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# A Riccati equation approach to anisotropy-based control problem for descriptor systems: state feedback and full information cases*

Olga G. Andrianova[†], Alexey A. Belov[†]

***Abstract*— The paper presents a solution of anisotropy-based suboptimal controller design problem for descriptor systems. The goal is to design a state feedback and full information control for the system such that the closed-loop system is admissible, and its anisotropic norm (mean anisotropy level is set) is bounded by a given positive real value. A numerical example is given.**

## I. INTRODUCTION

The solution of such problems as sensitivity reduction of the system or external disturbance rejection has become widespread in modern control theory. One of the determining factors for solving these problems is the nature of input disturbance.

In discrete-time $H_\infty$-based control approach input disturbances are represented as sequences with limited power, i.e. the sequences are square summable. The optimization goal is minimization of the objective function, which makes sense of the system's $L_2$-gain from the worst-case input disturbance to the controllable output. Suboptimal control problem is also solved. In this framework it is not necessary to find the minimal value of the objective function, but it is sufficient to restrict the norm of the transfer function by some positive scalar value to satisfy the designer's requirements. $H_\infty$-suboptimal state feedback and full information control problems for linear discrete-time normal systems are solved by Stoorvogel [1].

In LQG/$H_2$-optimal theory the Gaussian white noise sequence is considered as the input disturbance, and the control goal is to reduce the influence of the noise on the closed-loop system. The LQG/$H_2$-optimal control problem is solved in [2], [3], [4].

Being a general case of normal systems, descriptor systems have found wide applications in electrical, mechanical and economic systems [5]. Such systems are a prospective research field. Both these control theories may be generalized on the class descriptor systems. The solution of $H_\infty$-control problem in terms of generalized discrete-time algebraic Riccati inequalities (GDARI) is extended on the class of descriptor systems in [6]. LMI approach is also applied to $H_\infty$-control design in [7], [8]. The solution of LQG/$H_2$-optimization problem can be found in [9], [10].

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[†]The authors are with the Laboratory of Dynamics of Control Systems named after B.N. Petrov, Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia, 117997, Profsoyuznaya, 65, fax: +74953349340, e-mails: andrianovaog@gmail.com, a.a.belov@inbox.ru

Both theories have their disadvantages. The system, closed by LQG/$H_2$-controller, looses performance if it is effected by a strongly correlated random noise. On the contrary, the system with $H_\infty$-suboptimal controller requires greater control energy if the input disturbance is slightly correlated.

Anisotropy-based control theory allows to get rid of the disadvantages described herein above. The advantage of this theory lies in more precise knowledge of characteristics of the input disturbance — mean anisotropy level [11], [12], [13]. The quality criterion is described by anisotropic norm. In two limited cases anisotopic norm coincides with $H_\infty$-norm and the scaled $H_2$-norm. So, on the one hand, the solution of anisotropy-based control problem for discrete-time descriptor systems extends the existent results for normal systems, on the other hand, it generalizes the solutions of LQG/$H_2$- and $H_\infty$-suboptimal control problems for descriptor systems.

This work is devoted to state feedback (SF) and full information (FI) anisotropy-based suboptimal control problems for discrete-time descriptor systems. It is proved, that under some rank assumptions the solution to the stated problem may be expressed by generalized discrete-time algebraic Riccati equation (GDARE), constrained by a set of inequalities.

The paper is organized as follows. In the section II-A, the basics of linear discrete-time descriptor systems is introduced. The sections II-B and II-C deal with main anisotropy-based concepts and anisotropy-based analysis for descriptor systems. The sections III and IV are devoted to SF and FI problems solution. In the section V, a numerical example is considered.

## II. BACKGROUND

### A. Descriptor systems

The state-space representation of discrete-time descriptor systems is

$$\begin{aligned} Ex(k+1) &= Ax(k)+Bf(k), \\ y(k) &= Cx(k)+Df(k) \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively. $A,B,C,D$ are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, $\mathrm{rank}(E) = r < n$.

The transfer function of the system (1) is defined by the expression

$$P(z) = C(zE-A)^{-1}B+D,\ z \in \mathbb{C}.$$

$H_2$- and $H_\infty$-norms of descriptor systems (1) are defined as follows

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\big(P^*(e^{i\omega})P(e^{i\omega})\big)\,d\omega\right)^{\frac{1}{2}},$$

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{max}\left(P(e^{i\omega})\right).$$

Here, $P^*(e^{i\omega}) = P^{\mathrm{T}}(e^{-i\omega})$ is a conjugate system, $\sigma_{max}(A) = \sqrt{\max_j |\lambda_j(A^{\mathrm{T}}A)|}$ is a maximal singular value of the matrix $A$.

*Definition 1:* The system (1) is called admissible if it is

1) regular ($\exists\lambda : \det(\lambda E - A) \neq 0$),
2) causal ($\deg\det(\lambda E - A) = \operatorname{rank} E$),
3) stable ($\rho(E,A) = \max|\lambda|_{\lambda\in z|\{\det(zE-A)=0\}} < 1$).

For more information, see [5], [7], [14].

The admissibility of the system (1) can be checked, using the following theorem [15].

*Theorem 1:* A regular discrete-time descriptor system (1) is admissible if there exists a solution $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ of the generalized Lyapunov equation

$$A^{\mathrm{T}}XA - E^{\mathrm{T}}XE + C^{\mathrm{T}}C = 0, \tag{2}$$

satisfying the condition $E^{\mathrm{T}}XE \geqslant 0$.

Consider a state feedback control in the following form:

$$f(k) = F_c x(k) + h(k) \tag{3}$$

where $F_c \in \mathbb{R}^{m\times n}$ is a constant real matrix, $h(k)$ is a new input signal. The closed-loop system may be written in the form:

$$Ex(k+1) = (A + BF_c)x(k) + Bh(k). \tag{4}$$

*Definition 2:* The system (1) is called causal controllable if there exists a state feedback control in the form (3) such that the closed-loop system (4) is causal.

The causal controllability can be easily checked by the following rank criterion [5].

*Theorem 2:* The system (1) is causal controllable if

$$\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \operatorname{rank}(E) + n.$$

*Definition 3:* The system (1) is called stabilizable if there exists a state feedback control in the form $f(k) = F_{st}x(k)$ such that the pair $(E, A + BF_{st})$ is stable.

For more information, see [5].

### B. Mean anisotropy

Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary sequence of random $m$-dimensional vectors. It is shown that $W$ can be generated from the white Gaussian noise sequence $V = \{v(k)\}_{k\in\mathbb{Z}}$ with zero mean and identity covariance matrix by an admissible shaping filter $G(z) = C_G(zE_G - A_G)^{-1}B_G + D_G$ (i.e. the pair $(E_G, A_G)$ is admissible). Mean anisotropy of the signal is Kullback-Leibler information divergence from probability density function (p.d.f.) of the signal to p.d.f. of the Gaussian white noise sequence. It characterizes spectral color of the signal. Mean anisotropy of the sequence may be defined by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2}\,d\omega$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)\,(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(le^{i\omega})$ is a boundary value of the transfer function $G(z)$.

*Remark 1:* Mean anisotropy of the random sequence $W$, generated by shaping filter $G(z)$, is fully defined by its parameters, so the notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(W)$ are equivalent.

For more details, see [12].

### C. Anisotropy-based analysis

Consider an admissible linear discrete-time descriptor system $P$ written in a state-space representation

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k). \end{aligned} \tag{5}$$

$W = \{w(k)\}_{k\in\mathbb{Z}}$ is a stationary Gaussian sequence of $m$-dimensional random vectors with a known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ ($a \geqslant 0$) and zero mean.

For a given system $P$ with the input signal $W = \{w(k)\}_{k\in\mathbb{Z}}$ the root-mean-square gain is defined as [12], [13]

$$Q(P,W) = \frac{\|Y\|_{\mathscr{P}}}{\|W\|_{\mathscr{P}}} = \sqrt{\lim_{N\to\infty}\frac{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|y(k)|^2}{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|w(k)|^2}}$$

where

$$\|Y\|_{\mathscr{P}} = \sqrt{\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}|y(k)|^2}$$

is the power norm of the signal $Y = \{y(k)\}_{k\in\mathbb{Z}}$.

The sequence $\{w(k)\}_{k\in\mathbb{Z}}$ can be generated as [13]

$$w(k) = C_g x(k) + D_g v(k) \tag{6}$$

where $x(k)$ is the state of the system (5) (see fig. 1).

Using (6), we can choose $C_g$ and $D_g$ such that the filter $G$ with a state-space representation

$$\begin{aligned} Ex(k+1) &= (A + BC_g)x(k) + BD_g v(k), \\ w(k) &= C_g x(k) + D_g v(k) \end{aligned} \tag{7}$$

is admissible [16].

The power norms of outputs of the systems (5) and (7) are written as

$$\begin{aligned} \|W\|_{\mathscr{P}}^2 &= \lim_{k\to\infty}\left(\operatorname{tr}\operatorname{cov}(w(k)) + |\mathbf{E}w(k)|^2\right) = \|G\|_2^2, \\ \|Y\|_{\mathscr{P}}^2 &= \lim_{k\to\infty}\left(\operatorname{tr}\operatorname{cov}(y(k)) + |\mathbf{E}y(k)|^2\right) = \|PG\|_2^2. \end{aligned}$$

Finally, anisotropic norm of the system is defined in [13] as

$$|||P|||_a = \sup_{G:\overline{\mathbf{A}}(G)\leqslant a} Q(P,W) = \sup_{G:\overline{\mathbf{A}}(G)\leqslant a}\frac{\|PG\|_2}{\|G\|_2}. \tag{8}$$

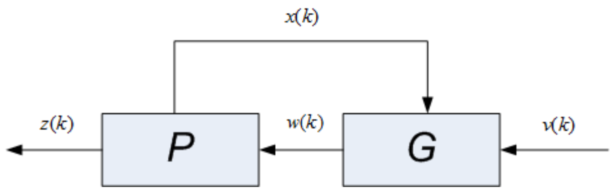


Fig. 1

The following theorem gives sufficient conditions to anisotropic norm boundedness for a descriptor system [16].

**Assumption 1:** We suppose that for the system (5) the condition

$$\operatorname{rank} E = \operatorname{rank}[\, E \quad B \,]$$

is satisfied.

*Theorem 3:* Let $P \in H_\infty^{p\times m}$ be an admissible system with the state-space representation (5) where $\rho(E,A) < 1$. For the given scalar quantities $a \geqslant 0$ and $\gamma > 0$ $a$-anisotropic norm is bounded by $\gamma$, that is

$$|||P|||_a \leqslant \gamma$$

if there exists a stabilizing solution $\widehat{R} = \widehat{R}^{\mathrm{T}}$ of the algebraic Riccati equation

$$\begin{aligned} E^{\mathrm{T}}\widehat{R}E &= A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, &(9)\\ L &= \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C), &(10)\\ \Sigma &= (I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, &(11)\end{aligned}$$

in addition

$$E^{\mathrm{T}}\widehat{R}E \geqslant 0,$$

and $q \in \left[0, \min(\gamma^{-2}, \|P\|_\infty^{-2})\right)$ satisfies the inequality

$$-\frac{1}{2}\ln\det\left((1-q\gamma^2)\Sigma\right) \geqslant a. \qquad (12)$$

## III. Problem statement

Consider a discrete-time descriptor system in the following form:

$$\begin{aligned} Ex(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), &(13)\\ z(k) &= Cx(k) + D_1 w(k) + D_2 u(k) \end{aligned}$$

where $w(k) \in \mathbb{R}^{m_1}$ and $z(k) \in \mathbb{R}^p$ are the input and output signals, respectively, $u(k) \in \mathbb{R}^{m_2}$ is the control vector. $A$, $B_1$, $B_2$, $C$, $D_1$, $D_2$ are constant real matrices of appropriate dimensions. $W = \{w(k)\}_{k\in\mathbb{Z}}$ is a stationary Gaussian sequence of $m$-dimensional random vectors with a known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ $(a \geqslant 0)$. The system is assumed to be causal controllable and stabilizable.

**Problem 1: state feedback control.** A state feedback (SF) anisotropy-based control problem is to find a static state feedback in the form

$$u_{SF}(k) = F_2 x(k)$$

such that the closed-loop system is admissible and $a$-anisotropic norm of its transfer function $P_{cl}^{SF}$ is limited by a known value $\gamma$ for a given mean anisotropy level $a$.

**Problem 2: full information control.** A full information (FI) anisotropy-based control problem is to find a feedback in the form

$$u_{FI}(k) = F_1 w(k) + F_2 x(k)$$

such that the closed-loop system is admissible and $a$-anisotropic norm of its transfer function $P_{cl}^{FI}$ is limited by a known value $\gamma$ for the given mean anisotropy level $a$.

## IV. Main result

### A. State feedback control

*Theorem 4:* Let the system (13) satisfy the following rank assumption:

$$\operatorname{rank} E = \operatorname{rank}[\, E \quad B_1 \,].$$

Suppose, $Ex(0) = 0$. Then the closed-loop system $P_{cl}^{SF}$ :

$$\begin{aligned} Ex(k+1) &= (A + B_2F_2)x(k) + B_1 w(k), &(14)\\ z(k) &= (C + D_2F_2)x(k) + D_1 w(k) \end{aligned}$$

is admissible, and the inequality $|||P_{cl}^{SF}|||_a \leqslant \gamma$ holds true if there exist a matrix $\Phi = \Phi^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a positive scalar $\eta > \gamma^2$, satisfying the following conditions:

$$\begin{aligned} &E^{\mathrm{T}}\Phi E \geqslant 0, &(15)\\ &B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \eta I_{m_1} < 0, &(16)\\ &B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2 > 0, &(17)\\ &-\tfrac{1}{2}\ln(\det((\eta-\gamma^2)(\eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}}D_1)^{-1})) \geqslant a, &(18)\\ &E^{\mathrm{T}}\Phi E = A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C - \\ &-(A^{\mathrm{T}}\Phi\overline{B} + S)(\overline{B}^{\mathrm{T}}\Phi\overline{B} + R)^{-1}(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}}) &(19)\end{aligned}$$

where $\overline{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $S = \begin{bmatrix} C^{\mathrm{T}}D_1 & C^{\mathrm{T}}D_2 \end{bmatrix}$,

$$R = \begin{bmatrix} D_1^{\mathrm{T}}D_1 - \eta I_{m_1} & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 \end{bmatrix}.$$

A state feedback controller is defined as

$$F_2 = -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2)^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}}C).$$

*Proof:* The proof consists of three steps. On the first step we prove that the system, closed by a given state feedback $u(k) = F_2 x(k)$, is admissible. On the second step the anisotropic norm boundedness by positive scalar $\eta$ is proven. On the third step the connection between $\eta$ and $\gamma$ is discussed.

Denote

$$\begin{aligned} M_1 &= B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \eta I_{m_1} < 0,\\ M_2 &= B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2 > 0,\\ N &= B_1^{\mathrm{T}}\Phi B_2 + D_1^{\mathrm{T}}D_2,\\ A_{cl} &= A + B_2F_2,\ C_{cl} = C + D_2F_2. \end{aligned}$$

Note that the matrix

$$M = \overline{B}^{\mathrm{T}}\Phi\overline{B} + R = \begin{bmatrix} M_1 & N \\ N^{\mathrm{T}} & M_2 \end{bmatrix} < 0.$$

Consider an auxiliary variable

$$F = \begin{bmatrix} \overline{F}_1 \\ F_2 \end{bmatrix} = -M^{-1}(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}}). \quad (20)$$

The expression (20) may be rewritten in the form $MF = -(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}})$. Consequently,

$$M_1\overline{F}_1 + (B_1^{\mathrm{T}}\Phi B_2 + D_1^{\mathrm{T}}D_2)F_2 = -(B_1^{\mathrm{T}}\Phi A + D_1^{\mathrm{T}}C).$$

Without loss of generality one can choose $\overline{F}_1$, satisfying the following conditions:

$$\begin{aligned} M_1\overline{F}_1 &= 0, \\ N^{\mathrm{T}}\overline{F}_1 &= 0. \end{aligned}$$

Hence,

$$(A + B_2F_2)^{\mathrm{T}}\Phi B_1 + (C + D_2F_2)^{\mathrm{T}}D_1 = 0. \quad (21)$$

The generalized algebraic Riccati equation (19) may be rewritten as

$$E^{\mathrm{T}}\Phi E = A^{\mathrm{T}}\Phi A + \begin{bmatrix} C \\ M^{1/2}F \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C \\ M^{1/2}F \end{bmatrix}. \quad (22)$$

Let $L = \begin{bmatrix} 0 & \overline{B}M^{-1/2} \end{bmatrix}$, then

$$A + L\begin{bmatrix} C \\ M^{1/2}F \end{bmatrix} = A + B_2F_2 = A_{cl}.$$

The equation (22) is equivalent to the generalized Lyapunov equation (2) for $E$ and $A_{cl}$, hence, the pair $(E, A_{cl})$ is admissible.

Now we show that $|||P_{cl}^{SF}|||_a \leqslant \gamma$. Following [17], we introduce a function

$$T(x(k)) \doteq x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) \geqslant 0.$$

Consider an auxiliary function

$$H(x(k), w(k)) \doteq T(x(k+1)) - T(x(k)) + ||z(k)||^2 - \eta||w(k)||^2.$$

It is shown in (23) (see the next page), that $H(x(k), w(k)) \leqslant 0$.

If we sum the expressions $H(x(k), w(k))$, defined by (23), from $k = 0$ to $k = \infty$, we get

$$\sum_{k=0}^{\infty} H(x(k), w(k)) =$$
$$= T(x(\infty)) - T(x(0)) + \sum_{k=0}^{\infty}(||z(k)||^2 - \eta||w(k)||^2) \leqslant 0.$$

The closed-loop system is stable, so $Ex(\infty) = Ex(0) = 0$, then

$$\sum_{k=0}^{\infty}(||z(k)||^2 - \eta||w(k)||^2) \leqslant 0$$

and, hence,

$$\sup_{W} \frac{\sum_{k=0}^{\infty}||z(k)||^2}{\sum_{k=0}^{\infty}||w(k)||^2} \leqslant \eta.$$

Consequently, $\sup\limits_{W:\mathbf{A}(W)\leqslant a} \frac{\sum_{k=0}^{\infty}||z(k)||^2}{\sum_{k=0}^{\infty}||w(k)||^2} \leqslant \eta$, it means, that $|||P_{cl}^{SF}|||_a^2 \leqslant \eta$.

Using denotations $\eta = q^{-1}$ and $\Phi = q^{-1}\widehat{R}$, it is straightforward to show that the inequality (18) coincides with the inequality (12). Besides, GDARE (9)–(11) for the closed-loop system (14) agrees with the equation (19). ■

*Remark 2:* Consider a limiting case when $a \to +\infty$. Transform the expression (18) in the following way:

$$-\ln\left(\det(\eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}}D_1)^{-1}\right) \geqslant 2a + m_1\ln(\eta - \gamma^2). \quad (24)$$

As $\eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}}D_1 \leqslant \eta I_{m_1}$, the inequality (24) may be rewritten as

$$-\ln\left(\det(\eta^{-1}I_{m_1})\right) \geqslant 2a + m_1\ln(\eta - \gamma^2).$$

So,

$$\eta \leqslant \frac{\gamma^2}{1 - e^{-2a/m_1}}$$

and

$$\gamma^2 < \eta \leqslant \frac{\gamma^2}{1 - e^{-2a/m_1}}. \quad (25)$$

We get $\eta \to \gamma^2$ for $a \to +\infty$ from the condition (25), and the inequality (18) becomes invalid. Substituting $\gamma^2$ instead of $\eta$, we get the conditions for $H_\infty$-control design from (15)–(19). So, $\lim\limits_{a \longrightarrow +\infty} |||P_{cl}^{SF}|||_a = \|P_{cl}^{SF}\|_\infty \leqslant \gamma$.

### B. Full information control

*Theorem 5:* Let the system (13) satisfy the following rank assumption:

$$\operatorname{rank} E = \operatorname{rank}\begin{bmatrix} E & B_1 \end{bmatrix}.$$

Suppose, $Ex(0) = 0$. Then the closed-loop system $P_{cl}^{FI}$ :

$$\begin{aligned} Ex(k+1) &= (A + B_2F_2)x(k) + (B_1 + B_2F_1)w(k), \quad (26) \\ z(k) &= (C + D_2F_2)x(k) + (D_1 + D_2F_1)w(k) \end{aligned}$$

is admissible, and the inequality $|||P_{cl}^{FI}|||_a \leqslant \gamma$ holds true if there exist a matrix $\Phi = \Phi^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a positive scalar $\eta > \gamma^2$, satisfying the following conditions

$$E^{\mathrm{T}}\Phi E \geqslant 0, \quad (27)$$
$$M_2 = B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2 > 0, \quad (28)$$
$$M_1 = B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \eta I_{m_1} - N^{\mathrm{T}}M_2^{-1}N < 0, \quad (29)$$
$$-\tfrac{1}{2}\ln\det\left((\eta - \gamma^2)(-M_1)^{-1}\right) \geqslant a, \quad (30)$$
$$E^{\mathrm{T}}\Phi E = A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C -$$
$$-(A^{\mathrm{T}}\Phi\overline{B} + S)(\overline{B}^{\mathrm{T}}\Phi\overline{B} + R)^{-1}(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}}) \quad (31)$$

where $N = B_2^{\mathrm{T}}\Phi B_1 + D_2^{\mathrm{T}}D_1$, $\overline{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$,

$$S = \begin{bmatrix} C^{\mathrm{T}}D_1 & C^{\mathrm{T}}D_2 \end{bmatrix}, R = \begin{bmatrix} D_1^{\mathrm{T}}D_1 - \eta I_{m_1} & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 \end{bmatrix}.$$

Moreover,

$$F_1 = -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2)^{-1}(B_2^{\mathrm{T}}\Phi B_1 + D_2^{\mathrm{T}}D_1), \quad (32)$$
$$F_2 = -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2)^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}}C).$$

*Proof:* The proof of conditions (27)–(29) and (31) is similar to the proof of the SF case. Here we prove the

$$H(x(k),w(k)) = x^{\mathrm{T}}(k+1)E^{\mathrm{T}}\Phi Ex(k+1) - x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) + ||Cx(k)+D_1w(k)+D_2u(k)||^2 - \eta||w(k)||^2 =$$
$$= \{\text{substituting } Ex(k+1) \text{ from (13)}\} =$$
$$= (Ax(k)+B_1w(k)+B_2u(k))^{\mathrm{T}}\Phi(Ax(k)+B_1w(k)+B_2u(k)) - x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) + ||Cx(k)+D_1w(k)+D_2u(k)||^2 - \eta||w(k)||^2 =$$
$$= \{u(k)=F_2x(k)\} = (Ax(k)+B_1w(k)+B_2F_2x(k))^{\mathrm{T}}\Phi(Ax(k)+B_1w(k)+B_2F_2x(k)) - x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k)+$$
$$+||Cx(k)+D_1w(k)+D_2F_2x(k)||^2 - \eta||w(k)||^2 = \{\Theta = (A^{\mathrm{T}}\Phi B_2 + C^{\mathrm{T}}D_2)F_2\} = w^{\mathrm{T}}(k)M_1w(k)+$$
$$+w^{\mathrm{T}}(k)((A+B_2F_2)^{\mathrm{T}}\Phi B_1 + (C+D_2F_2)^{\mathrm{T}}D_1)^{\mathrm{T}}x(k) + x^{\mathrm{T}}(k)((A+B_2F_2)^{\mathrm{T}}\Phi B_1 + (C+D_2F_2)^{\mathrm{T}}D_1)w(k)+$$
$$+x^{\mathrm{T}}(k)(A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C - E^{\mathrm{T}}\Phi E + F_2^{\mathrm{T}}M_2F_2 + \Theta + \Theta^{\mathrm{T}})x(k) = \{\text{using (21) and } A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C - E^{\mathrm{T}}\Phi E = -F_2^{\mathrm{T}}M_2F_2\} =$$
$$= w^{\mathrm{T}}(k)M_1w(k) - 2x^{\mathrm{T}}(k)((A^{\mathrm{T}}\Phi B_2 + C^{\mathrm{T}}D_2)M_2^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}}C))x(k) \leqslant 0. \quad (23)$$

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condition (30). Using denotations $\eta = q^{-1}$ and $\Phi = q^{-1}\widehat{R}$, we rewrite the inequality (12) for the closed-loop system (26)

$$-\frac{1}{2}\ln\left(\det\left((\eta-\gamma^2)(\eta I_{m_1} - (B_1+B_2F_1)^{\mathrm{T}}\Phi(B_1+B_2F_1) - (D_1+D_2F_1)^{\mathrm{T}}(D_1+D_2F_1))^{-1}\right)\right) \geqslant a. \quad (33)$$

Denote

$$\Psi = \eta I_{m_1} - (B_1+B_2F_1)^{\mathrm{T}}\Phi(B_1+B_2F_1) - (D_1+D_2F_1)^{\mathrm{T}}(D_1+D_2F_1). \quad (34)$$

Substituting $F_1$ from (32) into (34), we get

$$\Psi = \eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}}D_1 - N^{\mathrm{T}}M_2^{-1}N = M_1. \quad (35)$$

So, the inequality (33) agrees with the inequality (30). ■

*Remark 3:* As in the SF case it is easy to show that for $a \to +\infty$ the conditions of the theorem 5 coincide with $H_\infty$-control design problem.

## V. NUMERICAL EXAMPLE

Consider the following system:

$$E = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0.85 & -0.3 \\ 0.1 & 0.3 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.02 \\ 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0.35 & 0.09 \end{bmatrix},\ D_1 = 0.035,\ D_2 = 0.1.$$

It is easy to see, that $\operatorname{rank} E = \operatorname{rank}[E \quad B_1] = 1$. The system is causal, but unstable, $\rho(E,A) = 1.0556$.

Consider the full-information case. For the mean anisotropy $a = 0.5$ and $\gamma = 0.06$ the solution of the algebraic Riccati equation (31) is $\Phi = \begin{bmatrix} 1.0325 & 0.7228 \\ 0.7228 & 0.5059 \end{bmatrix}$, the scalar $\eta$ equals to 0.0056. The left-hand side of the condition (27) becomes $E^{\mathrm{T}}\Phi E = \begin{bmatrix} 0.8363 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to see that condition (27) holds true. The conditions (28), (29) become $M_2 = 0.0203 > 0$, $M_1 = -0.0055 < 0$. Finally, $-\frac{1}{2}\ln\det\left((\eta-\gamma^2)(-M_1)^{-1}\right) = 0.5$, which coincides with the given mean anisotropy level $a = 0.5$. Hence, the control law is defined as $u(k) = -0.2738w(k) + [2.4711 \ -0.9]x(k)$.

Now we find a state feedback control $u(k) = F_2x(k)$ and full information control $u(k) = F_1w(k) + F_2x(k)$ for the given mean anisotropy level $a$ and a scalar value $\gamma$, using the techniques from the proven theorems. The results of controller design are represented in Tables I, II and III.

TABLE I: SF control design for different values of mean anisotropy

| $a$ | 0.1 | 0.5 | 0.8 |
|---|---|---|---|
| $\gamma$ | 0.050 | 0.055 | 0.060 |
| $\|\|\|P_{cl}^{SF}\|\|\|_a$ | 0.0336 | 0.0422 | 0.0423 |
| $\rho(E,A_{cl})$ | 0.7611 | 0.7672 | 0.7631 |
| $\gamma^2$ | 0.0025 | 0.0030 | 0.0036 |
| $\eta$ | 0.0042 | 0.0038 | 0.0041 |
| $F_2$ | [2.3498, -0.9] | [2.2956, -0.9] | [2.3319, -0.9] |

TABLE II: FI control design for different values of mean anisotropy

| $a$ | 0.1 | 0.3 | 0.5 |
|---|---|---|---|
| $\gamma$ | 0.050 | 0.050 | 0.060 |
| $\|\|\|P_{cl}^{FI}\|\|\|_a$ | 0.0162 | 0.0208 | 0.0230 |
| $\rho(E,A_{cl})$ | 0.7247 | 0.7493 | 0.7477 |
| $\gamma^2$ | 0.0025 | 0.0025 | 0.0036 |
| $\eta$ | 0.0134 | 0.0054 | 0.0056 |
| $F_1$ | -0.2747 | -0.2737 | -0.2738 |
| $F_2$ | [2.6781, -0.9] | [2.4566, -0.9] | [2.4711, -0.9] |

## VI. CONCLUSIONS

In this paper, the problem of anisotropy-based suboptimal control for descriptor systems based on generalized algebraic Riccati equations is solved. The considered problem is a general case of LQG/$H_2$- and $H_\infty$-suboptimal control problems for normal and descriptor systems. It is shown that for $E = I_n$ the solution coincides with the result, obtained in [1]. Numerical experiments show that FI-controller allows

TABLE III: Comparison of SF and FI control design for $a = 0.5$ and $\gamma = 0.06$

| *Case* | $\eta$ | $\Phi$ | $\rho$ | $\|\|\|P_{cl}\|\|\|_a$ |
|---|---|---|---|---|
| *FI* | 0.0056 | $\begin{bmatrix} 1.0325 & 0.7228 \\ 0.7228 & 0.5059 \end{bmatrix}$ | 0.7477 | 0.0230 |
| *SF* | 0.0047 | $\begin{bmatrix} 1.0406 & 0.7284 \\ 0.7284 & 0.5099 \end{bmatrix}$ | 0.7553 | 0.0423 |

to reduce anisotropic norm of the system due to the known information about input disturbance. Information about complexity of GDARE numerical solutions can be found in [18] and the references within.

## REFERENCES


[1] A. A. Stoorvogel, The $H_\infty$ Control Problem: A State Space Approach, University of Michigan, Ann Arbor, 2000.

[2] K. J. Aström and B. Wittenmark, Computer Controlled Systems, Theory and Design, Prentice-Hall, 1984.

[3] B. Bamieh and J.B. Pearson, The $H_2$ problem for sampled-data systems, System and Control Letters, 1992, vol. 19, pp. 1–12.

[4] T. Chen and B.A. Francis, $H_2$-optimal sampled-data control, IEEE Transactions on Automatic Control, 1991, vol. 36, pp. 387–397.

[5] L. Dai, Singular Control Systems, Lecture Notes in Control and Information Sciences, New York, Springer-Verlag, 1989.

[6] C.-F. Yung, $H_\infty$ control for linear discrete-time descriptor systems: state feedback and full information cases, Proc. of the 17th IFAC World Congr., Seoul, Korea, 2008, pp. 10003–10008.

[7] S. Xu, J. Lam, Robust Control and Filtering of Singular Systems, Lecture Notes in Control and Information Sciences, Berlin, Springer-Verlag, 2006.

[8] M. Chadli, M. Darouach, Novel bounded real lemma for discrete-time descriptor systems: Application to $H_\infty$ control design, Automatica, 2012, vol. 48, pp. 449–453.

[9] L. Dai, Filtering and LQG problems for discrete-time stochastic singular systems, IEEE Trans. Automat. Control, 1989, vol. 34, pp. 1105–1108.

[10] T. Katayama, (J,J')-spectral factorization and conjugation for discrete-time descriptor systems, Circuits, systems, and signal processing, 1996, vol. 15, No. 5, pp. 649–669.

[11] I. G. Vladimirov, A. P. Kurdjukov, and A. V. Semyonov, Anisotropy of Signals and the Entropy of Linear Stationary Systems, Doklady Math., 1995, vol. 51, pp. 388–390.

[12] I. G. Vladimirov, A. P. Kurdjukov, and A. V. Semyonov, On computing the anisotropic norm of linear discrete-time-invariant systems, Proc. 13th IFAC World Congress, San-Francisco, 1996, pp. 179–184.

[13] P. Diamond, I. Vladimirov, A. Kurdjukov, and A. Semyonov, Anisotropy-based performance analysis of linear discrete time invariant control systems, International Journal of Control, 2001, vol. 74 (1), pp. 28–42.

[14] A. A. Belov, Synthesis of anisotropic controllers for descriptor systems, Ph. D. Thesis, Moscow, 2011.

[15] T. Stykel, Analisys and numerical solutions of generalised Lyapunov equations, Ph. D. thesis, Institut fur Mathematik, Techische Universitat Berlin, Berlin, 2002.

[16] O. Andrianova, A. Belov, Anisotropy-based bounded real lemma for linear discrete-time descriptor systems, Proc. 2013 International Conference on Process Control, Strbske Pleso, Slovakia, 2013, pp. 57–62.

[17] C.-F. Yung, C.-C. Wang, P.-F. Wu, H.-S. Wang, Bounded real lemma for linear discrete-time descriptor systems, Proc. of the 17th IFAC World Congr., Seoul, Korea, 2008, pp. 9982–9986.

[18] C. Oară, R. Andrei, Numerical solution to a descriptor discrete-time algebraic Riccati equation, Systems and Control Letters, 2013, No. 62(2), pp. 201–208.