Suboptimal Anisotropy-based Control Design for Discrete-Time Systems with Nonzero-Mean Input Disturbances

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***Abstract*—In this paper, linear discrete-time systems with Gaussian input disturbances are considered. Input sequences are characterized by nonzero mean and bounded mean anisotropy. Suboptimal control law, which guarantees stability of the closed-loop system and boundedness of its anisotropic norm, is designed.**

I. Introduction

Attenuation of disturbances, affecting on the control plant, is the one of the most important problems in modern control theory. Anisotropy-based analysis and control design theories deal with stationary Gaussian random sequences with zero mean, considered as input disturbance signals [1], [2]. Anisotropic norm defines the system's gain from the input disturbance to the output signal. In case of optimal control the objective is to design a control law, which internally stabilizes the closed-loop system and minimizes its anisotropic norm from the input disturbance to the controllable output [3]. Suboptimal anisotropy-based control goal is to find such control law for which anisotropic norm of closed-loop system is less than a prescribed value [4], [5].

But in some applications mathematical expectation of random disturbances can be nonzero due to equipment errors. Hence, taking into account the value of mathematical expectation of the input signal is significant. This approach requires development of additional mathematical tool, which allows to solve analysis and control design problems for nonzero-mean input random disturbances.

The concept of anisotropy-based analysis of linear discrete-time systems is first introduced in [6]–[8]. In [6] the concept of mean anisotropy of the random Gaussian sequence with nonzero mean is introduced. The paper [7] is devoted to anisotropic norm of systems with nonzero-mean input signals. The conditions of anisotropic norm boundedness in terms of matrix inequalities are derived in [8].

This paper deals with anisotropy-based state-space suboptimal control design problem for discrete-time systems affected by random Gaussian sequences with nonzero mean and bounded mean anisotropy level.

II. Basics of anisotropy-based control theory for nonzero-mean input signals

A. Anisotropy of a random vector

Anisotropy of the m-dimensional random vector w is introduced in [1] as the minimal value of the relative entropy of w with respect to the Gaussian m-dimensional vector with probability density function (p.d.f.)

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2}\exp\left(-\frac{x^T x}{2\lambda}\right), \quad x \in \mathbb{R}^m,$$

and is described by

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{E}_f\left[\ln\left(\frac{f}{p_{m,\lambda}}\right)\right] \tag{1}$$

where the function f is p.d.f. of w, $\mathbf{E}_f(X)$ is mathematical expectation of the m-dimensional random vector (sequence) X in sense of f.

We suppose w is the m-dimensional Gaussian random vector with nonzero mean ν and the covariance matrix S, which p.d.f. is given by [6]

$$f(x) = ((2\pi)^m|S|)^{-1/2}\exp\left(-\frac{1}{2}(x-\nu)^T S^{-1}(x-\nu)\right), \quad x \in \mathbb{R}^m.$$

By definition of anisotropy of the random vector (1),

$$\mathbf{A}(w) = -\frac{1}{2}\ln\det\left(\frac{mS}{\mathrm{tr}S + |\nu|^2}\right).$$

One can show that if $S = \gamma I_m$ and $\nu = 0$, then $\mathbf{A}(w) = 0$.

Consider a stationary sequence of random m-dimensional vectors $W = \{w(k)\}_{k\in\mathbb{Z}}$. Mean anisotropy of the stationary

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ergodic sequence W is defined by the following expression [1]:

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

where $W_{0:N-1}$ is an extended vector of the sequence

$$W_{0:N-1} = \begin{bmatrix} w(0) \\ \vdots \\ w(N-1) \end{bmatrix}.$$

It is known that the sequence W with can be generated [8], [9] by a stable linear discrete-time system.

Let the shaping filter have a state-space representation

$$\begin{aligned} x(k+1) &= \widehat{A}x(k) + \widehat{B}(v(k) + \mathcal{M}), \\ w(k) &= \widehat{C}x(k) + \widehat{D}(v(k) + \mathcal{M}) \end{aligned}$$

where $v(k)$ is a Gaussian white noise sequence with zero mean and identity covariance matrix and $\mathcal{M}$ is a mathematical expectation of the signal, $x(k) \in \mathbb{R}^n$, $v(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^m$. Then mean anisotropy $\overline{\mathbf{A}}(W)$ is determined by

$$\overline{\mathbf{A}}(W) = -\frac{1}{2} \ln\det \left(\frac{m(\Sigma + \Xi)}{\mathrm{tr}\Sigma + |\mathcal{G}\mathcal{M}|^2} \right)$$

where Σ and Ξ are found from the solutions of Lyapunov and Riccati equations P and R

$$\begin{gathered} \Sigma = \widehat{C}P\widehat{C}^T + \widehat{D}\widehat{D}^T, \\ P = \widehat{A}P\widehat{A}^T + \widehat{B}\widehat{B}^T, \end{gathered}$$

$$\begin{gathered} \Xi = \widehat{C}R\widehat{C}^T, \\ R = \widehat{A}R\widehat{A}^T - \Lambda(\Sigma + \Xi)^{-1}\Lambda^T, \\ \Lambda = \widehat{B}\widehat{D}^T + \widehat{A}(P + R)\widehat{C}^T, \end{gathered}$$

and

$$\mathcal{G} = (\widehat{D} + \widehat{C}(I_{n\times n} - \widehat{A})^{-1}\widehat{B}).$$

B. Anisotropic norm of the system

Consider a stable linear discrete-time system F given in a state-space representation

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), && (2) \\ y(k) &= Cx(k) + Dw(k) && (3) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are input and output signals, respectively. A, B, C and D are constant real matrices of appropriate dimensions. $W = \{w(k)\}_{k\in\mathbb{Z}}$ is a stationary Gaussian sequence of m-dimensional random vectors with a bounded mean anisotropy $\overline{\mathbf{A}}(W) \leqslant a$, $a \geqslant 0$ and known nonzero mean $\mathbf{E}w(\infty) = \mathcal{M}$, $|\mathcal{M}| < \infty$.

For a given system F with the input signal $W = \{w(k)\}$ a root mean-square gain is defined as

$$Q(F, W) = \frac{\|y\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}} = \sqrt{\lim_{N\to\infty} \frac{\frac{1}{N}\sum\limits_{k=0}^{N-1} \mathbf{E}|y(k)|^2}{\frac{1}{N}\sum\limits_{k=0}^{N-1} \mathbf{E}|w(k)|^2}} \qquad (4)$$

where

$$\|y\|_{\mathcal{P}} = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E}|y(k)|^2}$$

is the power norm of the signal $\{y(k)\}$.

Let the sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$ be represented in the form

$$w(k) = C_g x(k) + D_g(v(k) + \mu) \qquad (5)$$

where $x(k)$ is the state of the system (2)–(3), and μ is a known vector. Using (5), we obtain the admissible filter G

$$\begin{aligned} x(k+1) &= (A + BC_g)x(k) + BD_g(v(k) + \mu), && (6) \\ w(k) &= C_g x(k) + D_g(v(k) + \mu). && (7) \end{aligned}$$

Power norms of outputs of the systems (2)–(3) and (6)–(7) are

$$\begin{aligned} \|w\|_{\mathcal{P}}^2 &= \lim_{k\to\infty} \left(\mathrm{tr}\ \mathrm{cov}(w(k)) + |\mathbf{E}w(k)|^2\right) \\ &= \|G\|_2^2 + |\mathcal{M}|^2, \\ \|y\|_{\mathcal{P}}^2 &= \lim_{k\to\infty} \left(\mathrm{tr}\ \mathrm{cov}(y(k)) + |\mathbf{E}y(k)|^2\right) \\ &= \|FG\|_2^2 + |\mathcal{F}\mathcal{M}|^2 \end{aligned}$$

where

$$\mathcal{F} = F(1) = D + C(I_n - A)^{-1}B.$$

The root mean-square gain (4) of the system with nonzero-mean input signal is given by the following expression:

$$Q(F, W) = Q(F, G) = \sqrt{\frac{\|FG\|_2^2 + |\mathcal{F}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2}}.$$

Finally, anisotropic norm of the system is defined as [2]

$$|\!|\!|F|\!|\!|_a = \sup_{G:\overline{\mathbf{A}}(G)\leqslant a} Q(F, G).$$

Remark 1: A random sequence W is fully defined by its generating filter G, therefore, the notation $\overline{\mathbf{A}}(G)$ is used as equivalent to the notation $\overline{\mathbf{A}}(W)$.

For the system in a state-space representation (2)–(3) the following theorem defines conditions of anisotropic norm boundedness [8].

Theorem 1: Let $F \in \mathcal{H}_\infty{}^{p\times m}$ be a stable system represented by (2)–(3). Nonzero-mean input sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$ has a bounded mean anisotropy $\overline{\mathbf{A}}(W) \leqslant a$ and a known mean $\mathcal{M}$. Then a-anisotropic norm is bounded by $\gamma > 0$, i.e. $|\!|\!|F|\!|\!|_a < \gamma$ if there exist a scalar value

$$\eta \in \left(\max\left\{ \frac{\gamma^2 - |\mathcal{F}\mathcal{M}|^2}{1 - |\mathcal{M}|^2},\ \|F\|_\infty^2 \right\},\ \frac{\gamma^2 - |\mathcal{F}\mathcal{M}|^2}{1 - |\mathcal{M}|^2 - \mathrm{e}^{-\frac{2a}{m}}} \right)$$

and a matrix $\Phi = \Phi^{\mathrm{T}} > 0$ that satisfy the inequalities (8) and (9).

$$\begin{bmatrix} A^{\mathrm T}\Phi A - \Phi + C^{\mathrm T}C & A^{\mathrm T}\Phi B + C^{\mathrm T}D \\ B^{\mathrm T}\Phi A + D^{\mathrm T}C & B^{\mathrm T}\Phi B + D^{\mathrm T}D - \eta I_m \end{bmatrix} < 0, \quad (8)$$

$$\eta\left(1 - |\mathcal{M}|^2\right) + |\mathcal{F}\mathcal{M}|^2 - \mathrm{e}^{-\frac{2a}{m}} \det\left(\eta I_m - B^{\mathrm T}\Phi B - D^{\mathrm T}D\right)^{\frac{1}{m}} < \gamma^2. \quad (9)$$

III. Problem statement and main results

A linear discrete-time system F is given as

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \quad (10)$$

$$z(k) = Cx(k) + D_1 w(k) + D_2 u(k), \quad (11)$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $z(k) \in \mathbb{R}^q$ is the controllable output, $u(k) \in \mathbb{R}^{m_2}$ is the control signal, A, B_1, B_2, C, D_1 and D_2 are known matrices of appropriate dimensions.

The Gaussian sequence of m_1-dimensional random vectors $W = \{w(k)\}_{k\in\mathbb{Z}}$ is stationary with nonzero mean $\mathbf{E}w(\infty) = \mathcal{M}$, $|\mathcal{M}| < 1$ and bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$. Suppose, that the system (10)–(11) is stabilizable and detectible.

We consider the control law in the form

$$u(k) = Kx(k). \quad (12)$$

Problem 1: For the given plant F with a state-space representation as (10)–(11), for the known values of mathematical expectation $\mathcal{M}$ and the boundary value $a \geqslant 0$ of mean anisotropy of the input disturbance $W = \{w(k)\}_{k\in\mathbb{Z}}$, and for the given value $\gamma > 0$ **the problem is** to find the control law in the form (12), which internally stabilizes the closed-loop system and guarantees that its anisotropic norm is less than γ, i.e.

$$\|\!|F_{cl}|\!\|_a < \gamma.$$

Here F_{cl} is the system closed by the control law in the form (12).

The conditions (8) and (9) can't be applied directly to solving the stated problem because of cross product of the unknown matrix Φ and matrices of the closed-loop system realization, that depend on the parameters of the control law. In order to avoid these difficulties, transform (8) and (9), following [4].

First, formulate the conditions of Schur's lemma [10], which is important for matrix transformations.

Lemma 1: Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm T} & X_{22} \end{bmatrix}$$

where X_{11} and X_{22} are square matrices.

If $X_{11} > 0$, then $X > 0$ if and only if

$$X_{22} - X_{12}^{\mathrm T} X_{11}^{-1} X_{12} > 0.$$

If $X_{22} > 0$, then $X > 0$ if and only if

$$X_{11} - X_{12} X_{22}^{-1} X_{12}^{\mathrm T} > 0.$$

□

Now introduce $\Psi = \Psi^{\mathrm T} > 0$:

$$\eta I_m - B^{\mathrm T}\Phi B - D^{\mathrm T}D > \Psi. \quad (13)$$

In view of lemma 1, one can rewrite (13) as

$$\begin{bmatrix} \Psi - \eta I_m & * & * \\ B & -\Phi^{-1} & * \\ D & 0 & -I_p \end{bmatrix} < 0. \quad (14)$$

So, (9) is equivalent to the system of inequalities (14) and

$$\eta\left(1 - |\mathcal{M}|^2\right) + |\mathcal{F}\mathcal{M}|^2 - \left(\mathrm{e}^{-2a} \det\left(\Psi\right)\right)^{\frac{1}{m}} < \gamma^2. \quad (15)$$

Using lemma 1, transform (9) to

$$\begin{bmatrix} -\Phi & * & * & * \\ 0 & -\eta I_m & * & * \\ A & B & -\Phi^{-1} & * \\ C & D & 0 & -I_p \end{bmatrix} < 0. \quad (16)$$

Applying the new conditions (14), (15) and (16) to suboptimal control problem 1 for the system (10)–(11), closed by the control law (12), leads to the following theorem.

Theorem 2: For given scalar values $a \geqslant 0$ and $\gamma > 0$ the static state control law in the form (12), that stabilizes the closed-loop system (spectral radius of matrix $A + B_2K$ is $\rho(A + B_2K) < 1$) and guarantees $\|\!|F_{cl}|\!\|_a < \gamma$, exists if the following system of inequalities:

$$\eta\left(1 - |\mathcal{M}|^2\right) + |\mathcal{F}\mathcal{M}|^2 - \left(\mathrm{e}^{-2a} \det\left(\Psi\right)\right)^{\frac{1}{m_1}} < \gamma^2, \quad (17)$$

$$\begin{bmatrix} \Psi - \eta I_{m_1} & * & * \\ B_1 & -\Pi & * \\ D_{11} & 0 & -I_q \end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} -\Pi & * & * & * \\ 0 & -\eta I_{m_1} & * & * \\ A\Pi + B_2\Lambda & B_1 & -\Pi & * \\ C_1\Pi + D_{12}\Lambda & D_{11} & 0 & -I_q \end{bmatrix} < 0, \quad (19)$$

$$\eta > \frac{\gamma^2 - |\mathcal{F}\mathcal{M}|^2}{1 - |\mathcal{M}|^2} \quad (20)$$

$$\Psi > 0, \;\; \Pi > 0$$

is solvable with regard to the scalar value η, $m_1 \times m_1$-matrix Ψ, $n \times n$-matrix Π and $m_2 \times n$-matrix Λ.

If the system (17)–(19) is solvable, the static state regulator's matrix can be found as $K = \Lambda\Pi^{-1}$.

Remark 2: Straight application of (16) to the closed-loop system doesn't coincide with (19). The used matrix transformation is given in [4] in details.

Inequalities (18) and (20) contain an unknown value $|\mathcal{F}\mathcal{M}|^2$, which is difficult to transform into an applicable for LMI solvers form. Hence, the following algorithm is suggested. At the beginning we choose a desirable accuracy δ.

1) Set $\mathcal{M}_0 = 0$ and solve standard anisotropy-based control problem. Then we get value $\mathcal{F}$ for $\mathcal{M}_0 = 0$.
2) Choose $N_1 \in \mathbb{N}$ and $k = 1$. Set $\mathcal{M}_k = \dfrac{k\mathcal{M}}{N_1}$.
3) Solve anisotropy-based control design problem for $\mathcal{M}_k$ and $\mathcal{F}_{\mathcal{M}_{k-1}}$. Here
$$\mathcal{F}_{\mathcal{M}_{k-1}} = F_{cl}(1) = D + C(I_n - (A + B_2K_{k-1}))^{-1}B_1$$
where K_{k-1} is a controller found on a step $k-1$. Set $k := k+1$.
4) If $k \leqslant N_1$, go to step 2.
5) Choose $N_2 > N_1$ and repeat steps 1–4. If $|K_{N_1} - K_{N_2}| < \delta$, then $K = \dfrac{K_{N_1} + K_{N_2}}{2}$.

IV. Example

Consider a system in a state-space representation

$$A = \begin{bmatrix} -0.25 & 0.1 & 0 \\ -0.5 & 0.5 & 2 \\ -0.75 & -1 & -1.5 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.01 & -0.01 \end{bmatrix}, \quad D_2 = 0.$$

The mathematical expectation of the input disturbance is $\mathcal{M} = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}$.

We choose a desirable accuracy $\delta = 10^{-5}$, mean anisotropy level $a = 0.5$, and solve γ-optimal problem (find $\inf_K \gamma$). The spectral radius of matrix A is $\rho(A) = 1.1180$, i.e. the system is not stable.

For $\mathcal{M}_0 = 0$ the following results are obtained:
$K = \begin{bmatrix} 0.5118 & 0.3425 & 0.2036 \end{bmatrix}$, $|||F_{cl}|||_a = 0.9646$.

The results of control design using the algorithm above are

$$\Lambda = \begin{bmatrix} 78.1844 & -39.0337 & 7.0846 \end{bmatrix},$$
$$\Pi = \begin{bmatrix} 217.7290 & -108.8674 & 19.7954 \\ -108.8674 & 54.6851 & -10.0262 \\ 19.7954 & -10.0262 & 1.9094 \end{bmatrix},$$
$$K = \begin{bmatrix} 0.5042 & 0.3187 & 0.1568 \end{bmatrix},$$
$$\frac{\gamma^2 - |\mathcal{F}\mathcal{M}|^2}{1 - |\mathcal{M}|^2} = 0.8732, \qquad \|F_{cl}\|_\infty = 1.2266,$$
$$\frac{\gamma^2 - |\mathcal{F}\mathcal{M}|^2}{1 - |\mathcal{M}|^2 - e^{-2a/m_1}} = 5.9922, \qquad \eta = 1.8704.$$

Hence, the localization condition given by the theorem1 is satisfied. $|||F_{cl}|||_a = 0.9234$. The prescribed accuracy was reached on $N_2 = 20$ iterations.

V. Conclusion

Anisotropy-based state-space suboptimal control design problem for discrete-time systems affected by random Gaussian sequences with nonzero mean and known mean anisotropy level is solved. The numerical algorithm for solving control problem is suggested. It is shown that taking into account mathematical expectation of the input disturbance allows to tune the closed-loop system more precisely and perform better disturbance attenuation.

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