \Pi(N_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma(N_A^d)^{\mathrm{T}} & \Gamma(N_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & (N_B^d)^{\mathrm{T}} & (N_D)^{\mathrm{T}} \end{bmatrix}, \tag{35}$$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_p & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q+Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_q \end{bmatrix}, \tag{36}$$

where
$\Lambda_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Lambda_{21} = A_d\Gamma^{\mathrm{T}} + B_{ud}Z^{\mathrm{T}}\Omega^{\mathrm{T}}$,
$\Lambda_{31} = C_d\Gamma^{\mathrm{T}}$, $\Lambda_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$,
$\Lambda_{22} = \Pi A_d^{\mathrm{T}} + A_d\Pi^{\mathrm{T}} + \Phi Z B_{ud}^{\mathrm{T}} + B_{ud}Z^{\mathrm{T}}\Phi^{\mathrm{T}} - \Theta$,
$\Lambda_{32} = C_d\Pi^{\mathrm{T}}$, $\Lambda_{52} = B_{wd}^{\mathrm{T}}$, $\Lambda_{53} = D_{wd}^{\mathrm{T}}$.
$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$,
$\Omega = [\, I_r \ 0 \,]$, $\Gamma = [\, Q \ R \,]$.

The gain matrix can be obtained as

$$F = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix} \overline{V}^{-1}. \tag{37}$$

Proof. Show that controller (37) is a solution of anisotropy-based control problem for initial system (18)–(19). Indeed,

$$\begin{aligned} P_{cl}(z) &= C\overline{V}\overline{V}^{-1}(zE - A - B_uF)^{-1}\overline{W}^{-1}\overline{W}B_w + D_w = \\ &= C\overline{V}(z\overline{W}E\overline{V} - \overline{W}A\overline{V} - \overline{W}B_uF\overline{V})^{-1}\overline{W}B_w + D_w = \\ &= C_d(zE_d - A_d - B_{ud}F_d)^{-1}B_{wd} + D_{wd}, \end{aligned}$$

where $F_d = F\overline{V}$.

Introduce the following linear change of variables

$$\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = Z.$$

It implies that $[\, Q \ R \,] F_d^{\mathrm{T}} = [\, I_r \ 0 \,] Z$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z$. Substituting it into (25) we get Λ_{21} and Λ_{22} entries from (36), which coincide with the conditions of Theorem 4 for the system, dual to the system (18)–(19). So, according to Theorem 4, the closed-loop system (18)–(19) is admissible, and a-anisotropic norm of its transfer function is bounded by the given scalar γ.

As the inequality (33) holds, then the (1,1) entry implies matrix Q is invertible. We also suppose, that the matrix S is invertible. If it does not hold, there exists a scalar $\epsilon \in (0,1)$, such that the inequality (33) holds true for matrix $\bar{S} = S + \epsilon I_{n-r}$. So, we can use $\bar{S}$ instead of S.

As pointed out before, Q and S are invertible. So the feedback gain F_d for the closed-loop system is defined as $F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}$. Note that $F_d = F\overline{V}$. By the inverse change of variables we get F from (37).

The theorem is proved.

4. NUMERICAL EXAMPLE

Consider the following system:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.5 & 0.2 & 2 \\ 0.75 & -1 & -1.5 \end{bmatrix},\ B_u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0.4 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_w = \begin{bmatrix} 0.1 & -0.5 \\ 0 & 0.2 \end{bmatrix}.$$

The matrix A is assumed to be uncertain with $\Delta \in [-1;\ 1]$ and

$$M_A = [\, 0.1 \ -0.5 \ 0.05 \,]^{\mathrm{T}},\ N_A = [\, 0 \ 0.1 \ 0.1 \,].$$

The system is causal but not stable. The generalized spectral radius of the nominal system is $\rho(E, A) = 2.5$.

The design objective is to find $\min\gamma$ for such conditions of the theorem 5 hold.

For the case $a = 0$ we have a suboptimal $\mathcal{H}_2$ controller. The minimization of γ gives $\overline{\gamma}_{\min} = 0.4859$, the lower and upper bounds of $\|P_\Delta^{cl}(z)\|_a$ are $\underline{\gamma} = 0.4848$ and $\overline{\gamma} = 0.4859$ respectively. The controller parameters are

$$F = [\, 0.2704 \ 1.0014 \ 1.5014 \,].$$

For the case $a = 1.5$ we have $\overline{\gamma}_{\min} = 0.7026$, th boundary values of $\|P_\Delta^{cl}(z)\|_a$ are $\underline{\gamma} = 0.7009$ and $\overline{\gamma} = 0.6913$. The controller parameters are

$$F = [\, 2.7227 \ 1.0377 \ 1.2985 \,].$$

5. CONCLUSION

The paper deals with the state-feedback control design problem for discrete-time descriptor systems with norm-bounded uncertainties in presence of colored noise. It has been shown that the above problem can be solved via matrix inequality approach involving no parameter uncertainties. Thus, the derived result can be applied to design anisotropy-based controllers with guaranteed robust performance for descriptor systems.

REFERENCES

D.V. Balandin, M.M. Kogan. Synthesis of controllers on the basis of a solution of linear matrix inequalities and a search algorithm for reciprocal matrices. *Automation and Remote Control*, volume 66, No. 1, pages 74–91, 2005.

O.G. Andrianova and A.A. Belov. Anisotropy-based analysis for descriptor systems with norm-bounded parametric uncertainties. *Proc. of 2016 International Conference Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference)*, Moscow, Russia, pages 1–4, June 2016.

A.A. Belov, O.G. Andrianova. Anisotropy-based suboptimal state-feedback control design using linear matrix inequalities. *Automation and Remote Control*, volume 77, No. 10, pages 1741–1755, 2016.

S. Boyd, L.E. Ghaoui, E. Feron, and V. Balakrishnan *Linear Matrix Inequalities in Systems and Control Theory.* SIAM Studies in Applied Mathematics, Philadelphia, Pennsylvania, 1994.

M. Chadli, M. Darouach. Further Enhancement on Robust $\mathcal{H}_\infty$ Control Design for Discrete-Time Singular Systems. *IEEE Transactions on Automatic Control*, volume 59, No. 2, pages 494–499, 2014.

D. Coutinho, C.E. de Souza, K.A. Barbosa. Robust $\mathcal{H}_\infty$ control of discrete-time descriptor systems. *Proc. of European Control Conference*, Strasbourg, France, pages 1915–1920, July 2014.

L. Dai. *Singular Control Systems.* Lecture Notes in Control and Information Sciences, Springer-Verlag, New York, 1989.

P. Diamond, I. Vladimirov, A. Kurdyukov, and A. Semyonov. Anisotropy-based performance analysis of linear discrete time invariant control systems. *Int. Journ. of Control*, volume 74, No. 1, pages 28–42, 2001.

G.-R. Duan. *Analysis and Design of Descriptor Linear Systems.* Advances in Mechanics and Mathematics, Springer-Verlag, New York, 2010.

X. Ji, H. Su, and J. Chu. Robust state feedback $\mathcal{H}_\infty$ control for uncertain linear discrete singular systems. *IET Control Theory Appl.*, volume 1, No. 1, pages 195-200, 2007.

I.R. Petersen. A stabilization algorithm for a class of uncertain linear systems. *Systems & Control Letters*, volume 8, pages 351–357, 1987.

A.V. Semyonov, I.G. Vladimirov, and A.P. Kurdyukov. Stochastic approach to $\mathcal{H}_\infty$-optimization. *Proc. 33rd IEEE Conf. on Decision and Control*, Florida, USA, pages 2249–2250, December 1994.

I.G. Vladimirov, A.P. Kurdyukov, and A.V. Semyonov. Anisotropy of Signals and the Entropy of Linear Stationary Systems. *Doklady Math.*, volume 51, pages 388–390, 1995.

I.G. Vladimirov, A.P. Kurdyukov, and A.V. Semyonov. On computing the anisotropic norm of linear discrete-time-invariant systems. *Proc. 13th IFAC World Congress*, San Francisco, USA, pages 179–184, July 1996.

I.G. Vladimirov, P. Diamond, and P. Kloeden. Anisotropy-based performance analysis of finite horizon linear discrete time varying systems. *Automation and Remote Control*, volume 8, pages 1265–1282, 2006.

S. Xu, J. Lam. *Robust Control and Filtering of Singular Systems.* Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 2006.