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Anisotropy-based analysis for descriptor systems with nonzero-mean input signals*

Olga G. Andrianova†, Alexander P. Kurdyukov†, Alexey A. Belov†, and Arkadiy Yu. Kustov†

***Abstract*— The paper presents a novel concept of anisotropy-based analysis for descriptor systems with nonzero-mean random input signals. The algorithm for mean anisotropy computation of the Gaussian stationary random sequence with nonzero mean is obtained. The equations for anisotropy norm computation (in the frequency domain) for descriptor systems are developed. Numerical examples are given.**

I. INTRODUCTION

The theory of optimal stochastic robust control for linear discrete-time systems was established in Russia since 1994 to 2008 [1], [2]. This theory allows to design control, that minimizes specified norm of the closed-loop system (anisotropic norm). Since 2008 to 2012, the theory of suboptimal stochastic robust control, that provides boundedness of anisotropic norm for the referred above systems, was developed [3]. At present these mathematical theories find application in different control and filtration problems.

The created theory lies between the classical H_2-optimal and H_∞-optimal control theories (and suboptimal as well) in some sense. The basic concepts of these theories are anisotropy of the random signal and mean anisotropy of the input sequence. Anisotropy of the vector describes so-called "spectral color" of the signal as the distinction between its probability density function (p.d.f.) and p.d.f. of the Gaussian white noise. Mean anisotropy of the sequence is the time averaging anisotropy of the sequence.

In classical anisotropy-based theory the input signal is supposed to be the signal with zero mean and certain "spectral color" [1]. But in real technical systems the input signal can be a stochastic signal with nonzero mean. That is why the extension of anisotropic theory on the class of signals with nonzero mean has a practical interest. For linear normal systems the the basics of anisotropy-based theory with nonzero mean input signal were introduced in [4].

Classical anisotropy-based theory was generalized on linear discrete-time descriptor systems. Optimal control problem was solved in [5], [6], norm boundedness conditions were obtained in [7], [8]. This paper presents the extension of anisotropic theory on the class of descriptor systems in assumption that the "colored" Gaussian noise has nonzero mean.

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†The authors are with the Laboratory of Dynamics of Control Systems, Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia, 117997, Profsoyuznaya, 65, fax: +74953349340, e-mails: andrianovaog@gmail.com, akurd@ipu.ru, a.a.belov@inbox.ru, arkadiykustov@gmail.com

The paper is organized as follows. In the first section, the basics of linear discrete-time descriptor systems is introduced. The second section deals with anisotropy-based concepts, applied to the Gaussian stationary random sequences with nonzero mean. The third section is devoted to anisotropy-based analysis of descriptor systems in frequency domain. Numerical examples are given.

II. BASICS OF DESCRIPTOR SYSTEMS THEORY

In linear case discrete-time descriptor systems are written as

$$\begin{cases} Ex_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k \end{cases} \tag{1}$$

where $x_k \in R^{n_1}$ is the state, $u_k \in R^m$ is the control signal and $y_k \in R^p$ is the output signal. A, B, C, D and E are constant real matrices of appropriate dimensions.

For the system (1) we suppose that $\operatorname{rank}(E) = n < n_1$. Such systems are called descriptor or singular.

The system is assumed to be admissible if it is regular ($\exists \lambda \neq 0 : \det(\lambda E - A) \neq 0$), causal ($\deg \det(zE - A) = \operatorname{rank} E$), and stable ($\rho(E, A) = \max |\lambda|_{\lambda \in \{z \mid \det(zE - A) = 0\}} < 1$). For more information, see [9].

It is known from matrix theory, that there are two non-singular matrices Q and U such that $QEU = \operatorname{diag}(I_n, 0)$. By the following transformation of coordinates $U^{-1}x_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}$, $x_{1,k} \in R^n$, $x_{2,k} \in R^{n_1 - n}$ the system (1) can be written in the following equivalent form:

$$\begin{cases} x_{1,k+1} = A_{11}x_{1,k} + A_{12}x_{2,k} + B_1 u_k, \\ 0 = A_{21}x_{1,k} + A_{22}x_{2,k} + B_2 u_k, \\ y_k = C_1 x_{1,k} + C_2 x_{2,k} + Du_k \end{cases} \tag{2}$$

where matrices A_{ij}, B_i, C_i $(i, j = 1, 2)$ satisfy $QAU = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $CU = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$.

Matrices Q and U can be found from the singular value decomposition (SVD) [5]

$$E = X \operatorname{diag}(S, 0) Z^{\mathrm{T}}$$

where X and Z are real orthogonal matrices, S is a diagonal matrix of the order n. S consists of nonzero singular values of the matrix E. So,

$$\begin{aligned} Q &= \operatorname{diag}(S^{-1/2}, I_{n_1 - n}) X^{\mathrm{T}}, \\ U &= Z \operatorname{diag}(S^{-1/2}, I_{n_1 - n}). \end{aligned}$$

Definition 1: The system (2) is called SVD canonical form of the system (1).

For the system (1) in SVD canonical form (2) the following lemma [10] holds true:

Lemma 1: The system (1) is

1) causal if and only if A_{22} is nonsingular;
2) admissible if and only if it is casual and $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$.

Definition 2: The transfer function of the system (1) is given by the expression $P(z) = C(zE - A)^{-1}B + D$.

Definition 3: Let $L_2^{p\times m}(\Gamma)$ (where Γ is a unit circle on the complex plane) be the Hilbert space of matrix-valued functions $P : \Gamma \to C^{p\times m}$ that have bounded $L_2^{p\times m}(\Gamma)$-norm

$$\|P\|_{L_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\big(P^*(e^{i\omega})P(e^{i\omega})\big)\, d\omega\right)^{1/2}.$$

A subspace of $L_2^{p\times m}(\Gamma)$ which consists of all rational transfer functions that have no poles in the exterior of the closed unit disk is denoted by H_2. The H_2-norm of a transfer function $P(z) \in H_2$ is defined by

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\omega})|^2 d\omega\right)^{1/2}.$$

III. ANISOTROPY OF THE RANDOM VECTOR MEAN ANISOTROPY OF THE GAUSSIAN SEQUENCE WITH NONZERO MEAN

In this section, we consider the concept of anisotropy of the random vector with nonzero mean and give definition of mean anisotropy of the Gaussian nonzero-mean sequence, generated by the shaping filter in descriptor form.

A. Anisotropy of the random vector

Anisotropy of m-dimensional random vector w is introduced in [1] as the minimal value of the relative entropy of w with respect to the Gaussian m-dimensional vector with p.d.f.

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{x^T x}{2\lambda}\right), \quad x \in R^m,$$

and is described by

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{E}_f \ln \frac{f(x)}{p_{m,\lambda}(x)} \tag{3}$$

where the function f is p.d.f. of w.

We suppose w is m-dimensional Gaussian random vector with nonzero mean μ and covariance matrix S, which p.d.f. is given by

$$f(x) = ((2\pi)^m|S|)^{-1/2}\, e^{-\frac{1}{2}(x-\mu)^T S^{-1}(x-\mu)},\ x \in R^m.$$

By definition of anisotropy (3) of the random vector,

$$\mathbf{A}(w) = -\frac{1}{2}\ln\det\left(\frac{mS}{\operatorname{tr}S + |\mu|^2}\right).$$

One can show that if $S = \gamma I_m$ and $\mu = 0$, then $\mathbf{A}(w) = 0$.

B. Mean anisotropy of the Gaussian sequence with nonzero mean

Let W be a stationary sequence of random m-dimensional vectors. Mean anisotropy of the stationary ergodic sequence $W = \{w_k\}$ is defined in [1] by the following expression

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

where $W_{0:N-1}$ is an extended vector of the sequence:

$$W_{0:N-1} = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix}.$$

Let the sequence $W = \{w_k\}$ be generated from the Gaussian white noise $V = \{v_k\}$ by an admissible shaping filter

$$G \sim \begin{cases} E_g x_{k+1} = A_g x_k + B_g(v_k + \mu), \\ w_k = C_g x_k + D_g(v_k + \mu) \end{cases} \tag{4}$$

where $E_g \in R^{n_1\times n_1}$, $A_g \in R^{n_1\times n_1}$, $B_g \in R^{n_1\times m}$, $C_g \in R^{m\times n_1}$, $D_g \in R^{m\times m}$. Besides, $\det(D_g) \neq 0$, $\operatorname{rank} E_g = n < n_1$ and $|\mu| < \infty$. Connection between mean anisotropy $\overline{\mathbf{A}}(W)$ of the sequence W and state-space representation (4) of shaping filter is given by the following theorem.

Theorem 1: For a given state-space representation (4) of the shaping filter G mean anisotropy $\overline{\mathbf{A}}(W)$ is determined by

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\det\left(\frac{m(\Sigma + \Xi)}{\operatorname{tr}\Sigma + |\mathcal{M}|^2}\right)$$

where Σ and Ξ are connected with the solutions of Lyapunov and Riccati equations P and R by formulas

$$\Sigma = \widehat{C}P\widehat{C}^T + \widehat{D}\widehat{D}^T,$$
$$P = \widehat{A}P\widehat{A}^T + \widehat{B}\widehat{B}^T,$$

$$\Xi = \widehat{C}R\widehat{C}^T,$$
$$R = \widehat{A}R\widehat{A}^T - \Lambda(\Sigma + \Xi)^{-1}\Lambda^T,$$
$$\Lambda = \widehat{B}\widehat{D}^T + \widehat{A}(P + R)\widehat{C}^T$$

with matrices

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$

$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \widehat{D} = D_g - C_2A_{22}^{-1}B_2,$$

connected with matrices A_{ij}, B_i, C_i $(i, j = 1, 2)$ of SVD canonical form of the system (4), and the vector $\mathcal{M}$ is represented by

$$\mathcal{M} = (\widehat{D} + \widehat{C}(I_{n\times n} - \widehat{A})^{-1}\widehat{B})\mu.$$

Proof: The system (4) in SVD canonical form is given by

$$\begin{cases} x_{k+1}^1 = A_{11}x_k^1 + A_{12}x_k^2 + B_1(v_k + \mu), \\ 0 = A_{21}x_k^1 + A_{22}x_k^2 + B_2(v_k + \mu), \\ w_k = C_1x_k^1 + C_2x_k^2 + D_d(v_k + \mu) \end{cases}$$

where $x_k^1 \in R^n$, $x_k^2 \in R^{n_1-n}$. According to Lemma 1, $\det A_{22} \neq 0$, then

$$x_k^2 = -A_{22}^{-1}(A_{21}x_k^1 + B_2(v_k + \mu)). \tag{5}$$

Substituting x_k^2 into the first and the third subsystems of (5), one can get

$$\begin{cases} x_{k+1}^1 = \widehat{A}x_k^1 + \widehat{B}(v_k + \mu), \\ w_k = \widehat{C}x_k^1 + \widehat{D}(v_k + \mu) \end{cases} \quad (6)$$

where

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}\ , \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2\ ,$$

$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}\ , \quad \widehat{D} = D_g - C_2A_{22}^{-1}B_2\ .$$

Applying Theorem 1 from [4] to the system (6), we finish the proof. ■

Remark 1: The random sequence W is fully defined by its generating filter G, therefore, the notation $A(G)$ is used as equivalent to the notation $A(W)$.

Example 1: Let the shaping filter (4) be formed by the following matrices

$$E_g = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \quad B_g = \begin{bmatrix} 0.03 \\ 0.10 \\ 0.07 \end{bmatrix},$$
$$A_g = \begin{bmatrix} 0.7649 & 0.7572 & -0.0581 \\ -0.0424 & 0.2854 & 0.2218 \\ 0.7706 & 0.6003 & 0.7157 \end{bmatrix},$$
$$C_g = \begin{bmatrix} 1 & 2 & 1.5 \end{bmatrix}, \quad D_g = [0.5]\,,$$

and vector $\mu = [0.1]$; $\operatorname{rank} E = 2$, $m = 1$. The system in SVD canonical form is defined by matrices:

$$\widehat{A} = \begin{bmatrix} 0.7187 & 0.0253 \\ 0.9639 & -0.3064 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} -0.1213 \\ -0.2673 \end{bmatrix},$$
$$\widehat{C} = \begin{bmatrix} -2.2291 & 0.9010 \end{bmatrix}, \quad \widehat{D} = [0.7324]\,.$$

So, the vector $\mathcal{M} = [0.065]$. Solving Lyapunov and Riccati equations from the Theorem 1, we obtain

$$\Sigma = [0.5873]\,, \quad \Xi = [-0.0510]\,.$$

Consequently,

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\det\left(\frac{m(\Sigma + \Xi)}{\mathrm{tr}\Sigma + |\mathcal{M}|^2}\right) = 0.049.$$

IV. ANISOTROPIC NORM

Consider an admissible linear discrete-time descriptor system written in a state-space representation

$$P \sim \begin{cases} Ex_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + Dw_k \end{cases} \quad (7)$$

where $x_k \in R^{n_1}$ is the state, $w_k \in R^m$ and $z_k \in R^p$ are input and output signals, respectively. E, A, B, C, D are constant real matrices of appropriate dimensions. We suppose that matrix E is singular, i.e. $\operatorname{rank}(E) = n < n_1$. $W = \{w_k\}$ is the stationary Gaussian sequence of m-dimensional random vectors with a given mean anisotropy level $\overline{\mathbf{A}}(W) = a \geq 0$ and known nonzero mean $\mathbf{E}w_\infty = \mathcal{M}$, $|\mathcal{M}| < \infty$.

For a given system P with the input signal $W = \{w_k\}$ the mean-square gain is defined as [11]

$$Q(P, W) = \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}} = \sqrt{\lim_{N\to\infty} \frac{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|z_k|^2}{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|w_k|^2}} \quad (8)$$

where

$$\|y\|_{\mathcal{P}} = \sqrt{\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|y_k|^2}$$

is the power norm of the signal $\{y_k\}$.

Let the sequence $\{w_k\}$ be represented in the form

$$w_k = C_g x_k + D_g(v_k + \mu) \quad (9)$$

where x_k is the state of the system (7), and μ is a known vector. Using (9), we obtain the admissible filter

$$G \sim \begin{cases} Ex_{k+1} = (A + BC_g)x_k + BD_g(v_k + \mu), \\ w_k = C_g x_k + D_g(v_k + \mu). \end{cases} \quad (10)$$

The power norms of outputs of the systems (7) and (10) are written as

$$\begin{aligned} \|w\|_{\mathcal{P}}^2 &= \lim_{k\to\infty}\left(\mathrm{tr}\,\mathrm{cov}(w_k) + |\mathbf{E}w_k|^2\right) \\ &= \|G\|_2^2 + |\mathcal{M}|^2, \\ \|z\|_{\mathcal{P}}^2 &= \lim_{k\to\infty}\left(\mathrm{tr}\,\mathrm{cov}(z_k) + |\mathbf{E}z_k|^2\right) \\ &= \|PG\|_2^2 + |\mathcal{P}\mathcal{M}|^2 \end{aligned}$$

where

$$\mathcal{P} = P(1) = D + C(E - A)^{-1}B.$$

The mean-square gain (8) for the system with nonzero-mean input signal is given by the following expression:

$$Q(P, W) = Q(P, G) = \sqrt{\frac{\|PG\|_2^2 + |\mathcal{P}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2}}\,. \quad (11)$$

Finally, anisotropic norm of the system is defined as [2]

$$|\!|\!|P|\!|\!|_a = \sup_{G:\overline{\mathbf{A}}(G)\leq a} Q(P, G). \quad (12)$$

Theorem 2: Consider the system, defined by (7) (with the transfer function $P(z) = C(zE - A)^{-1}B + D$). Let W be the sequence of nonzero-mean m-dimensional Gaussian random vectors, generated by an admissible shaping filter G in the form (4), with mean anisotropy $\overline{\mathbf{A}}(W) = a$ and $\mathbf{E}w_\infty = \mathcal{M}$. Then anisotropic norm of the descriptor system (7) can be computed in a frequency domain as

$$|\!|\!|P|\!|\!|_a = \sup_{q\in[0;\|P\|_\infty^{-2})}\{N(q) \mid A(q) = a\} \quad (13)$$

where

$$
\begin{aligned}
A(q) &= \frac{m}{2}\left(\ln\left(\Phi(q)+\frac{1}{m}|\mathcal{M}|^2\right)-\Psi(q)\right), \\
N(q) &= \sqrt{\frac{\Phi(q)-1+\frac{q}{m}|\mathcal{PM}|^2}{q\Phi(q)+\frac{q}{m}|\mathcal{M}|^2}}, \\
\Phi(q) &= \frac{1}{2\pi m}\int_{-\pi}^{\pi}\mathrm{tr}S(q,\omega)d\omega\,, \qquad (14) \\
\Psi(q) &= \frac{1}{2\pi m}\int_{-\pi}^{\pi}\ln\det S(q,\omega)d\omega\,, \qquad (15) \\
S(q,\omega) &= (I_m - q\Lambda(\omega))^{-1}, \quad q\in[0;\|F\|_\infty^2).
\end{aligned}
$$

Here $\Lambda(\omega)=\widehat{P}^*(\omega)\widehat{P}(\omega)$, $\widehat{P}(\omega)=\lim_{r\to 1}P(re^{j\omega})$, and $\mathcal{P}=P(1)$.

Besides, $N(0)=\sqrt{\frac{\|P\|_2^2+|\mathcal{PM}|^2}{m+|\mathcal{M}|^2}}$.

Proof: It is shown in [12] that

$$\overline{\mathbf{A}}(W)=\mathbf{A}(w_0)+\mathbf{I}(w_0;(w_k)_{k<0}) \qquad (16)$$

where $\mathbf{I}(w_0;(w_k)_{k<0}) = \lim_{s\to-\infty}\mathbf{I}(w_0;W_{s:-1})$ is the Shannon mutual information [13] between w_0 and the prehistory $(w_k)_{k<0}$ of the sequence W.

Now, suppose that the stationary random sequence W is Gaussian. Then

$$\mathbf{I}(w_0;(w_k)_{k<0})=\frac{1}{2}\ln\det(\mathrm{cov}(w_0)\mathrm{cov}(\tilde{w}_0)^{-1}) \qquad (17)$$

where

$$\tilde{w}_0=w_0-\mathbf{E}(w_0|(w_k)_{k<0}) \qquad (18)$$

is the error of the mean-square optimal prediction of w_0 with the prehistory $(w_k)_{k<0}$, provided by the conditional expectation.

The covariance matrix of the prediction error (18) and the spectral density $S(\omega)=\widehat{G}^*(\omega)\widehat{G}(\omega)$ are connected by the Kolmogorov-Szegö formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\det\widehat{G}^*(\omega)\widehat{G}(\omega)d\omega=\ln\det\mathrm{cov}(\tilde{w}_0). \qquad (19)$$

Using (17)–(19) and the Szegö limit theorem [14], mean anisotropy (16) of the stationary Gaussian random sequence W may be computed in terms of the spectral density $S(\omega)=\widehat{G}^*(\omega)\widehat{G}(\omega)$ and H_2-norm of the shaping filter G as

$$
\begin{aligned}
\overline{\mathbf{A}}(G) &= -\frac{1}{4\pi}\ln\det\frac{m\mathrm{cov}(\tilde{w}_0)}{\|G\|_2^2+|\mathcal{M}|^2} \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{m\widehat{G}^*(\omega)\widehat{G}(\omega)}{\|G\|_2^2+|\mathcal{M}|^2}d\omega. \qquad (20)
\end{aligned}
$$

Using (11) and (12), we have:

$$\|P\|_a^2=\sup_{G:\overline{\mathbf{A}}(G)\le a}\frac{\|PG\|_2^2+|\mathcal{PM}|^2}{\|G\|_2^2+|\mathcal{M}|^2}=$$

$$=\sup_{\|G\|_2^2\le 1+|\mathcal{M}|^2}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}(\Lambda(\omega)S(\omega))d\omega+|\mathcal{PM}|^2:\ \overline{\mathbf{A}}(G)\le a,\ \|G\|_2^2+|\mathcal{M}|^2<\gamma\right\}$$

where γ is a positive real constant.

Construct a Lagrange function as

$$\mathcal{L}=\|PG\|_2^2+|\mathcal{PM}|^2-\alpha_1(\|G\|_2^2+|\mathcal{M}|^2)-\alpha_2\overline{\mathbf{A}}(G).$$

Using the definitions of H_2-norm and anisotropic norm of descriptor systems, we get

$$\mathcal{L}=\int_{-\pi}^{\pi}\Big(\mathrm{tr}(\Lambda(\omega)S(\omega)-\alpha_1 S(\omega))+ +\frac{1}{2}\alpha_2\ln\det S(\omega)\Big)d\omega+|\mathcal{PM}|^2-\alpha_1|\mathcal{M}|^2. \qquad (21)$$

Find an extremum point of the function (21) from the condition:

$$\frac{\partial\mathcal{L}}{\partial S(\omega)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\left(\Lambda(\omega)-\alpha_1 I_m+\frac{\alpha_2}{2}S^{-1}(\omega)\right)d\omega=0.$$

Hence,

$$\Lambda(\omega)-\alpha_1 I_m+\frac{\alpha_2}{2}S^{-1}(\omega)=0. \qquad (22)$$

Let $q=\frac{1}{\alpha_1}$, $\sigma=\frac{\alpha_2}{2\alpha_1}$, then the equation (22) can be rewritten as

$$q\Lambda(\omega)-I_m+\sigma S^{-1}(\omega)=0.$$

The expression

$$S(\omega)=\sigma(q\Lambda(\omega)-I_m)^{-1} \qquad (23)$$

defines the spectral density for the worst case of the input disturbance. Without loss of generality we consider $\sigma=1$ [2].

Substituting (23) into (20) and using notations (14)-(15), we obtain

$$\overline{\mathbf{A}}(G)=\frac{m}{2}\left(\ln\left(\Phi(q)+\frac{1}{m}|\mathcal{M}|^2\right)-\Psi(q)\right)=A(q).$$

Finally, substituting the spectral density $S(q,\omega)$ into (12), we have

$$P(Q,G)^2=\frac{m\frac{\Phi(q)-1}{q}+|\mathcal{PM}|^2}{m\Phi(q)+|\mathcal{M}|^2}=N^2(q).$$

The function $\Phi(q)$ satisfies the following properties: $\lim\limits_{q\to 0+0}\Phi(q)=1$, $\lim\limits_{q\to 0+0}\frac{\Phi(q)-1}{q}=\|P\|_2^2$. So, $N(0)=\sqrt{\frac{\|P\|_2^2+|\mathcal{PM}|^2}{m+|\mathcal{M}|^2}}$.

■

Example 2: Let the system P be described by

$$E = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.7 & -0.3 \\ 0.1 & 0.3 \end{bmatrix}, B = \begin{bmatrix} -0.02 \\ 0.07 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.50 & 0.09 \end{bmatrix}, D = [0.035].$$

The transfer function of the system is

$$P(z) = \frac{0.235}{9z-8} + 0.014.$$

The spectral density of the system is

$$\Lambda(\omega) = \frac{0.031(1+\cos\omega)}{-144\cos\omega + 145}.$$

The spectral density of the worst case shaping filter is

$$S(q,\omega) = \frac{144\cos\omega + 145}{(-144-0.031q)\cos\omega + 145 - 0.031q}.$$

Fig. 1 and Fig. 2 present $A(q)$ and $N(q)$ plots for different values of $\mathcal{M}$, respectively.

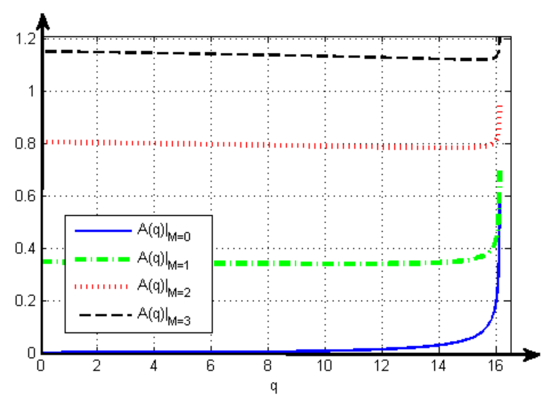

Fig. 1. $A(q)$ for different values of $\mathcal{M}$.

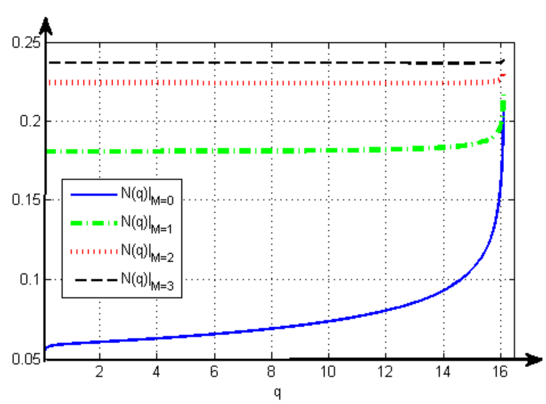

Fig. 2. $N(q)$ for different values of $\mathcal{M}$.

For large values of $\mathcal{M}$ functions $A(q)$ and $N(q)$ lose their monotony (see Fig. 3). The set $\{N(q)\,|_{A(q)=a}\}$ can be empty or contain several values of $N(q)$. Therefore, the anisotropic norm is defined as a supremum function by (13).

When $a = 0.34$, the anisotropic norm of the system is equal to $|||P|||_a = 0.1841$ for $\mathcal{M} = 0$, $|||P|||_a = 0.1823$ for $\mathcal{M} = 1$ and cannot be computed for $\mathcal{M} = \{2, 3\}$ (see Fig. 4 and Fig. 5).

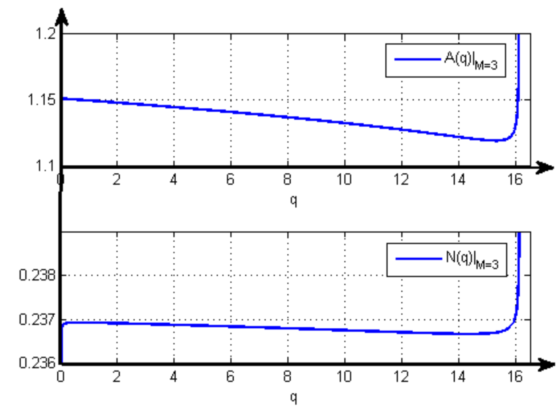

Fig. 3. $A(q)$ and $N(q)$ for $\mathcal{M} = 3$.

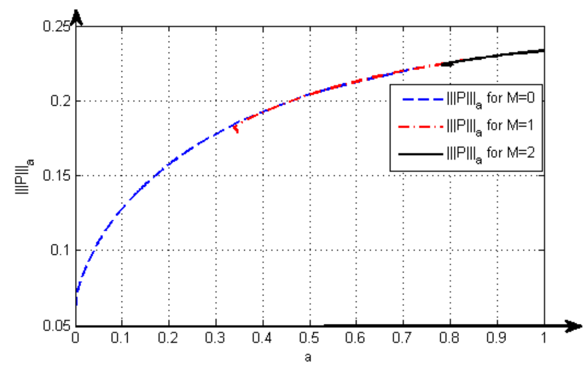

Fig. 4. $N(A(q))$ for different $\mathcal{M}$.

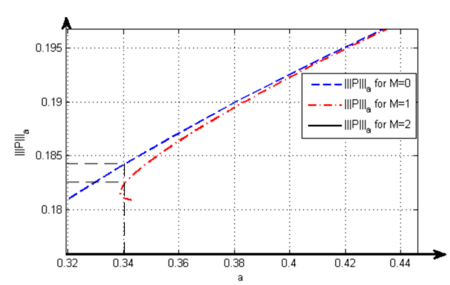

Fig. 5. $N(A(q))$ plots for different $\mathcal{M}$ (magnified).

V. CONCLUSIONS

In this paper, the problem of anisotropy-based analysis for descriptor systems with Gaussian stationary random nonzero-mean sequences as input signals was solved. The concept of mean anisotropy of nonzero-mean sequence, generated by the shaping filter in descriptor form, generalizes the existent anisotropy-based theory. The presented results for anisotropic norm computation in frequency domain can be suitable for analysis problems solution, but it can be difficult to compute anisotropic norm in frequency domain for large scale systems. The further problem is to develop the methods anisotropic norm computation in time domain.

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