Dynamic Network Formation Models with Strategic Interactions

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Introduction - different approaches to network formation

- The literature on network formation is basically divided into:
  - random network approach – mainly dynamic, the reason why a link is formed is pure chance (for surveys, see e.g., Vega-Redondo 2007, Jackson 2008, Pin and Rogers 2016) –
    How emerging networks match real-world networks (small worlds, high clustering, short average path lengths, ...)
  - strategic network approach – the reason why a link is formed is strategic interaction, aiming at maximization of a utility function (e.g., Jackson and Wolinsky 1996; for surveys see e.g. Jackson 2008, Bloch and Dutta 2010, Mauleon and Vannetelbosch 2016)

Strategic network formation

- There are many settings in which choice plays a central role in determining relationships.
- Individuals have discretion in which relationships they form and maintain and how much effort/time they devote to different links.
- The models of strategic formation answer the question why the networks take a particular structure rather than how they take it.
- Many works consider the formation of networks in a static setting (agents form a network once and for all with payoffs being generated only once).
- We focus on the formation of (undirected) networks in a dynamic setting (networks evolve over time, the structure of links can change), where agents do not bargain over payoffs.

Surveys on the related literature

- Bramoullé, A. Galeotti, B. Rogers (2016), The Oxford Handbook of the Economics of Networks, Oxford University Press
- M.O. Jackson (2008) Social and Economic Networks, Princeton University Press
- F. Vega-Redondo (2007) Complex Social Networks, Cambridge University Press
- F. Bloch and B. Dutta (2010) Formation of networks and coalitions, In: Benhabib, Bisin and Jackson, Handbook of Social Economics
- A. Mauleon and Vannetelbosch (2016) Network formation games, In: Bramoullé et al. (2016)
- F. Vega-Redondo (2016) Links and actions in interplay, In: Bramoullé et al. (2016)

Network notations and basic concepts - undirected networks

- $N = \{1, 2, \dots, n\}$ - set of agents (players, nodes)
- Network g – set of pairs $\{i, j\}$ (denoted ij), with $i, j \in N$, $i \neq j$.
- Link ij describes a relationship between i and j
- i and j are directly connected iff $ij \in g$
- Degree $d_i(g)$ of i counts the number of links i has in g, i.e.,

$$d_i(g) = |\{j \in N \mid ij \in g\}|$$

- Path $=$ sequence of links $i_1 i_2, \cdots, i_{K-1} i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \cdots, K-1\}$, and all nodes are distinct.

Network notations and basic concepts (cont. 1)

- $g + ij$ - network obtained by adding link ij to g
- $g - ij$ - network obtained by deleting link ij from g
- g and g' are adjacent if $g' = g + ij$ or $g' = g - ij$
- Let g^N be the set of all subsets of N of size 2
- A value function $v : G \to \mathbb{R}$ specifies the aggregate value of each network, where $G := \{g | g \subseteq g^N\}$
- An allocation rule $Y : G \times V \to \mathbb{R}^N$ specifies the payoff corresponding to each (g, v), where V - set of all value functions.

Network notations and basic concepts (cont. 2)

- In some applications: $v(g) = \sum_{i \in N} u_i(g)$ and $Y_i(g, v) = u_i(g)$, where $u_i : G \to \mathbb{R}$ is the utility of player i from network g
- Geodesic between two nodes is a shortest path between them.
- $d(i, j; g)$ = geodesic distance between i and j in g
  If there is a path between i and j in g, then

  $d(i, j; g)$ = the number of links in a shortest path between i and j

$$d(i, j; g) = \min_{\text{paths } P \text{ from } i \text{ to } j} \sum_{(k,l) \in P} g_{kl}.$$

  If there is no path between i and j in g, we set $d(i, j; g) = \infty$.

Network notations and basic concepts (cont. 3)

- Network can change over time either by agents creating new links or/and destroying existing ones:
  - two-sided network formation - A link between two agents can be formed only if both agree to add the link, while a single agent can sever an existing link (e.g. Jackson and Watts 2002)
  - one-sided network formation - an agent does not need another agent's permission to form a link with him (e.g. Bala and Goyal 2000)
- A network $g \in G$ is pairwise stable (PS) w.r.t. v and Y if:
  1. $\forall\ ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$ and
  2. $\forall\ ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$
- When g is not PS, it is said to be defeated by g' if either $g' = g + ij$ and (ii) is violated for ij, or $g' = g - ij$ and (i) is violated for ij.
- A network $g \subseteq g^N$ is efficient (E) if $v(g) \geq v(g')$ *for all* $g' \subseteq g^N$.

The symmetric connections model by Jackson & Wolinsky

- M.O. Jackson, A. Wolinsky (1996) A strategic model of of social and economic networks, *Journal of Economic Theory*
- The utility of player i from network g:

$$u_i(g) = \sum_{j \neq i} \delta^{d(i,j;g)} - c d_i(g)$$

  with $0 < \delta < 1$ benefit term, $c > 0$ costs for a link.
- The unique efficient network in the symmetric connections model is:
  1. the complete network g^N if $c < \delta - \delta^2$
  2. a star g^* if $\delta - \delta^2 < c < \delta + \frac{(n-2)\delta^2}{2}$
  3. no links if $\delta + \frac{(n-2)\delta^2}{2} < c$.
- Pairwise stability - For:
  1. $c < \delta - \delta^2$, the unique PS network is the complete graph g^N.
  2. $\delta - \delta^2 < c < \delta$, a star g^* is PS (not necessarily the unique one).
  3. $\delta < c$, any PS network which is non-empty is such that every player has at least two links (and thus is inefficient).

Strategic network formation in a dynamic context

- Which network structures will arise in dynamic models of network formation where self-interested individuals can form and severe links?
- Myopic agents versus non-myopic (forward looking) agents
- Myopic agents - each pair of active agents in any period t choose their actions only by looking at their t-period payoff.
- The assumption of myopic agents is a common assumption in situations where agents have limited information about the payoffs and incentives of others; e.g., Watts (2001), Jackson and Watts (2002), Bala and Goyal (2000).

A dynamic version of the connections model (Watts 2001)

- A. Watts (2001) A dynamic model of network formation, *Games and Economic Behavior*
- First study of the dynamic evolution of networks, limited to the context of the J-W model and a particular deterministic dynamic, myopic agents
- Random ordering over links. At any point in time, any link is as likely as any other to be identified:
  - If the link has not yet been added $\rightarrow$ if at least one player involved would benefit from adding it and the other would be at least as well off, then the link is added.
  - If the identified link has already been added $\rightarrow$ it is deleted if either player would (myopically) benefit from its deletion.

A dynamic version of the connections model (Watts 2001)

- PS network that will be reached in the symmetric connections model under this dynamic process:
  - If $c > \delta$, the process ends at the empty network (even if there are non-empty networks strictly preferred by all players to $g^{\emptyset}$).
  - If $\delta - c > \delta^2$, the efficient complete network will be reached.
- Assume that $\delta - \delta^2 < c$. For $3 < n < \infty$, there is a positive probability $0 < p(star) < 1$ that the formation process will converge to a star. However, as n increases, $p(star)$ decreases, and as n goes to infinity, $p(star)$ goes to 0.

Myopic agents versus non-myopic (forward looking) agents

- Watts (2001) shows that the dynamic process of network formation does not always converge to the efficient network.
- Myopic agents can end up stuck in inefficient network structures, as they cannot exploit even very high increasing returns to network formation.
- Two ways of avoiding this problem:
  - keeping the assumption of myopic behavior, but allowing for the possibility of exogenous "shocks" causing a link to form, and helping the network formation process; e.g., Jackson and Watts (2002), Tercieux and Vannetelbosch (2006), Feri (2007), ...
  - assuming that agents are farsighted (can suffer an initial loss to gain in the future), e.g., Watts (2002), Dutta et al. (2005), ...

The evolution of social and economic networks (Jackson and Watts 2002)

- M. Jackson and A. Watts (2002) The evolution of social and economic networks, *Journal of Economic Theory*
- A dynamic formation and stochastic evolution of networks
- Main differences with respect to Watts (2001):
  - agents remain myopic, but a wider collection of network models is admitted
  - introducing the concept of improving path
  - adding random perturbations to the basic deterministic dynamic and examining the distribution over networks as the level of random perturbations goes to 0.
- The stochastic dynamic process refines the prediction of the deterministic process (robustness check which networks are most stable in the face of small perturbations).

The evolution of social and economic networks (Jackson and Watts 2002)

- A (myopic) improving path $g \to g'$ from network g to network g' is a finite sequence of adjacent networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either
  (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or
  (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_l(g_k + ij) > Y_l(g_k)$ for $l = i, j$
- PS of a network is equivalent to saying that the network has no improving paths emanating from it.
- Time is divided into discrete time periods $\{1, 2, \ldots\}$. In any period t, g_t is the historically given graph. A pair i and j meet randomly with probability p_{ij} in period t. The selected pair can decide to:
  - form the link ij if $ij \notin g_t$ (the link forms if both i and j agree)
  - either i or j can unilaterally break the link ij if $ij \in g_t$.

The evolution of social and economic networks (Jackson and Watts 2002)

- Process of mutation: in any period t, after the action is taken, there is a probability $\epsilon > 0$ that a mutation (tremble) occurs, and the link is deleted if present and added if absent.
- This process defines a Markov chain (where the states are the networks existing at the end of every period), which has a unique corresponding stationary distribution.
- As $\epsilon \to 0$, this stationary distribution converges to a unique limiting distribution.
- A network is stochastically stable if it is in the support of this limiting distribution.
- Intuitively: a stochastically stable network is one that is observed infinitely many more times than others when the probability of mutations is infinitely small.

The evolution of social and economic networks (Jackson and Watts 2002)

- For a given g, let $im(g) = \{g' \mid g' \to g\}$
- The resistance $r(p)$ of a path $p = \{g_1, \cdots, g_K\}$ is

$$r(p) = \sum_{i=1}^{K-1} I(g_i, g_{i+1})$$

where $I(g_i, g_{i+1}) = 0$ if $g_i \in im(g_{i+1})$ and $I(g_i, g_{i+1}) = 1$ otherwise.

- Resistance keeps truck of how many mutations must occur along a specific path to follow that path from one network to another.
- Let $r(g', g) = min\{r(p) \mid p \textit{ is a path from } g' \textit{ to } g\}$, $r(g, g) = 0$

The evolution of social and economic networks (Jackson and Watts 2002)

- Given a network g, a g-tree $t(g)$ is a directed graph which has as vertices all networks and has a unique directed path from g' to g.
- $g'g'' \in t(g)$ iff there is a directed edge connecting g' to g'' in $t(g)$.
- Let $T(g)$ denote all the g-trees.
- The resistance of a network g is computed as

$$r(g) = \min_{t(g)\in T(g)} \sum_{g'g''\in t(g)} r(g', g'')$$

- $r(g)$ measures how many mutations are needed in order to get away from the network to an improving path leading to another network.
- The set of stochastically stable networks is the set $\{g \mid r(g) \leq r(g') \textit{ for all } g'\}$ (networks which minimize resistance).

A characterization of stochastically stable networks (Tercieux and Vannetelbosch 2006)

- O. Tercieux and V. Vannetelbosch (2006), A characterization of stochastically stable networks, *International Journal of Game Theory*
- They provide a refinement of PS, p-pairwise stability (p-PS), to characterize the stochastically stable networks without requiring the "tree construction" and the computation of resistance.
- A network is p-pairwise stable if when we add a set of links to this network (or sever a set of links), then if we allow agents to successively create or delete links, they will come back to the initial network.
- The parameter $p \in [0, 1]$ indicates the fraction of links that can be modified, e.g.:
  - $p = 0$: no link may be added or severed (pairwise stability)
  - $p = 1$: all links may be modified.

A characterization of stochastically stable networks (Tercieux and Vannetelbosch 2006)

- When a $\frac{1}{2}$-PS network exists, it is unique and coincides with the unique stochastically stable network.
- An extension: p-pairwise stable set, unique, always exists, coincides with the $\frac{1}{2}$-PS network when it exists.
- Under Jackson and Watts (2002) stochastic process, the only networks that will arise with a significant frequency in the long run (i.e., stochastically stable networks) are in the $\frac{1}{2}$-PS set.

A non cooperative model of network formation (Bala and Goyal 2000)

- V. Bala and S. Goyal (2000), A non cooperative model of network formation, *Econometrica*
- Each agent in $N = \{1, \dots, n\}$ possesses some information and can communicate it with other people, via the setting up of pairwise links.
- A strategy of i is $g_i = (g_{i,1}, \dots, g_{i,i-1}, g_{i,i+1}, \dots, g_{i,n})$, where $g_{i,j} \in \{0, 1\}$ for $i \neq j$.
- A strategy profile $g = (g_1, \dots, g_n)$
- i has a link with j if $g_{i,j} = 1$, which can allow for:
  - one-way communication - if i can access j's information but not vice-versa, g is a directed network
  - two-way communication - if both i and j can access each other's information, g is a undirected network,
    the closure of g is $\overline{g} = cl(g)$ defined as $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$.

A non cooperative model of network formation (Bala and Goyal 2000)

- $N(i;g)$ set of all agents whose information i accesses through a (sequence of) link(s), $N^d(i;g) = \{j \in N \mid g_{i,j} = 1\}$
- $\mu_i(g) = \#N(i;g)$ = number of agents observed by i ("benefit")
  $\mu_i^d(g) = \#N^d(i;g)$ = number of agents with whom i has formed a link ("cost")
- The payoff function:
  - one-way flow: $\Pi_i(g) = \Phi(\mu_i(g), \mu_i^d(g))$
  - one-way flow model with decay:

$$\Pi_i(g) = 1 + \sum_{j \in N(i;g) \setminus i} \delta^{d(i,j;g)} - \mu_i^d(g)c$$

  - two-way flow: $\overline{\Pi}_i(g) = \Phi(\mu_i(\overline{g}), \mu_i^d(g))$
  - two-way flow model with decay:

$$\overline{\Pi}_i(g) = 1 + \sum_{j \in N(i;\overline{g}) \setminus i} \delta^{d(i,j;\overline{g})} - \mu_i^d(g)c$$

A non cooperative model of network formation (Bala and Goyal 2000)

- The dynamic process - a version of naive best response dynamics:
  - The game is repeated in each time period $t = 1, 2, \ldots$.
    In each $t \geq 2$ each agent observes the network of the previous period.
  - With probability $p_i \in (0, 1)$, i maintains the strategy chosen in the previous period, with probability $1 - p_i$ he chooses a myopic pure strategy best response to the strategy of all other agents in the previous period.
  - If there is more than one best response, choice with positive probability.

A non cooperative model of network formation (Bala and Goyal 2000)

- The absence of decay - the dynamic process converges to a limit network which is a strict Nash network:
  - for the one-way flow model: in all cases - empty, wheel (every agent bears an equal share of the cost)
  - for the two-way flow model: for a broad class of payoff functions: empty, center-sponsored star (the center bears the full cost)
- Low levels of decay - the set of strict Nash equilibria expands in both models, but many are extensions of the wheel and center-sponsored star, and also appear as limits of simulated paths of the dynamic process.

Stochastic stability in networks with decay (Feri, 2007)

- F. Feri (2007) Stochastic stability in networks with decay, *Journal of Economic Theory*
- The two-way flow network with decay (Bala and Goyal 2000)
- The process of network formation in a dynamic framework where agents can form and delete links and occasionally make mistakes.
- For a sufficiently large number of agents, a full characterization of stochastically stable networks for almost all possible values of parameters: complete, star or empty network according to link costs.

Payoffs in network formation models

- Stability concepts typically consider payoffs that are:
  - either immediately obtained from adding or deleting a link: myopic stability concepts, like e.g. pairwise stable sets (Jackson and Wolinsky 1996), pairwise myopically stable set (Herings et al. 2009)
  - or derived in the final (stable) network: farsighted stability concepts, like e.g. pairwise farsightedly stable set (Herings et al. 2009), farsightedly consistent set of networks (Page et al. 2005), von Neumann-Morgenstern pairwise farsightedly stable set
  - or intermediate payoffs (weighting of payoffs at all steps, including as special cases myopia and placing weight only on the final network), e.g. Dutta et al. (2005), Teteryatnikova (2017).

Non-myopic formation of circle networks (Watts 2002)

- A. Watts (2002) Non-myopic formation of circle networks, *Economic Letters*
- A dynamic version of Jackson and Wolinsky model (1996) with non-myopic agents
- Network Dynamics: agents initially unconnected, they meet over time and have the opportunity to form and sever links with each other:
  - In each period, i and j meet, and if they are unlinked, they can form a link with each other if both agree.
  - At the same time any agent can sever any of his existing links with the restriction that no agent can simultaneously form and sever a link.
  - The agents meet in a specific order (order of play).
  - After every pair of agents has met, the order of play starts over.

Non-myopic formation of circle networks (Watts 2002)

- Agents are forward seeking and discount the future at rate $0 \leq a \leq 1$.
- If agent i knows that in periods t, $t+1$, $t+2$, ... he will be a member of networks g_t, g_{t+1}, g_{t+2}, ..., respectively, then at the beginning of period t, i's non-myopic payoff u_i^t will be equal the summation of his discounted myopic payoffs:

$$u_i^t = u_i(g_t) + au_i(g_{t+1}) + a^2 u_i(g_{t+2}) + \ldots$$

where $u_i(g)$ is the utility function as in the connections model:

$$u_i(g) = \sum_{j \neq i} \delta^{d(i,j;g)} - \sum_{j:ij \in g} c$$

- Watts (2001): if agents are myopic ($a = 0$) and $c > \delta$, then no links will form even though all agents may be better off in a connected network.

Non-myopic formation of circle networks (Watts 2002)

Watts (2002) shows that:

- Under $c > \delta$, if some conditions on $(c - \delta)$ (with n, a, δ) hold true, then there exists an order of play for which the formation of a circle network is supported as a subgame perfect equilibrium.
- As the number of players increases, it becomes more likely that such an order of play exists.
- The order of play must be such that the circle forms as quickly as possible, which will minimize the number of periods where an agent receives a payoff of $\delta - c$.
- If the order of play is such that it takes more than n periods for the circle to form, then the formation of the circle is not ensured.

Farsighted network formation (Dutta et al. 2005)

- B. Dutta, S. Ghosal, D. Ray (2005) Farsighted network formation, *Journal of Economic Theory*
- A model of dynamic network formation when agents are farsighted: they evaluate a "current" move in terms of its consequences on the entire discounted stream of payoffs.
- Network formation process:
  - At any date, a pair of agents i and j is randomly chosen (with uniform probability) and endowed to take actions at that date.
  - Each of these agents can unilaterally sever any existing link with any other agent, and they can bilaterally form a link between the two of them if one doesn't exist.
  - These actions create a (possibly) new network, and then one-period payoffs are received according to the given allocation rule.
  - The current period then ends, and the whole process begins again *ad infinitum.*

Farsighted network formation (Dutta et al. 2005)

- A state is defined as a pair $s = (g, ij)$ where g is the current network and ij is the chosen active pair.
- A Markov strategy for any agent i is a probability distribution over possible actions at each state s in which i is an active agent.
- A strategy profile μ precipitates for each s some probability measure $\lambda(s)$ over the feasible set $F(s)$ of future networks starting from s.
- A Markov process is induced on the set S of states; while $\lambda(s)$ describes the movement to a new network, the given random choice of active agents moves the system to a new active pair.
- An equilibrium process of network formation is a strategy profile μ such that there is no active pair at any state s which can benefit (unilaterally or bilaterally) by departing from $\mu(s)$.

Farsighted network formation (Dutta et al. 2005)

- A state is strongly absorbing if it is absorbing (no deviation from that state) and the process converges to that state from every other state (the dynamic counterpart of a stable network in the static setting).
- Link monotonicity (LM) - an agent's payoff is increasing in the number of his own links
- Under LM, for all δ, there is some equilibrium μ^* such that the complete network g^N is strongly absorbing.
- Increasing returns to link creation (IRL) - along "increasingly connected" networks, there is a threshold network for which the worth turns nonnegative and then both aggregate and individual payoffs increase with extra links
- Under IRL (which implies the unique efficiency of g^N), for all δ sufficiently large, g^N is strongly absorbing for some equilibrium.

Networks and farsighted stability, Page et al. (2005)

- F. Page, M. Wooders and S. Kamat (2005) Networks and farsighted stability, *Journal of Economic Theory*
- farsightedly consistent directed networks
- supernetwork - collection of directed networks (the nodes) which provides (via the arcs connecting the nodes) a network representation of agents preferences and the rules governing network formation
- A subset of directed networks F_G is farsightedly consistent if given any network $G_0 \in F_G$ and any deviation to network $G_1 \in G$ by coalition S_1 (via adding, subtracting, or replacing arcs) there exists further deviations leading to some network $G_2 \in F_G$ where the initially deviating coalition S_1 is not better off - possibly worse off.
- For any supernetwork, there exists a nonempty subset of farsightedly consistent directed networks.

Farsighted stability, Jackson (2008)

- A farsighted improving path is a sequence of networks that can emerge when players add or delete links based on the improvement that the end network offers relative to the current network:
  - Each network in the sequence differs by one link from the previous one.
  - If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network.
  - If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network.
- A network is farsightedly pairwise stable if there is no farsighted improving path emanating from it.
- Some drawbacks: may fail to exist, it does not require that a farsighted improving path ends at a network that is stable itself

Farsightedly stable networks, Herings et al. (2009)

- P.J.J. Herings, A. Mauleon and V. Vannetelbosch (2009) Farsightedly stable networks, *Games and Economic Behavior*
- A set of networks G is pairwise farsightedly stable if the following conditions are satisfied:
  1. All possible pairwise deviations from any network $g \in G$ to a network outside G are deterred by a credible threat of ending worse off.
  2. There exists a farsighted improving path from any network outside the set leading to some network in the set (external stability condition).
  3. There is no proper subset of G satisfying the first two conditions.
- A pairwise farsightedly stable set of networks exists.

Cautious farsighted stability in network formation

- M. Teteryatnikova (2017), Cautious farsighted stability in network formation games with streams of payoffs, Working Paper
- A new notion of farsighted pairwise stability for dynamic network formation which always exists and includes two features:
  - consideration of intermediate payoffs, arbitrary (and heterogeneous) preferences and arbitrary weighting of payoffs
  - cautiousness.
- Cautious path stable set of networks (CPS): minimal set G satisfying external stability so that for any network outside G there exists a surely improving path relative to G leading to some network in the set, and no proper subset of G satisfies this condition.
- Any CPS satisfies internal stability, so that for any pair of networks in G, there does not exist a surely improving path (relative to G) between them.
- Efficient networks do not always belong to a CPS.

The co-evolution of links and actions in coordination games

- Modeling approach that jointly accounts for the adjustment of links and actions, it contributes to two research areas: network formation games and equilibrium selection/coordination problems.
- For survey, see e.g. F. Vega-Redondo (2016) Links and actions in interplay, In: Bramoullé et al. (2016)
- Network can change over time either by agents creating new links or/and destroying existing ones, in addition to their ability to adjust actions:
  - two-sided network formation - undirected networks (e.g. Jackson and Watts 2002)
  - one-sided network formation - directed networks (e.g. Goyal and Vega-Redondo 2005, Hojman and Szeidl 2006)

On the formation of interaction networks in social coordination games, Jackson and Watts (2002)

- M.O. Jackson, A. Watts (2002), On the formation of interaction networks in social coordination games, *Games and Economic Behavior*
- Endogenous choice of partners in social coordination games, with the possibility of mutation (on actions and links). In any period t:
  1. One link ij is chosen at random, and this is the only link that can be formed or severed at the given time (as in Watts 2001). After the addition/deletion choice, with a probability $\gamma \in (0,1)$ the choice is reversed by a tremble. The process determines a network g^t.
  2. One agent is randomly selected to adjust their strategy. The agent chooses the strategy that is a best response to g^t and to the previous action profile. After the choice is made, with a probability $\epsilon \in (0,1)$ the choice is reversed by a tremble. This determines a strategy profile a^t.

On the formation of interaction networks in social coordination games, Jackson and Watts (2002)

3. Agents play the coordination game with their neighbors and receive a payoff $u_i(g^t, a^t)$ determined by the coordination game played on a fixed network:

$$u_i(g; a_1^t, \ldots, a_n^t) = \sum_{i \neq j} \pi_{ij}(g)[v_i(a_i^t, a_j^t) - c(d_i)]$$

where $a_i^t \in \{A, B\}$ action, $v_i(a_i^t, a_j^t)$ payoff, $c(d_i)$ cost of maintaining each link, $\pi_{ij}(g) = 1$ if $ij \in g$ and $\pi_{ij}(g) = 0$ if $ij \notin g$.

- The process determines a finite state, irreducible, aperiodic Markov chain, and thus has a unique stationary distribution $\mu^{\gamma,\epsilon}$ over states, where states are now network/strategy configurations.

On the formation of interaction networks in social coordination games, Jackson and Watts (2002)

- A network/strategy configuration (g, a) is stochastically stable if it is in the support of this limiting distribution.
- The previous literature shows that:
  (1) with fixed interaction networks society coordinates on the risk-dominant equilibrium
  (2) with endogenous interaction patterns society coordinates on the efficient equilibrium.
- Jackson and Watts (2002) show that (1) may fail and we can see multiple stochastically stable states of play, and (2) may fail with endogenous networks, as stochastically stable states can include coordination on equilibria that are neither efficient nor risk-dominant.

Network formation and social coordination, Goyal and Vega-Redondo (2005)

- S. Goyal and F. Vega-Redondo (2005) Network formation and social coordination, *Games and Economic Behavior*
- A dynamic model: at regular intervals, agents choose links (one-sided network formation) and actions to maximize (myopically) their respective payoffs. Occasionally, they make errors. The interest is in the nature of long-run outcomes, when the probability of these errors is small.
- $N = \{1, \ldots, n\}$, directed network: $g = (g_1, \ldots, g_n)$, where $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$ set of links formed by i, $g_{i,j} \in \{1, 0\}$, i forms a link with j if $g_{i,j} = 1$
- $N(i; g) = \{j \in N \mid g_{i,j} = 1\}$
- Strategy of i: $s_i = (g_i, a_i)$, where $a_i \in \{A, B\}$ action of i

Network formation and social coordination, Goyal and Vega-Redondo (2005)

- Payoff of i: $\Pi_i(s_i, s_{-i}) = \sum_{j \in N(i;g)} \pi(a_i, a_j) - \#N(i;g)c$
- Dynamics:
  - At each $t = 1, 2, \ldots$, the state of the system is given by the strategy profile $s(t) = [(g_i(t), a_i(t))]_{i=1}^n$, S_i - the set of strategies of i
  - At every t, there is a probability $p \in (0, 1)$ that any given agent gets a chance to revise his strategy, and then he selects a new strategy

$$s_i(t) \in \arg\max_{\boldsymbol{s_i} \in \boldsymbol{S_i}} \Pi_i(s_i, s_{-i}(t-1))$$

- The strategy revision process defines a Markov chain on $S_1 \times \ldots \times S_n$ which has several absorbing states (they coincide with strict Nash equilibria of the one-shot game).

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- In the static setting:
  - A variety of networks (complete, empty, partially connected networks) can be supported at NE of the static (strategic-form) game.
  - The society can coordinate on different actions (conformism as well as diversity with regard to actions is possible at equilibrium).
- In the dynamic model:
  - Provided the costs of link formation are not too high, any network that occurs at stochastically stable states (i.e., robust to be observed a significant fraction of time in the long run) must be complete.
  - in the long-run states (where the social network is complete), agents always coordinate on the same action (social conformism), BUT:
    - if the costs of link formation are below a certain threshold, agents coordinate on the risk-dominant action;
    - if the costs are above the threshold, agents coordinate on the efficient action at all stochastically stable states.

The co-evolution of links and actions (cont.)

- D. Hojman and A. Szeidl (2006) Endogenous networks, social games and evolution, *Games and Economic Behavior*
  - one-sided model of network formation similar to Goyal and Vega-Redondo (2005), but with the assumption that the partners of any agent are not just his direct neighbors but also indirect contacts.
- The co-evolution of links and actions under strategic complementarities (e.g., König et al. 2014)
- The co-evolution of conventions and connections in coordination games in a setting where agents instead of choosing their neighbors directly, do it indirectly by selecting a "location" (e.g., Ely 2002, Bhaskar and Vega-Redondo 2004)
- The interplay of network-based interaction and link adjustment may play a key role in cooperation (literature not studied here).

Network evolution based on centrality, König and Tessone (2011)

- M. König and C.J. Tessone (2011) Network evolution based on centrality, *Physical Review E*
- Model of network evolution when the creation and removal of links are based on the position of nodes in the network measured by their centrality.
- The network evolution is independent of the particular centrality measure used.
- During the complete evolution, the emerging network structures are characterized by nestedness: the neighborhood of a node is contained in the neighborhood of the nodes with higher degrees.

Network evolution based on centrality, König and Tessone (2011)

- Dynamics:
  - Nodes initially connected by an arbitrary network. Each node has its centrality.
  - A node is randomly selected and modifies its neighborhood: with probability $p \in [0, 1]$ it creates a link to the node with the highest centrality it is not already connected to. With probability $1 - p$, a link of the selected node decays, and then the node removes the link to the neighbor with the lowest centrality.
  - If the node is connected to all the other nodes in the network (respectively it is isolated), and it has to create (respectively remove) a link, nothing happens.
- The model is an extension of König et al. (2014)

Nestedness in networks: A theoretical model and some applications, König et al. (2014)

- M. König, C.J. Tessone and Y. Zenou (2014) Nestedness in networks: A theoretical model and some applications, *Theoretical Economics*
- They use stochastic stability to identify the networks to which the network formation process converges (nested split graphs).
- Network formation process: two-stage game on two different time scales:
  - All agents simultaneously choose their effort level in a fixed network structure. It is a game of Ballester, Calvo-Armengol and Zenou (2006) with local complementarities where players have linear-quadratic payoff functions. The simultaneous move game has a unique Nash equilibrium which is characterized by the Bonacich centrality.
  - Agents receive linking opportunities at a given rate and decide with whom they want to form a link.